The Weyl Equation

Benjamin Ostergaard

April 29, 2020

Abstract

We show how to form Dirac spinors and how to derive the Weyl equation. We start by forming Dirac spinors by stacking two Weyl spinors to account for parity. We then construct a group theoretic framework for the Weyl equation by investigating the transformation properties of Weyl spinors before constructing the Weyl Lagrangian and deriving the Weyl equation. We finish by deriving the Majorana mass term.

1 Introduction

This paper is adapted from the second half of chapter VII.3 of *Group Theory for Physicists in a Nutshell* by Anthony Zee. The Weyl equation is used to describe all spin ½ particles before the Higgs mechanism gives them mass. In that manner it can be regarded as a prerequisite to the Dirac equation from a group theoretic standpoint. We have already seen the Weyl spinors u and v corresponding to the (½, 0) and (0, ½) representations respectively. We have also seen how they transform the same for rotations and oppositely for boosts.

2 Weyl spinor to Dirac spinor

We know that electrons are described by a 4-component Dirac spinor. We cannot describe electrons, or any other spin ½ particle, due to parity violations. Under parity, $\boldsymbol{x} \to -\boldsymbol{x}$, and $\boldsymbol{p} \to -\boldsymbol{p}$, hence $\boldsymbol{J} \to \boldsymbol{J}$ $\boldsymbol{K} \to -\boldsymbol{K}$. This implies $\boldsymbol{J}_+ \leftrightarrow \boldsymbol{J}_-$. Physically, this implies that under parity transformations, the representations swap, (½, 0) $\leftrightarrow$ (0, ½).

We can describe an electron by using both of these 2-dimensional irreducible representations to form (½, 0) $\oplus$ (0, ½) which we call the Dirac spinor. We can then stack the two component spinors u and v form a 4-component Dirac spinor[1].

3 Group Theory leads us to the Weyl equation

We consider the Weyl spinors u and v separately. The equation that a Weyl spinor satisfies is called the Weyl equation. Since the Weyl equation violates

[1] When students see the Dirac equation for the first time, they might conclude that the spinors have 4 components because of the four-vectors seen in relativity. We can see here that this is merely a coincidence.

parity it was considered to be only of mathematical interest until 1956 when Lee and Yang proposed parity violation. The Weyl equation could then be used to describe neutrinos, which are massless[2]. It is now known that all spijn ½ particles, known as fermions, violate parity, thus they are all described by the Weyl equation before they acquire mass through the Higgs mechanism.

As we already know, $u^\dagger u$ is invariant under rotations, and $u^\dagger \sigma u$ transforms like a 3-vector. $u^\dagger u$ is not invariant under boosts. Instead we have,

$$u^\dagger u \rightarrow u^\dagger e^{\cdot} u \simeq u^\dagger (I + \ \cdot) u = u^\dagger u + u^\dagger u \tag{1}$$

where φ is infinitesimal. We can define the object $\omega^\mu \equiv (u^\dagger u,\ u^\dagger \sigma u)$ which transforms like a 4-vector.

We work with infinitesimal boosts in, without loss of generality, in the z direction. We have $u \rightarrow (I + \varphi \frac{\sigma_z}{2})u$, which we can write as $u = \frac{\sigma_z}{2} u$, where the infinitesimal φ has been absorbed. We take the hermitian conjugate and get $= \frac{\sigma_z}{2} u^\dagger$. Applying to $u^\dagger u$, we get,

$$(u^\dagger u) = (u^\dagger)u + u^\dagger \delta u = u^\dagger \sigma_z u \tag{2}$$

Then,

$$(u^\dagger \sigma_i u) = (u^\dagger)\sigma_i u + u^\dagger \sigma_i u = u^\dagger \tfrac{1}{2}\{\sigma_i,\ \sigma_z\} u = {}_{iz} u^\dagger u \tag{3}$$

That is $u^\dagger u$ and $u^\dagger \sigma_z u$ transform into each other while the x and y components are unaffected.

Let us introduce the forth Pauli matrix $\sigma^0 \equiv I$ so that we can write them as a 4-vector,

$$\sigma^\mu = (I,\ ) = (I,\ \sigma_1,\ \sigma_2,\ \sigma_3) \tag{4}$$

From a group theoretic perspective, what we are doing is simple. Under conjugation $\boldsymbol{J}_+ \leftrightarrow \boldsymbol{J}_-$ Thus, since $u \sim (\frac{1}{2},\ 0)$, then, $u^\dagger \sim (0,\ \frac{1}{2})$. Note that $u = u(t,\ x,\ y,\ z)$. Which we can Fourier transform into momentum space to get, $u = u(p)$.

