University of Rochester

Group Theory for Physicists II

Professor Sarada Rajeev

The Spin Representations of SO(2n)

David Mayrhofer

PHY 391 Independent Study Paper

April 29$^{\text{th}}$, 2020

1 Introduction

At this point in the course, we had already shown that $SO(4)$ is isomorphic to the group $SU(2) \otimes SU(2)$. Therefore, one can define two subgroups of $SO(4)$ that are isomorphic to $SU(2)$. We can choose a representation of $SO(4)$ that makes is isomorphism to $SU(2)$ clear. The two representations of $SU(2)$ that are contained in $SO(4)$ are called the spinor representations of $SO(4)$. We build off of a previous lecture to define these spinor representations more generally for $SO(2n)$.

2 Overview of Previous Definitions

We have determined that the generators of $SO(2n)$ can be formed using combinations of $2n$ total γ matrices. These γ matrices are defined as follows

$$\gamma_{2k-1} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \tag{1}$$

$$\gamma_{2k} = \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{\text{Occurs } k-1 \text{ times}} \otimes \sigma_2 \otimes \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\text{Occurs } n-k \text{ times}}, \tag{2}$$

where $k = 1, \cdots, n$. The σ_i are the Pauli matrices, and 1 is the 2×2 identity matrix. The generators of $SO(2n)$ are defined as

$$\tau_{jk} = -\frac{i}{2}[\gamma_j, \gamma_k] = \begin{cases} -i\gamma_j\gamma_k, & j \neq k \\ 0, & j = k. \end{cases} \tag{3}$$

Also in a previous section we defined an additional matrix γ_F as

$$\gamma_F = (-i)^n \gamma_1\gamma_2 \cdots \gamma_{2n} \tag{4}$$

Since these τ_{ij} and γ_i formed by direct products, we can also write the states which these matrices act on as $|\epsilon_1\epsilon_2 \cdots \epsilon_n\rangle$. Then when we apply γ_F, we get

$$\gamma_F|\epsilon_1\epsilon_2 \cdots \epsilon_n\rangle = \left(\prod_{j=1}^{n} \epsilon_j\right) |\epsilon_1\epsilon_2 \cdots \epsilon_n\rangle. \tag{5}$$

Note that $\prod_{j=1}^{n} \epsilon_j = 1$ if there are an even number of negative ϵ_j and -1 if there are an odd number of negative ϵ_j. The right-handed spinor S^+ consists of all of the states in the former situation, whereas the left-handed spinor S^- is composed of the latter.

3 Examples

3.1 SO(2)

We have already defined the γ matrices for $SO(2)$ as

$$\gamma_1 = \sigma_1,\ \gamma_2 = \sigma_2 \implies \gamma_F = -i\gamma_1\gamma_2 = \sigma_3 \tag{6}$$

$SO(2)$ has only one generator, which we can write as

$$\tau_{12} = -i\gamma_1\gamma_2 = -i\sigma_1\sigma_2 = \sigma_3 \tag{7}$$

We note that the states for this system correspond to $|+\rangle$ and $|-\rangle$. The right and left handed spinor representations must then be $e^{i\theta}$ and $e^{-i\theta}$.

3.2 SO(4)

We now work through the relevant terms for $SO(4)$ using the formalism in section 2. We can write the γ_i as

$$\begin{aligned} &\gamma_1 = \sigma_1 \otimes \sigma_3,\ \gamma_2 = \sigma_2 \otimes \sigma_3,\ \gamma_3 = 1 \otimes \sigma_1,\ \gamma_4 = 1 \otimes \sigma_2 \\ &\gamma_F = \gamma_1\gamma_2\gamma_3\gamma_3 = \sigma_3 \otimes \sigma_3 \end{aligned} \tag{8}$$

We can then calculate each of the generators of $SO(4)$

$$\begin{aligned} &\tau_{12} = -i\gamma_1\gamma_2 = -i\left(\sigma_2 \otimes \sigma_3\right) = \sigma_3 \otimes 1, \\ &\tau_{31} = -\sigma_1 \otimes \sigma_2,\ \tau_{23} = \sigma_2 \otimes \sigma_2 \\ &\tau_{14} = -\sigma_1 \otimes \sigma_1,\ \tau_{24} = -\sigma_2 \otimes \sigma_1 \\ &\tau_{34} = 1 \otimes \sigma_3 \end{aligned} \tag{9}$$

