The Lindbladian Master Equation

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Abstract

This paper is the summary of a lecture given to the members of the Kapitza society at the University of Rochester in February 2022.
The issue of the Markovianity of the time evolution of non-closed system is discussed, along with some requirement for the validity of a Markovian approximation.
Finally, the paper develops some ideas surrounding the solution of linear differential equations, and uses them to develop an elementary treatment of the Lindbladian formalism, deriving the Lindbladian Master Equation. The case of a continuously monitored environment is discussed.

The content of this paper was mostly drawn from J. Preskill's lecture notes in quantum information theory. I claim no originality in the content reported below.

1 Introduction

Time evolution in quantum theory is generally described by a unitary transformation acting on the full Hilbert space or density matrix representing a quantum system. Such a transformation is usually obtained from the system's Hamiltonian by solving the related Schrodinger equation.
In practice, however, we don't have access to the full quantum system in general: the most common example is the interaction of the system under study with the environment, which is defined as anything that said system interacts with other than itself.
When considering a part of a quantum mechanical system, time evolution is not unitary nor Markovian anymore, and its treatment requires new tools.
In this paper we will focus on how this is done through the Lindbladian formalism. It turns out that, under the assumption of Markovianity, it is possible to obtain the time evolution of the accessible part of the system by solving a first order differential equation, just like in the closed system case.
In particular, one can derive a generalized version of the Hamiltonian, the Lindbladian, that describes the time evolution of the system through an equation analogous to Schrodinger's. This time evolution, however, will not be unitary

in general.

This paper presents a derivation of the Lindbladian Master Equation, starting from a discussion of Markovianity in Section 2, followed by a review of solutions of first order linear differential equations in Section 3. Finally, the LME is derived in Section 4.

2 Markovian approximation

One ingredient necessary for the existence of a differential equation that describes the evolution of a system is the requirement that its state at time $t + dt$ be completely determined by the state at time t. A system that satisfies this condition is called *Markovian*.

Markovianity is usually built in in quantum mechanics: the Schroedinger equation is a first order (in time) differential equation, and thus it is Markovian by definition, guaranteeing that the time evolution of a closed quantum mechanical system always has this property.

In the case we are interested in, however, there is no such guarantee: just like a unitary evolution of system and environment doesn't guarantee the unitary evolution of the system alone, a global Markovian evolution doesn't guarantee that the system evolves in that way.

In facts, we are guaranteed the opposite: the fluctuation-dissipation theorem tells us that the information that flows from the system into the environment will always flow back. Therefore, the state of the system at a time $t + dt$ might be influenced by its state at any prior time.

Despite this theoretical issues, there are many instances in which the system can be treated as if it was Markovian in practice, by making a so-called Markovian approximation.

To understand the validity of such an approximation one must consider three time scales involved in the scenario where a system A interacts with the environment E.

The first time scale, which we denote $\Delta t_{\mathbf{env}}$, is the one needed for the environment to "forget" the information that flowed out of the system. In practice, this means that after a time $t \gg \Delta t_{\mathbf{env}}$ we may ignore whatever information flowed out of the system.

The other physical quantity one needs is the time scale at which the system evolves, $\Delta t_{\mathbf{sys}}$. For the Markovian approximation to work we need, at the very least, $\Delta t_{\mathbf{sys}} \gg \Delta t_{\mathbf{env}}$.

After a more careful analysis, it turns out that we need to consider one more quantity to ensure that the approximation is indeed valid. Suppose, for example, that we wanted to simulate the time evolution of A. To do so, we

would have to coarse-grain time by sampling at a rate $(\Delta t_{\mathbf{coarse}})^{-1}$. Clearly, if $\Delta t_{\mathbf{coarse}} \gg \Delta t_{\mathbf{env}}$ then the evolution can be considered Markovian, but we also need $\Delta t_{\mathbf{sys}} \gg \Delta t_{\mathbf{coarse}}$, or we won't be able to faithfully simulate the system itself.

We therefore see that the Markovian approximation can only be used if there exists a time scale $\Delta t_{\mathbf{coarse}}$ satisfying:

$$\Delta t_{\mathbf{sys}} \gg \Delta t_{\mathbf{coarse}} \gg \Delta t_{\mathbf{env}} \tag{1}$$

[1] Mentions an atom interacting with an electromagnetic field as an example where this simulation is useful, and the case of an electron spin interacting with nuclear spins in a superconductor as a questionable one.

