The Lorentz Group and Relativistic physics.

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Abstract

In this paper, we introduce the Lorentz group which is key in our understanding of relativistic physics. We start our discussion by introducing transformations and in particular the Galilean transformation. We transition into the Lorentz transformation which we derive in three ways, each of which builds our understanding of the Lorentz group. Finally, we formally define the Lie algebra $SO(m,n)$ by introducing its generators and determining how they commute. Our discussion will emphasize the group theoretic aspects instead of the physics of special relativity.

1 Relative motion and transformations

The laws of physics for observers in relative motion are related. We denote the transformation that takes the laws of physics as experienced by observer 1 to those seen by observer 2 by $T(1 \to 2)$.
A fundamental postulate of physics states that:

$$T(2 \to 3)T(1 \to 2) = T(1 \to 3) \tag{1}$$

This postulate asserts that the relativity of motion defines a group where the transformation T reflects our understanding of space and time. Its form has been a fundamental question in twentieth-century physics. The two forms of concern were the Galilean and the Lorentzian transformation.

2 Galilean Transformation and Addition group

The two dimensional representation

$$D(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \tag{2}$$

denotes the addition group where

$$D(u)D(v) = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u+v & 1 \end{pmatrix} = D(u+v)$$

$D(u)$ also represents the Galilean group of non-relativistic physics such as:

$$D(u) \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

Where:

$$\{ \begin{matrix} t' = t \\ x' = ut + x \end{matrix} \tag{3}$$

t is the time coordinate while x is the space coordinate. Equations 3 relate the time and space coordinates of two observers in uniform motion with velocity u with respect to each other. We can see that the representation in equation 2 translates to the addition of relative velocity.

We now know that time is not absolute which means that $t = t'$ is incorrect. This was historically known as the fall of absolute time. The realization that electromagnetism is not invariant under the Galilean transformation led Einstein to formulate special relativity. He showed that the Galilean transformation is in fact an approximation of the Lorentzian transformation in the non -relativistic regime. Before then, it wasn't known that the speed of light is universal (that it does not depend on the observer) which was the key to modifying the Galilean transformation. In the next chapter, we outline three derivations of the Lorentz transformation.

3 Derivations of the Lorentz transformation

3.1 Derivation 1 (brute force)

In Einstein's thought experiment, light is bouncing off between two mirrors, separated by a distance y, moving in a direction that is perpendicular to the separation. We consider two frames with two sets of coordinates, a primed and a non-primed. In the primed frame, two events that give off light happen at the same place at different times, and in the non-primed frame, the events happen wherever and whenever. If we are considering a $(1+1)$ dimensional spacetime, then we can call the spatial separation between the two events Δx, and temporal separation Δt, for the non primed coordinates. Similarly, for the primed coordinates, we have $\Delta x' = 0$ and $\Delta t'$.
By a simple Pythagorean theorem calculation, the total distance traveled by the light beam in the non primed frame would be $2\sqrt{y^2 + (\Delta x/2)^2}$ which is equivalent to $c\Delta t$ where c is the speed of light. Therefore, for the non-primed frame, we have:

$$\mathrm{c^2 \Delta t^2 - \Delta x^2 = (2y)^2}$$

In the primed frame, the total distance that light traveled is just $2y$ which implies:

$$\mathrm{c^2 \Delta t'^2 = (2y')^2}$$

As a result, the interval measured in any frame is the same as the interval measured in the frame where the events happen in the same place (primed). Thus, the interval is the same for any frame. We can also note that $\mathrm{(2y)^2}$ is constant since y is unaffected by the motion of the mirrors in any frame (since its perpendicular to the separation of the mirrors), thus the spacetime interval is the same in every frame.

