The Boltzmann Equation for Photons and Dark Matter

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Abstract

We derive the Boltzmann equations for the cosmic distributions of photons and cold dark matter, respectively. To do this, we begin with the unintegrated Boltzmann equation and use previous work on the time derivative and collision terms for photons, and work with zeroth and first-order terms to find the final, Fourier-space Boltzmann equation. We perform a similar procedure for dark matter, but with no collision terms, obtaining two equations instead.

1 Introduction

We wish to understand the anisotropies of the cosmic photon and dark matter distributions. To do this, we need to start with the unintegrated Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C[f] \tag{1}$$

The left side of the equation is the change in the distribution function, f. The right side contains all possible collision terms. Using this framework, we will derive the actual Boltzmann equations for photons and dark matter, which govern the evolution of perturbations to the distribution.

We are going to use the following perturbed Friedman-Robertson-Walker metric:

$$g_{00}(t,\vec{x}) = -1 - 2\Psi(t,\vec{x})$$
$$g_{0i}(t,\vec{x}) = 0$$
$$g_{ij}(t,\vec{x}) = a^2\delta_{ij}(1 + 2\Phi(t,\vec{x}))$$

where $a = a(t)$ is the expansion coefficient, and Ψ and Φ are temporal and spatial perturbation terms, respectively. Since these perturbations are small at the times and scales we are interested in, we will drop them in second order and higher.

2 Photons

We have previously found the right side of (1) to be

$$C[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}] \tag{2}$$

where p is the momentum; $f^{(0)}$ is the zeroth-order piece of the photon distribution function f, which turns out to be the Bose-Einstein distribution; Θ is a first-order perturbation to f with Θ_0 being the *monopole* part of the perturbation; $\vec{v}$ is the bulk velocity of the electron distribution; n_e is the density of electrons; and σ_T is the Thomson cross-section.

We also found the left side of (1) to be

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] \tag{3}$$

Notice that this is entirely first-order! While we did find an equation for the zeroth-order piece, we know that it must be equal to 0, since all the terms in (2) are first-order.

Now we can plug (2) and (3) into (1), obtaining the following:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b] \tag{4}$$

It is convenient to write this in terms of the conformal time, η:

$$\eta \equiv \int_0^t \frac{dt'}{a(t')} \tag{5}$$

To write these derivatives in terms of η, we'll use a product rule:

$$\frac{\partial h}{\partial \eta} = \frac{\partial h}{\partial t}\frac{\partial t}{\partial \eta}$$

where h is any function.

Now to find $\partial\eta/\partial t$, we'll take the derivative of (5):

$$\begin{aligned} 1 &= \frac{\partial}{\partial \eta}\left[\int_0^t \frac{dt'}{a(t')}\right] \\ &= \frac{1}{a(t)}\frac{\partial t}{\partial \eta} - \frac{1}{a(0)}\frac{\partial 0}{\partial \eta} + \int_0^t \frac{\partial}{\partial \eta}\left(\frac{1}{a(t')}\right) dt' \end{aligned}$$

a is independent of η, and the second term is a derivative with respect to 0, so we're just left with the first term. Thus,

$$\frac{\partial t}{\partial \eta} = a$$
$$\rightarrow \frac{\partial h}{\partial \eta} = \frac{\partial h}{\partial t} a$$

There are many terms in (4) that are of this form (to see this, just move the a to the other side). So we can rewrite (4) as

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b] \tag{6}$$

where a dot indicates derivative with respect to η, the conformal time.

This is a linear PDE coupling Θ to Ψ, Φ, and $\vec{v}_b$, which is difficult to solve. We can Fourier transform this expression so we're dealing with ODEs instead, which are easier to solve. The other benefit of this is that, since we are dealing with a smooth universe, the only $\vec{x}$ dependence is in the perturbations themselves, which can act as black boxes for now. This means each Fourier mode evolves independently, and thus our final set of equations are uncoupled.

We will use the following Fourier transform definition:

$$\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \widetilde{\Theta}(\vec{k}) \tag{7}$$

Before writing the final equation, we will define two more quantities which will be useful. First, define μ as the cosine of the angle between the perturbation wavevector $\vec{k}$ and the photon direction $\hat{p}$:

$$\mu \equiv \frac{\vec{k} \cdot \hat{p}}{k} \tag{8}$$

$\vec{k}$ always points in the direction the temperature is changing, so when $\mu = 1$, the photon is travelling in the direction of temperature change as well. We will assume $\widetilde{\vec{v}_b} \cdot \hat{p} = \widetilde{v}\mu$, ie the velocity points in the same direction as $\vec{k}$.

We will also define the optical depth:

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a \tag{9}$$

Roughly, we are taking the probability of interaction ($n_e \sigma_T$), scaling it by the time constant a, and integrating that over the time period of interest, which should give us the distance a photon can travel without interacting. Note that τ is defined such that $\dot{\tau} = -n_e \sigma_T a$.