We will assume that the evolution of the systems under consideration is Markovian from now on. This assumption is necessary in the derivation of the LME presented in Section 4.

3 Solving linear differential equations

The aim of this paper is to construct the equivalent of a Hamiltonian for open systems. To understand the requirements for this object, it is useful to quickly review the theory of linear differential equations in general, and of the Schroedinger equation in particular.
Consider the following linear differential equation:

$$\frac{d}{dt}v = Av$$
$$v(0) = 0 \tag{2}$$

Where $v(0)$ is an element of some vector space V, and A is linear operator from V to itself.
Recall that (2) has a formal solution given by:

$$v(t) = e^{At}v(0) \tag{3}$$

Where $e^A = \sum\limits_{n=0}^{\infty} \frac{1}{n!}A^n$.
Clearly, one has to check that the exponential in (3) is a well defined object. In the cases we are going to consider, A is a bounded operator, and thus e^A is indeed well-defined.
The simplest example is when $V = \mathbb{R}$ and A is just a real number. Then the solution to (2) is the usual exponential growth/decay.
A more interesting example is the Schroedinger equation: here V is some Hilbert space and $A = -iH$ is the negative imaginary units times the (time-independent) Hamiltonian of the system. Then (2) becomes:

$$\frac{d}{dt}v = -iHv \tag{4}$$

With solution

$$v(t) = e^{-iHt}v(0) \tag{5}$$

Here e^{-iHt} is well defined as long as H is a bounded operator. Furthermore, since Hamiltonians are hermitian, e^{-iHt} is a unitary operator on V.

We see that this works even for the time evolution of the density matrix. In this case, A is the superoperator $-i[H, \cdot]$ which takes the density operator ρ to $-i[H, \rho]$. Note that we replaced $v \to \rho$ to follow the usual convention for density operators.

$$\frac{d}{dt}\rho = -i[H, \cdot]\rho \tag{6}$$

Which is solved by:

$$\rho(t) = e^{-i[H, \cdot]}\rho(0) \tag{7}$$

It can be shown that $e^{-i[H, \cdot]}\rho(0) = e^{iHt}\rho(0)e^{-iHt}$.

In the more general case of an open system, we are looking for a superoperator $\mathcal{L}$, called Lindbladian, that relates the derivative of the density matrix $\rho(t)$ to itself:

$$\frac{d}{dt}\rho = \mathcal{L}[\rho] \tag{8}$$

Which should then be solved by:

$$\rho(t) = e^{\mathcal{L}t}\rho(0) \tag{9}$$

Unlike the Hamiltonian in (6), we don't have requirements on $\mathcal{L}$ other than the fact that it should reduce to H in the case of an isolated system. For example, since $\mathcal{L}$ is not necessarily hermitian, $e^{\mathcal{L}t}$ will not be unitary in general.

Finding $\mathcal{L}$ explicitly will be the topic of the next section.

4   Master equation

We are now ready to derive the Lindbladian Master Equation for an open system.

First of all, recall that any superoperator (including $e^{\mathcal{L}t}$) has an operator-sum representation. In our case:

$$\rho(t) = e^{\mathcal{L}t}\rho(0) = \sum_{\mu \geq 0} M_\mu(t)\rho(0)M_\mu^\dagger(t)$$

Expanding $\rho(t)$ to first order, we get:

$$\rho(t + dt) = \rho(t) + \dot{\rho}dt = \sum_{\mu \geq 0} M_\mu \rho(t) M_\mu^\dagger \tag{10}$$

Note that the first equality relies on the Markovian assumption.