One can verify that these satisfy the commutation relations for $SO(4)$. Here we calculate $[\tau_{12}, \tau_{23}]$ as an example.

$$[\tau_{12}, \tau_{23}] = [\sigma_3 \otimes 1, \sigma_2 \otimes \sigma_2] = [\sigma_3, \sigma_2] \otimes \sigma_2 = -2i\sigma_1 \otimes \sigma_2 = 2i\tau_{31} \tag{10}$$

To determine how we can form the spinor representations of $SO(4)$, we determine how these generators act on the states associated with this algebra. We first calculate the affect of $\tau_{!2}$ on $|{++}\rangle$ and $|{--}\rangle$ explicitly.

$$\begin{aligned} \tau_{12}\,|{++}\rangle &= \sigma_3\,|+\rangle \otimes 1\,|+\rangle = |{++}\rangle \\ \tau_{12}\,|{--}\rangle &= \sigma_3\,|-\rangle \otimes 1\,|-\rangle = -\,|{--}\rangle \end{aligned} \tag{11}$$

We see here that $\tau 12$ acts on the $|{++}\rangle$ and $|{--}\rangle$ in a way that is similar to how σ_3 acts on $|+\rangle$ and $|-\rangle$. We will find a similar relationship for the remaining generators when acting on these state. For each of the generators, we find

$$\begin{aligned}
\tau_{23}\left|{+}{+}\right\rangle &= -\left|{-}{-}\right\rangle\ ,\ \tau_{23}\left|{-}{-}\right\rangle = -\left|{+}{+}\right\rangle, \\
\tau_{31}\left|{+}{+}\right\rangle &= -\left|{-}{-}\right\rangle\ ,\ \tau_{31}\left|{-}{-}\right\rangle = -\left|{+}{+}\right\rangle, \\
\tau_{14}\left|{+}{+}\right\rangle &= -i\left|{-}{-}\right\rangle\ ,\ \tau_{14}\left|{-}{-}\right\rangle = i\left|{+}{+}\right\rangle, \\
\tau_{24}\left|{+}{+}\right\rangle &= -i\left|{-}{-}\right\rangle\ ,\ \tau_{24}\left|{-}{-}\right\rangle = i\left|{+}{+}\right\rangle, \\
\tau_{34}\left|{+}{+}\right\rangle &= \left|{+}{+}\right\rangle\ ,\ \tau_{34}\left|{-}{-}\right\rangle = -\left|{-}{-}\right\rangle.
\end{aligned} \tag{12}$$

We immediately see that these generators come in pairs. The generators τ_{12} and τ_{34} acting on these states both have the same effect on $\left|{+}{+}\right\rangle$ and $\left|{-}{-}\right\rangle$. Similarly, τ_{31} and τ_{24} can be paired together, as well as τ_{23} and τ_{14}. We can form three operators for$SU(2)$ from these pairs of generators,

$$\frac{1}{2}\left(\tau_{12}+\tau_{34}\right), \quad \frac{1}{2}\left(\tau_{31}+\tau_{24}\right), \quad \frac{1}{2}\left(\tau_{23}+\tau_{14}\right). \tag{13}$$

These acting on the states $\left|{+}{+}\right\rangle$ and $\left|{-}{-}\right\rangle$ represent $SU(2)$. We can do a similar thing by taking

$$\frac{1}{2}\left(\tau_{12}-\tau_{34}\right), \quad \frac{1}{2}\left(\tau_{31}-\tau_{24}\right), \quad \frac{1}{2}\left(\tau_{23}-\tau_{14}\right) \tag{14}$$

with $\left|{+}{-}\right\rangle$ and $\left|{-}{+}\right\rangle$ to form another representation of $SU(2)$. These are the S^+ and S^- representations, respectively. We note that the operators associated with S^+ acting on the states associated with S^- return 0. The operators of S^- with the states of S^+ behave the same way.

4 Real, Pseudoreal, or Complex

4.1 The Condition for Reality or Complexity

We are interested in the behavior of these spinor representations under complex conjugation. We recall that a representation is real or pseudoreal if $D(g)^* = CD(g)C^{-1}$ for some unitary matrix C and complex otherwise. Last semester, we proved that this condition is equivalent with the requirement that $\zeta^T C\psi$ is invariant under transformations $D(g)$, where ζ and ψ are states that are acted on by the representations of the group. Additionally, we know that the representation is real for S symmetric and pseudoreal for S antisymmetric.