In the infinitesimal limit,

$$\left(cdt'\right)^2 - \left(dx'\right)^2 = (cdt)^2 - (dx)^2 \tag{4}$$

We replace the Galilean transformation from Equation 3 by:

$$\left\{ \begin{array}{c} t' = w\left(t + \zeta ux/c^2\right) \\ x' = k(ut + x) \end{array} \right. \tag{5}$$

Where w, k, and ζ are three unknown dimensionless functions of $\frac{u}{c}$. We plug 5 into 4 and solve for w, k, and ζ. We obtain:

$$ct' = \frac{ct + ux/c}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{6}$$

and,

$$x' = \frac{ut + x}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{7}$$

To recover the Galilean transformation, we can simply take c to ∞. From now on, we will use $c = 1$ since it is more convenient to use ct than t in equation 6 for instance.

3.2 Derivation 2

Let dx denote the column vector $\begin{pmatrix} dt \\ dx \end{pmatrix}$ and let η represent the 2-by-2 matrix known as the Minkowski metric such that : $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$ for $(1+1)$ dimensional spacetime.
Therefore, we can write $(dt)^2 - (dx)^2$ as $dx^T \eta dx$.
We also denote the Lorentz transformation by L such that: $\begin{pmatrix} dt' \\ dx' \end{pmatrix} = L \begin{pmatrix} dt \\ dx \end{pmatrix}$
As a result, we can write equation 4 of the invariant space interval as

$$dx'^T \eta dx' = dx^T L^T \eta L dx = dx^T \eta dx$$

.
For an arbitrary dx, this implies that:

$$L^T \eta L = \eta \tag{8}$$

Consider an infinitesimal transformation with:

$$L \simeq I + i\varphi K$$

Where φ is an infinitesimal real parameter and k is a generator that we will solve for. Notice that this is similar to our discussion of the rotation group with the equivalence: $R \to L$ and $I \to \eta$ and thus, $R \simeq I + i\vec{\theta}\vec{J} \to L \simeq I + i\varphi K$.

Plugging in equation 8, we obtain $K^T\eta + \eta K = 0$ which implies $K^T\eta = -\eta K$. We know that $\eta = \sigma_3$ and that $\{\sigma_1, \sigma_3\} = 0$ therefore

$$iK = \sigma_1$$

.
Thus, an immediate solution is:

$$iK = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

3.3 Derivation 3

We introduce the light cone coordinates: $x^{\pm} = t \pm x$. Notice that if the spacetime interval is left invariant if either dx^+ or dx^+ is multiplied and the other one divided by the same quantity. We obtain the Lorentz transformation if we let this quantity be e^{φ}. For instance, we have $t = \frac{1}{2}\left(x^+ + x^-\right)$ which becomes

$$t = \frac{1}{2}\left(e^{\varphi}x^+ + e^{-\varphi}x^-\right) = \frac{1}{2}\left(e^{\varphi}(t+x) + e^{-\varphi}(t-x)\right) = \cosh\varphi t + \sinh\varphi x \tag{10}$$

We can say that the Lorentz transformation stretches and compresses the light cone coordinates by the same quantity.

In summary, the Lorentz tranformation , to leading order, is:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} \simeq (I + i\varphi K)\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ \varphi & 1 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} \tag{11}$$

Where $\begin{pmatrix} 1 & \varphi \\ \varphi & 1 \end{pmatrix}$ is an approximate representation of the additive group since:

$$\begin{pmatrix} 1 & \varphi \\ \varphi & 1 \end{pmatrix}\begin{pmatrix} 1 & \chi \\ \chi & 1 \end{pmatrix} = \begin{pmatrix} 1+\varphi\chi & \varphi+\chi \\ \varphi+\chi & 1+\varphi\chi \end{pmatrix}$$

which is the addition matrix if we ignore the quadratic terms.