From (10) we see that M_μ must either be equal to the identity to zeroth order,

be of order $\sqrt{dt}$, or both. By lumping all the terms equal to the identity to zeroth order into M_0, we get:

$$M_0 = I + O(dt)$$

$$M_\mu = \sqrt{dt}L_\mu, \ \mu = 1, 2, 3... \tag{11}$$

We can go one step further and express M_0 as:

$$M_0 = 1 + (-iH + K)dt$$

Where H, K, L are of order zero in dt and H and K are hermitian.
As we will see, H is labeled this way because it has to equal the Hamiltonian of the system, although it's not obvious.
First, we see can use the normalization condition on the Kraus operators to find K:

$$1 = \sum_{\mu \geq 0} M_\mu^\dagger M_\mu = 1 + \left(-iH + iH + K + K + \sum_{\mu > 0} L_\mu^\dagger L_\mu \right) dt + O(dt^2)$$

And hence

$$2K + \sum_{\mu > 0} L_\mu^\dagger L_\mu = 0$$

Concluding that

$$K = -\frac{1}{2} \sum_{\mu > 0} L_\mu^\dagger L_\mu$$

And therefore

$$M_0(t)\rho M_0^\dagger(t) = \rho + dt \left(-i[H, \rho] + K\rho + \rho K \right) \tag{12}$$

Note how the term $-i[H, \rho]$ arises naturally here. In the case where the system is isolated, we wouldn't have the terms in K, thus reducing to the usual time evolution.

If we plug (11) and (12) into (10) we obtain:

$$\sum_{\mu \geq 0} M_\mu(t)\rho(0)M_\mu^\dagger(t) \sim \rho + dt \left(-i[H, \rho] + \sum_{\mu > 0} \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu \right) \right) \tag{13}$$

By combining (10) and (13) and comparing the first order terms we get:

$$\dot{\rho} = -i[H, \rho] + \sum_{\mu > 0} \left(L_\mu \rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu \right) = \mathcal{L}[\rho] \tag{14}$$

(14) is known as the Lindbladian master equation.
It is important to check that the time evolution of a density matrix is still a

density matrix. In particular, if $\rho(t)$ is a density operator, $\rho(t + dt)$ should be too.

Since H and ρ are assumed to be hermitian, we see that $-i[H, \rho]$ must be hermitian too. Furthermore, the product of an operator by its hermitian conjugate is also hermitian: $\left(L^\dagger L\right)^\dagger = L^\dagger \left(L^\dagger\right)^\dagger = L^\dagger L$.

Finally, $\left(L\rho L^\dagger\right)^\dagger = \left(L^\dagger\right)^\dagger (L\rho)^\dagger = L\rho^\dagger L^\dagger = L\rho L^\dagger$. Thus (14) preserves the hermiticity of ρ.

Note that $tr(\dot\rho) = 0$ is automatically enforced by the cyclical and linear properties of the trace:

$$tr(H\rho - \rho H) = tr(H\rho) - tr(H\rho) = 0$$

$$tr\left(L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu\right) = tr\left(L_\mu \rho L_\mu^\dagger\right) - \frac{1}{2}tr\left(L_\mu^\dagger L_\mu \rho\right) - \frac{1}{2}tr\left(\rho L_\mu^\dagger L_\mu\right) = 0$$

And thus (14) is trace preserving.

It can also be shown that (14) also preserves positivity [1] and thus it maps density matrices to density matrices.

4.1   Monitoring the environment

The master equation derived above describes a system interacting with an unobserved environment. In that case, we had to keep track of all possible changes to the system and the state. Here we examine the case in which the environment is continuously monitored. In this instance, quantum jumps are detected and thus affect the state of the system, so time evolution will be altered.

Note that the the probability of having a quantum jump associated to L_μ, $\mu \geq 1$ between times t and $t + dt$ is equal to $dt(\langle\psi(t)| L_\mu^\dagger L_\mu |\psi(t)\rangle$.

If there is a jump to the state associated to L_μ, then the new (normalized) state will be:

$$\psi(t + dt) = \frac{L_\mu |\psi(t)\rangle}{|L_\mu |\psi(t)\rangle|}$$

While, if there is no jump, the state will be:

$$\psi(t + dt) = \frac{M_0 |\psi(t)\rangle}{|M_0 |\psi(t)\rangle|}$$

This is known as Stochastic Schroedinger Equation. Unlike the Lindbladian Master Equation (14), the SSE is not deterministic: time evolution has a stochastic nature, as it depends on whether a given jump occurs.

On the other hand, the SSE is easier to simulate, as one only needs to deal with a d-dimensional state vector instead of a d^2-dimensional density matrix. One can then simulate the SSE many times and average over the results to recover the time evolution of the LME.

References

[1] J. Preskill. *Lecture Notes for Physics 229: Quantum Information and Computation.* California Institution of Technology, Pasadena, 1998.