For $SO(2n)$, we can write an arbitrary element as $D(g) = e^{\frac{i}{4}\omega_{ij}\tau_{ij}}$, where Einstein summation is used. Then the condition for the representation to be real or pseudoreal is

$$\zeta^T C\psi \to \zeta^T e^{\frac{i}{4}\omega_{ij}\tau_{ij}^T} C e^{\frac{i}{4}\omega_{ij}\tau_{ij}}\psi \approx \zeta^T C\psi + \frac{i}{4}\omega_{ij}\zeta^T\left(\tau_{ij}^T C + C\tau_{ij}\right)\psi. \tag{15}$$

The condition for invariance is then

$$\tau_{ij}^T C = -C\tau_{ij} \tag{16}$$

Since the τ_{ij} are Hermitian, this is equivalent with

$$C^{-1}\tau_{ij}^T C = C^{-1}\tau_{ij}^* C = -\tau_{ij} \tag{17}$$

We note that any C that satisfies this condition is not unique. For instance, if we have some C that does satisfy the above condition, $\gamma_F C$ will also be invariant, as γ_F commutes with all of the γ matrices.

4.2 Determining C

We determine a general form for C using an induction argument. We use $SO(4)$ as our base case. For $SO(4)$, we make a make an inspired guess[1] and choose $C = i\tau_2 \otimes \tau_1$. We check to see if Eq. (16) is satisfied for a specific τ_{ij}

$$\tau_{12}^T C = i\left(\sigma_3^T \otimes 1\right)(\sigma_2 \otimes \sigma_1) = i(\sigma_3\sigma_2 \otimes \sigma_1) = -i(\sigma_2\sigma_3 \otimes \sigma_1) = -C\tau_{12} \tag{18}$$

Checking the remaining τ_{ij} are left as an exercise.

With the base case satisfied, we then make the inductive step. Label the conjugation matrix for $SO(2n)$ as C_n. Our goal is then to find C_{n+1} in terms of C_n. We do this by first assuming that $C_{n+1} = C_n \otimes \kappa$, where κ is some 2×2 matrix. Earlier we had determined the form for the generators of $SO(2(n+1))$ in terms of the generators of $SO(2n)$. These relations are rewritten here for convenience.

$$\tau_{ij}^{(n+1)} = -i\gamma_i^{(n+1)}\gamma_j^{(n+1)} = \tau_{ij}^{(n)} \otimes 1 \tag{19}$$

$$\tau_{i,2n+1}^{(n+1)} = \gamma_i^{(n)} \otimes \sigma_2 \tag{20}$$

$$\tau_{i,2n+2}^{(n+1)} = -\gamma_i^{(n)} \otimes \sigma_1 \tag{21}$$

$$\tau_{2n+1,2n+2}^{(n+1)} = 1 \otimes \sigma_3, \tag{22}$$

where the superscripts denote the dimension of SO(2n) with which the object is associated. Inserting Eq.(22) into Eq.(17), we find that κ must either be σ_1 or $i\sigma_2$, where we have added an i to the σ_2 to ensure its arguments are real. Doing the same Eq. (20) and Eq. (21), we get

$$C_{n+1}^{-1}\tau_{i,2n+1}^{(n+1)\,T} C_{n+1} = C_n^{-1}\gamma_i^{(n)\,T} C_n \otimes (\kappa^{-1}\sigma_2\kappa) = -\gamma_i^{(n)} \otimes \tau_2 \tag{23}$$

$$C_{n+1}^{-1}\tau_{i,2n+2}^{(n+1)\,T} C_{n+1} = -C_n^{-1}\gamma_i^{(n)\,T} C_n \otimes (\kappa^{-1}\sigma_2\kappa) = \gamma_i^{(n)} \otimes \tau_1 \tag{24}$$

Equating the first terms in the direct products, we find as a condition for C_n

$$C_n^{-1}\gamma_i^{(n)\,T} C_n = (-1)^n \gamma_i^{(n)} \tag{25}$$

Combining this with the expression for κ, we find

$$C_{n+1} = \begin{cases} C_n \otimes \sigma_1 & \text{for } n \text{ odd} \\ C_n \otimes i\sigma_2 & \text{for } n \text{ even} \end{cases} \tag{26}$$

[1] The reader is welcome to credit whatever source they so desire for this inspiration.