4 Reconstruction of Finite Transformations

We build up to finite transformations the same way we did with rotations in a previous lecture using the group's multiplicative structure. Reproducing:
We replace $L \simeq (I + i\varphi K)$ by $L = e^{i\varphi K}$. We expand the exponential as a Taylor series and separate the expansion into even and odd terms to obtain the finite transformation as follows:

$$
\begin{aligned}
L(\varphi) = e^{i\varphi K} &= \sum_{n=0}^{\infty} \varphi^n (iK)^n / n! = \left(\sum_{k=0}^{\infty} \varphi^{2k}/(2k)!\right) I + \left(\sum_{k=0}^{\infty} \varphi^{2k+1}/(2k+1)!\right) iK \\
&= \cosh\varphi I + \sinh\varphi iK \\
&= \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}
\end{aligned} \tag{12}
$$

Where we used $(iK)^2 = I$.

As a result, we have:

$$
\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{13}
$$

This implies that $x' = \sinh\varphi t + \cosh\varphi x$. By comparison with equation 7 we have:

$$
u = \frac{\sinh\varphi}{\cosh\varphi} = \tanh\varphi
$$

which means $\cosh\varphi = \frac{1}{\sqrt{1-u^2}}$ and $\sinh\varphi = \frac{u}{\sqrt{1-u^2}}$ in line with equations 6 and 7.

The matrix $\begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$ is the exact representation of the additive group. We can see that by writing $e^{i\varphi_1 K} e^{i\varphi_2 K} = e^{i(\varphi_1+\varphi_2)K}$. Note that the matrix from 11 is the small angle approximation of $\begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$.

5 SO(m,n)

In the previous sections, we noticed parallels between our discussion of the Lorentz transformations in $(1+1)$ dimensional spacetime and our discussion of rotation in 2 dimensional space. Consider the following:

Rotations	Lorentz transformations
$d\vec{x}' = Rd\vec{x}$ where: $d\vec{x} = \left(dx^1, dx^2, \cdots, dx^N\right)$	$d\vec{x}' = Ld\vec{x}$ where $d\vec{x} = (dx^1, dx^2, \cdots,\ dx^{(m+n)})$
$ds^2 = \sum_{i=1}^{N} \left(dx^i\right)^2$ is unchanged	$ds^2 = \sum_{i=1}^{m} \left(dx^i\right)^2 - \sum_{i=m+1}^{m+n} \left(dx^i\right)^2$ is unchanged

The invariant quantity for the rotation transformation represents the distance squared between two nearby points while the invariant quantity of the Lorentz transformation represents the generalized distance squared. This set of Lorentz transformations defines the group $SO(m,n)$ that satisfies all the group axioms.

Prior to this section we worked with the $(1+1)$ dimensional spacetime which is $SO(1,1)$ which analytically continues from $SO(2)$ if you set $t = iy$ yielding $-dt^2 + dx^2 = dy^2 + dx^2$. By setting $\varphi = i\theta$, we can continue the Lorentz transformation

$$t' = \cosh\varphi t + \sinh\varphi x, x' = \sinh\varphi t + \cosh\varphi x$$

to

$$y' = \cos\theta y + \sin\theta x, x' = -\sin\theta y + \cos\theta x$$

which is the rotation transformation.

6 SO(3,1)

$SO(3,1)$ refers to the $(3+1)$ dimensional spacetime with 3 spatial coordinates. We already know that we can boost or rotate in/w.r.t each spatial coordinate. The Lie Algebra $SO(3,1)$ consists of six generators: three rotational generators J_x, J_y, J_z and three Lorentz boost generators K_x, K_y, K_z. From the rotation group lecture, we know the explicit form of J_x, J_y, and J_z.
Let $J_x \equiv -i\mathcal{J}_x, J_y \equiv -i\mathcal{J}_y, J_z \equiv -i\mathcal{J}_z$ which makes them antisymmetric and hermitian. Recall:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

For the Lorentz boost generators, we generalize 9 to:

$$iK_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad iK_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad iK_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

iK_j is real symmetric and therefore hermitean. Thus, K_j is imaginary symmetric and thus antihermitean which makes $L = e^{i\varphi K} = e^{\varphi(iK)}$ not unitary but obeys equation 8.