4 The Weyl Lagrangian

In this context there are two important concepts about Lagrangians. The first is that if the Lagrangian is invariant under a symmetry transformation then there is a symmetry in the physics. The second is that taking the derivative of the Lagrangian with respect to a dynamical variable and setting it equal to zero gives the equation of motion for that variable.

We must construct a Lagrangian density that is invariant under Lorentz transformations. Write,

$$= i u^\dagger \sigma^\mu \partial_\mu u \tag{5}$$

[2] Recently, neutrinos were discovered to have a tiny, but nonzero mass making the mystery behind the physics that much more interesting.

We then take the derivative with respect to $u^\dagger$ and set it equal to zero, we get

$$\sigma^\mu \partial_\mu u = 0 \tag{6}$$

We then Fourier transform into momentum space and get,

$$\sigma^\mu p_\mu u = 0 \tag{8}$$

To make things more explicit we can write these equations in non-relativistic notation,

$$= iu^\dagger(\tfrac{\partial}{\partial t} + \ \cdot\ )u \tag{9}$$

$$(\tfrac{\partial}{\partial t} + \ \cdot\ )u = 0 \tag{10}$$

$$(E - \ \cdot\ )u = 0 \tag{11}$$

We then act on (11) with $(E + \ \cdot\ )$ from the left and obtain $E^2 = \boldsymbol{p}^2$ implying that the mass must be zero. (10) and (11) are the Weyl equation and they describe a spin ½ particle with $m = 0$ and $E = |\boldsymbol{p}|$.

Of course we can do the same thing with v. But we must lower the index of σ^μ by dotting with $\eta_{\mu\nu}$. This will change the sign on the $\boldsymbol{\sigma}$ terms, We get,

$$= iu^\dagger(\tfrac{\partial}{\partial t} - \ \cdot\ )v \tag{12}$$

$$(\tfrac{\partial}{\partial t} - \ \cdot\ )v = 0 \tag{13}$$

$$(E + \ \cdot\ )v = 0 \tag{14}$$

5 Handedness of Spinors

Since our particles are massless it makes sense to choose the direction of motion as the quantization axis for spin angular momentum. We define helicity as $h \equiv \ \cdot \frac{}{||}$. Since $E = |\boldsymbol{p}|$ this implies that the particle described by u has helicity +1 while the particle described by v has helicity -1. Note that under reflection, the helicity flips sign as $\boldsymbol{p} \rightarrow -\boldsymbol{p}$ and $\boldsymbol{J} \rightarrow \boldsymbol{J}$. In the diagram below we can see that the helicity flips if the mirror is placed parallel *or* perpendicular to a particle's direction of motion. Hence, Weyl particles violate parity.

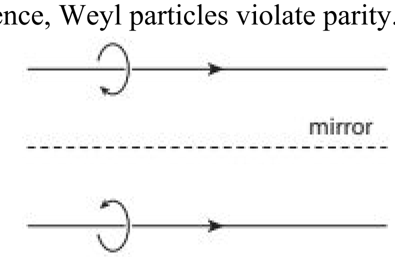

Figure 1

6 Majorana Mass term

We consider the possibility that there is another invariant term we can add to the Lagrangian for u. Consider $u^T\sigma_2 u.$ checking , we transpose, $u \to e^{i\cdot} u$ resulting in $u^T \to u^T e^{i\cdot^T}$. Note that ${}^T{}_2 =- {}_2$. Thus $u^T{}_2 u \to u^T e^{i\cdot^T}{}_2 e^{i\cdot} u = u^T{}_2 u$. It is invariant. It should be noted that $u^T{}_2 u$ vanishes identically since σ_2 is an antisymmetric 2x2 matrix. However, to proof of which is beyond this paper, the components of u must be anticommuting Grassman numbers, $u_1 u_2 =- u_2 u_1$.

We then add the term $mu^T{}_2 u$ to the Lagrangian for m constant. We also had the hermitian conjugate to keep the Lagrangian hermitian. We get,

$$= u^\dagger(\tfrac{\partial}{\partial t} + \ \cdot\ )u - \tfrac{1}{2}(mu^T{}_2 u + m^* u^\dagger{}_2 u^*) \qquad (15)$$

Again we take the derivative with respect to $u^\dagger$ and obtain,

$$i(\tfrac{\partial}{\partial t} + \ \cdot\ )u = m^*{}_2 u^* \qquad (16)$$

Now we act $(\frac{\partial}{\partial t} - \ \cdot\ )$ on (16) from the left and we get $\partial^2 u =- |m|^2 u$. The Weyl particle has become massive[3]! We could similarly add a mass term for v.

[3]We can keep the mass term real by redefining $u \to e^{i\alpha}u$ for some α.