We can also write out C_n explicitly now as

$$C_n = i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \cdots \tag{27}$$

4.2.1 Symmetric or Antisymmetric?

We naturally want to know if C_n is symmetric or symmetric, as such a feature distinguishes between real and pseudoreal representations. We do so by defining some set $\{a_n\} \in \mathbb{Z}$ such that $C_n^T = (-1)^{a_n} C_n$. Combine this with Eq. (27) to obtain the recursion relation $(-1)^{(a_{n+1})} = (-1)^{a_n+n+1}$. Solving this, we find $a_n = \frac{1}{2}n(n+1)$. Now, for the relationship between C_n^T and C_n we have

$$C_n^T = (-1)^{\frac{1}{2}n(n+1)} C_n \tag{28}$$

Therefore, we see that for $n = 2, 5, 6, 9, \cdots$ the matrix C_n is antisymmetric and for $n = 3, 4, 7, 9, \cdots$ the matrix C_n is symmetric.

4.2.2 Alternate Definition of C

We can also determine C by looking at the symmetry of the γ_i. It is easy to show that $\gamma_i^T = (-1)^{i+1}\gamma_i$, i.e. the γ_i are antisymmetric when i is even and symmetric when i is odd. We can rewrite Eq. (16) as

$$\begin{aligned} (\gamma_i\gamma_j)^T C = \gamma_j^T \gamma_i^T C = -\gamma_i^T \gamma_j^T C = -C\gamma_i\gamma_j \\ \implies \gamma_i^T \gamma_j^T C = C\gamma_i\gamma_j \end{aligned} \tag{29}$$

The lefthand side of the equation reduces to the following cases:

$$\gamma_i^T \gamma_j^T = \begin{cases} \gamma_i\gamma_j & \text{if one of } i \text{ or } j \text{are odd} \\ -\gamma_i\gamma_j & \text{if both } i \text{ and } j \text{ are odd or even} \end{cases} \tag{30}$$

If the first case is true, then Eq. (29) is $\gamma_i\gamma_j C = -C\gamma_i\gamma_j$. If the second case is true, then it is $\gamma_i\gamma_j C = C\gamma_i\gamma_j$. Both of these conditions are satisfied when C is a product of even γ_i. For example, in the case of $n = 3$, $C = \gamma_2\gamma_4\gamma_6$.

4.3 The Conjugate Spinor

We return to the task of determining the reality of the spinor representations. First we define projection operators $P_\pm$ as

$$P_\pm = \frac{1}{2}(1 \pm \gamma_F) \tag{31}$$

These projection operators will project out only the coordinates associated with the right or left-handed spinor representation from some spinor ψ. Then, when we transform some spinor, we have $\psi \to e^{\frac{i}{4}\omega_{ij}\tau_{ij}} P_\pm \psi$. The conjugate of ψ will then transform as $\psi^* \to e^{-\frac{i}{4}\omega_{ij}\tau_{ij}^*} P_\pm^* \psi^*$. The question of whether or not the spinor representation is real or complex then reduces to determining if there is a C such that

$$C^{-1}e^{-\frac{i}{4}\omega_{ij}\tau_{ij}^*}P_{\pm}^* = e^{\frac{i}{4}\omega_{ij}\tau_{ij}}P_{\pm}C \tag{32}$$

If there was, then the conjugate spinor ψ_c would transform as

$$\psi_c = C^{-1}\psi^* \to C^{-1}e^{-\frac{i}{4}\omega_{ij}\tau_{ij}^*}P_{\pm}^*\psi^* = e^{\frac{i}{4}\omega_{ij}\tau_{ij}}P_{\pm}C^{-1}\psi^*, \tag{33}$$

which is the same way ψ transforms. We can rearrange the above relationship slightly to get

$$C^{-1}\tau_{ij}^*\frac{1}{2}(1 \pm \gamma_F)^*C = -\tau_{ij}\frac{1}{2}(1 \pm \gamma_F) \tag{34}$$

To simplify this expression, we use the fact that we can write $\gamma_F = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$ along with Eq. (27) to show

$$C_n^{-1}\gamma_F C_n = (-1)^n\gamma_F \tag{35}$$

This, along with Eq. (16), can be used to show

$$C^{-1}\tau_{ij}^*\frac{1}{2}(1 \pm \gamma_F)^*C = C^{-1}\tau_{ij}^T\frac{1}{2}(1 \pm \gamma_F)C = \tau_{ij}\frac{1}{2}\left(1 \pm (-1)^n\gamma_F\right) = \begin{cases} -\frac{1}{2}\tau_{ij}P_{\pm} & \text{if } n \text{ is even} \\ -\frac{1}{2}\tau_{ij}P_{\mp} & \text{if } n \text{ is odd} \end{cases} \tag{36}$$

This means that when n is even, taking a complex conjugate of S^+ just maps the representation to itself, and likewise of S^-. However, when n is odd, complex conjugation takes S^+ to S^- and S^- to S^+.