In the following discussion, we will use differential operators instead of matrices, as introduced in previous lectures. For our purposes, we have

$$J_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \quad iK_x = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad iK_y = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} \quad \text{, and} \quad iK_z = t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

As demonstrated in a previous lecture, the rotational generators commute as follows:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \tag{16}$$

We try commuting a rotational generator with a Lorentz boost generator as follows:

$$\begin{aligned}[J_z, iK_x] &= i\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\right] \\ &= i\left(\left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial t}\right] - \left[x\frac{\partial}{\partial y}, t\frac{\partial}{\partial x}\right]\right) \\ &= i\left(y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}\right) = i\left(iK_y\right)\end{aligned} \tag{17}$$

In general,

$$[J_i, K_j] = i\varepsilon_{ijk}K_k \tag{18}$$

Therefore, the J' s rotate the three K's into one another. We expect the commutators between K's to produce J' s since the algebra closes, we try:

$$\begin{aligned}[K_x, K_y] &= (-i)^2\left[t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}\right] \\ &= \left(\left[y\frac{\partial}{\partial t}, t\frac{\partial}{\partial x}\right] - \left[x\frac{\partial}{\partial t}, t\frac{\partial}{\partial y}\right]\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -iJ_z\end{aligned} \tag{19}$$

In general,

$$[K_i, K_j] = -i\varepsilon_{ijk}J_k \tag{20}$$

We learned in a previous lecture that $SO(4)$ falls apart into two pieces which is also manifested in $SO(3,1)$ since this latter can be obtained analytically from $SO(4)$. We start by defining:

$$J_{\pm,i} = \frac{1}{2}\left(J_i \pm iK_i\right)$$

which is hermitian since J_i and K_i are both hermitian. Using equations 16, 18, and 20 we write:

$$\begin{aligned}[J_{+,i}, J_{-,j}] &= \left(\frac{1}{2}\right)^2[J_i + iK_i, J_j - iK_j] \\ &= \left(\frac{1}{2}\right)^2\left([J_i, J_j] - i\,[J_i, K_j] - i\,[J_j, K_i] + [K_i, K_j]\right) \\ &= \left(\frac{1}{2}\right)^2 i\varepsilon_{ijk}\left(J_k - iK_k + iK_k - J_k\right) = 0\end{aligned} \tag{21}$$

and,

$$
\begin{aligned}
[J_{+,i}, J_{+,j}] &= \left(\frac{1}{2}\right)^2 [J_i + iK_i, J_j + iK_j] \\
&= \left(\frac{1}{2}\right)^2 ([J_i, J_j] + i\,[J_i, K_j] - i\,[J_j, K_i] - [K_i, K_j]) \\
&= \left(\frac{1}{2}\right)^2 i\varepsilon_{ijk}\,(J_k + iK_k + iK_k + J_k) = i\varepsilon_{ijk} J_{+,k}
\end{aligned} \tag{22}
$$

Using a similar calculation, we also have:

$$
[J_{-,i}, J_{-,j}] = i\varepsilon_{ijk} J_{-,k} \tag{23}
$$

We verified that the six generators $J_{\pm,i}$ divide into two sets of three generators each: the $J_{+}s$ and the $J_{-}s$ where each set of generators commute past the other one. Therefore, the Lorentz algebra falls apart into two pieces. This observation is important in the upcoming discussion of the Dirac equation.

7 Conclusion

We started by introducing the role of transformations and began our discussion with the Galilean transformation. We showed how Einstein helped us transition into the Lorentz transformations which we outlined the derivation for. Finally, we formulated the Lorentz Lie algrebra and discussed some of its properties. This lecture is but the begining of our journey towards discussing the Dirac equation.

References

Zee, A. (2016). Group theory in a nutshell for physicists. Princeton and Oxford: Princeton University Press.