This is consistent with thinking of the S^+ and S^- as states represented by Dirac notation. Recall that $S^{\pm}$ are associated with states $|\epsilon_1 \cdots \epsilon_n\rangle$ where $\prod_j \epsilon_j = \pm 1$. Since we know from Eq. (27) C is an alternating product of σ_1 and σ_2, each ϵ_j is flipped when applying C to the state. Therefore, when n is even, an even number of states are flipped, and the total product of the ϵ_j is unchanged. This means that C acting on some state associated with $S^{\pm}$ maps it to $S^{\pm}$. When n is odd, the sign of product of the ϵ_j is flipped, so $S^{\pm}$ is mapped to $S^{\mp}$.

All that remains is to then determine which representations are real and which are pseudoreal. Since we already determined the symmetry or antisymmetry of C, however, this becomes fairly trivial. We already know that our candidates are strictly for n even, so we can take $n = 2k$. From Eq. (28), we then know $C^T = (-1)^{k(2k+1)}C = (-1)^k C$. Therefore, we can summarize the reality of $SO(2n)$ in the following table.

$SO(4k+2)$	complex
$SO(8(k+1))$	real
$SO(8k+4)$	pseudoreal

5 Spinor Representation for Odd Dimensional SO(n)

Thus far, we have only considered the case of the even dimensional $SO(n)$. It is natural to ask how one can generate these spinor representations for the odd dimensional case. There are two options. The first option is to start from a higher dimensional algebra and decrease dimension by one to get $SO(2n-1)$. The second option is then to start from a lower dimensional $SO(2n)$ and move to $SO(2n+1)$.

The first option is extraordinarily simple. If we are given $SO(2n)$, to form the γ_i for $SO(2n-1)$ we simply take the same γ_i as $SO(2n)$, but leave out γ_{2n}. There is a subtlety here, however. The γ_F for $SO(2n-1)$ will still include γ_{2n}, as this is needed to commute with all of the other γ_i. There is no problem with this, as γ_{2n} can still be defined for $SO(2n-1)$ even if it is not considered to be one of the γ_i.

The second option is similarly straightforward. To obtain the necessary γ_i for $SO(2n+1)$, just add the γ_F to the set of γ_i for $SO(2n)$. Alternatively, one could also add

$$\gamma_{2n+1} = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_3, \tag{37}$$

which satisfies the same symmetry properties needed for the additional γ matrix.

5.1 C in SO(2n+1)

To find C, we will use a similar definition to what we used in $SO(2n)$, but with some small changes. First, we must note that C will be half as big in $SO(2n-1)$. However, C will have the same dimensions in $SO(2n+1)$ as in $SO(2n)$. We can see how this is so by recalling that we could write C as a product of the even γ_i. When adding one additional matrix, we are not changing the number of even γ_i, so C will be the same in $SO(2n)$ and $SO(2n+1)$.

6 System of n Fermions

There is an interesting relationship between the spinor representations of $SO(2n)$ and a system with n fermions. This has already been hinted at in the use of the Dirac notation to describe the states on which the generators of $SO(2n)$ act.

6.1 Review of Fermionic Systems

It is well known that fermions (particles of half-integer spin) obey the Pauli Exclusion Principle. That is, given some quantum state for a fermion, no more than one fermion can occupy that state. Therefore, for a system with one possible state, we can describe the system as either $|0\rangle$ or $|1\rangle$. This is a contrast from bosons (particles of integer spin), which can have any number of particles occupying a single state. However, in spite of this difference, we are still encouraged to define creation and annihilation operators for the fermionic states in the same spirit as we do for the bosonic states. In the case of fermions, we will define two operators as follows

$$b\,|1\rangle = |0\rangle\,, \quad b^\dagger\,|0\rangle = |1\rangle \tag{38}$$

$$b\,|0\rangle = 0, \quad b^\dagger\,|1\rangle = 0. \tag{39}$$

Following the same naming conventions as in the bosonic case, we call b the annihilation operator, and $b^\dagger$ the creation operator. Combing these relations, we find that

$$\left(bb^\dagger + b^\dagger b\right)|0\rangle = |0\rangle\,, \tag{40}$$

$$\left(bb^\dagger + b^\dagger b\right)|1\rangle = |1\rangle\,. \tag{41}$$

This means that $\{b, b^\dagger\} = 1$, where the curly braces indicate the anti-commutator.[2] We can define a number operator, $N = b^\dagger b$, which will indicate how many particles are in the state. Also, since we can neither put more than one particle in a state nor annihilate the vacuum state, we must also have

$$\left(b^\dagger\right)^2 = 0, \quad b^2 = 0. \tag{42}$$

We can find matrix representations of these operators as well if we write the states of the systems as column vectors.

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{43}$$

$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{44}$$

Looking at these matrix representations, we see that we can also write these creation and annihilation operators in terms of the Pauli matrices

$$b^\dagger = \frac{\sigma_1 + i\sigma_2}{2},\ b = \frac{\sigma_1 - i\sigma_2}{2} \tag{45}$$

Based on the earlier work on $SO(2)$, we would expect to be able to connect that group to this system in some way. Recall the generator for $SO(2)$ is σ_3. We can write this generator in terms of these operators as

$$\sigma_3 = 2N - 1 = 2b^\dagger b - 1 = b^\dagger b - bb^\dagger. \tag{46}$$

This is a linear combination of a multiplication of these creation and annihilation operators. We expect, then, when we go to the case of n fermions that we can form the generators of $SO(2n)$ from linear combinations of these creation and annihilation operators.

6.2 Fermionic System with n States

Previously we had considered the case of a system with a single energy state available. Now take the case of a system with n possible states. We write such a system as $|\epsilon_1 \cdots \epsilon_n\rangle$. We note that while previously we had defined the ϵ_j as ± 1, here we are defining them as . While this notation is used to draw the comparison to the number of fermions in a state, the two labels provide equivalent descriptions.

[2]Compare to the canonical bosonic relationship $[a, a^\dagger] = 1$ obtained when setting $\hbar = \omega = 1$.

Now there are a total of n operators which satisfy

$$\{b_i, b_j^\dagger\} = \delta_{ij}\ ,\ \ \{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0 \tag{47}$$

The number operator for the j^{th} state is defined as $N_j = b_j^\dagger b_j$. We have for this operator acting on the state $N_j\,|\epsilon_1 \cdots \epsilon_n\rangle = \epsilon_j\,|\epsilon_1 \cdots \epsilon_n\rangle$.

We now set out to determine the generators of $SO(2n)$ from these operators. As mentioned before, we expect the generators to come from linear combinations of $b_i b_j$, $b_i^\dagger b_j^\dagger$, $b_i^\dagger b_j$, and $b_i b_j^\dagger$. However, note that due to the canonical commutation relation the b_i and $b_i^\dagger$ satisfy, the difference between $b_i^\dagger b_j$, and $b_i b_j^\dagger$ is only a constant term. Therefore, we only consider the first three bilinear operators. By using these anticommutation relations, we can determine that there are a total of $n(n-1)/2$ unique operators for each $b_i b_j$ and $b_i^\dagger b_j^\dagger$. However, there are n^2 terms for $b_i^\dagger b_j$. These add up to $2n(2n-1)/2$, which is equal to the number of generators for $SO(2n)$.

7 Conclusion

These spinor representations of $SO(2n)$ have numerous uses in physics. Perhaps the most obvious and well known uses are in the Weyl and Dirac equations, where the state of the particle of interest is represented by a spinor. However, it also has important applications in condensed matter physics. One example of this is the 2-d Ising model. While Onsager provided the original solution to the 2-d Ising model using Lie theory, Kaufman was able to simplify the solution significantly by relating the problem to a system of many free fermions. By using this spinor representation along with a few other tools, Kaufman was able to successful determine the partition function for the 2-d Ising model exactly.

8 References

[1] Zee, A., *Group Theory in a Nutshell for Physicists*, Chapter VII.1, Princeton University Press, 2016.

[2] Rajeev, Sarada. PHY 511 Field Theory Lecture Notes. 2019.

[3] Kaufamn, Bruria, Phys. Rev. **76**, 1232 (1949)