Now we will take the Fourier transform of (4). Note that

$$\begin{aligned} F\left[\hat{p}^i \frac{\partial h}{\partial x^i}\right] &= i\hat{p}^i \cdot \vec{k}\,\widetilde{h} \\ &= ik\mu\widetilde{h} \end{aligned}$$

where h is any function. Thus, by taking the Fourier transform, we obtain

$$\dot{\widetilde{\Theta}} + ik\mu\widetilde{\Theta} + \dot{\widetilde{\Phi}} + ik\mu\widetilde{\Psi} = -\dot{\tau}\left[\widetilde{\Theta}_0 - \widetilde{\Theta} + \mu\tilde{v}\right] \tag{10}$$

This is the Boltzmann equation for photons.

3 Cold Dark Matter

The main difference between the distribution for dark matter and that for photons is that dark matter doesn't interact with any of the other constituents of the universe; thus, there are no collision terms. Also, dark matter is non-relativistic, which changes kinematics applied to the distribution function. Specifically, since dark matter is massive, we now have

$$g_{\mu\nu}P^\mu P^\nu = -m^2 \tag{11}$$

Also, the energy is now

$$E = \sqrt{p^2 + m^2} \tag{12}$$

where $p^2 = g_{ij}P^iP^j$. For the most part, E will replace p in the photon equations.

Now we wish to find the components of P^μ. First,

$$\begin{aligned} P^2 = -m^2 &= -(1+2\Psi)(P^0) + p^2 \\ \rightarrow \quad E^2 &= (1+2\Psi)(P^0) \\ \rightarrow \quad P^0 &= \frac{E}{\sqrt{1+2\Psi}} \approx E(1-\Psi) \end{aligned}$$

Now for the spatial components. Write $P^i = C\hat{p}^i$. Then

$$\begin{aligned} p^2 &= g_{ij}\hat{p}^i\hat{p}^jC^2 \\ &= a^2\delta_{ij}(1+2\Phi)\hat{p}^i\hat{p}^jC^2 \\ &= a^2(1+2\Phi)C^2 \\ \rightarrow \quad C &= \frac{p}{a\sqrt{1+2\Phi}} \approx \frac{p}{a}(1-\Phi) \end{aligned}$$

Now, the total time derivative of the dark matter distribution function is

$$\frac{\mathrm{d}f_{\mathrm{dm}}}{\mathrm{d}t} = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial f_{\mathrm{dm}}}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} \tag{13}$$

We drop the last term since it's the product of two first-order terms.

After working through the algebra, which is identical to that performed for the photon distribution other than the addition of an E term, we obtain the Boltzmann equation for non-relativistic matter:

$$\frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{\mathrm{dm}}}{\partial x^i} - \frac{\partial f_{\mathrm{dm}}}{\partial E}\left[\frac{da/dt}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0 \tag{14}$$

In this linear, non-relativistic treatment, we will drop terms second-order in $p/E \approx v$.

We will take moments of (14) and deal with them individually. First, multiply by the momentum phase-space volume, $d^3p/(2\pi)^3$, and integrate:

$$\begin{aligned}\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}}\frac{p\hat{p}^i}{E} \\ - \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right]\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{p^2}{E} - \frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\hat{p}^i p = 0\end{aligned} \tag{15}$$

The integral over the direction vector is non-zero only for the perturbed part of f_{dm}. Thus, the integral in the last term is first-order, and it multiplies the first-order term $\partial\Psi/\partial x^i$. Thus, we can drop the last term.

We can further simplify this by noting that the dark matter density is

$$n_{\mathrm{dm}} = \int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}} \tag{16}$$

and the velocity is defined as

$$v^i \equiv \frac{1}{n_{\mathrm{dm}}}\int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}}\frac{p\hat{p}^i}{E} \tag{17}$$

We can write the first two terms in terms of n_{dm} and v_i. To evaluate the third term, we will need to integrate by parts.

Write

$$\frac{\partial f_{\mathrm{dm}}}{\partial E} = \frac{\partial f_{\mathrm{dm}}}{\partial p}\frac{\partial p}{\partial E}$$

Note that

$$\begin{aligned}\frac{\partial E}{\partial E} &= 1 = \frac{p}{\sqrt{p^2+m^2}}\frac{\partial p}{\partial E} \\ &= \frac{p}{E}\frac{\partial p}{\partial E} \\ \rightarrow \quad \frac{\partial p}{\partial E} &= \frac{E}{p}\end{aligned}$$

We can use this to first integrate over the solid angle in momentum phase space, and then integrate by parts:

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3} p \frac{\partial f_{\rm dm}}{\partial p} &= 4\pi \int_0^\infty \frac{dp}{(2\pi)^3} p^3 \frac{\partial f_{\rm dm}}{\partial p} \\ &= -3\left(4\pi \int_0^\infty \frac{dp}{(2\pi)^3} p^2 f_{\rm dm}\right) \\ &= -3 n_{\rm dm}\end{aligned}$$

With these simplifications, (15) becomes

$$\frac{\partial n_{\rm dm}}{\partial t} + \frac{1}{a}\frac{\partial (n_{\rm dm} v^i)}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right] n_{\rm dm} = 0 \tag{18}$$

Interestingly, the first two terms here are essentially the fluid-dynamical continuity equation, adjusted by the temporal scale factor a. The third term is a correction from the FRW metric and the perturbations we included.

To evaluate this expression more, we need to split into zeroth and first-order components, which we can deal with individually. Since velocity and the perturbations are first-order, the zeroth-order equation is

$$\frac{\partial n^{(0)}_{\rm dm}}{\partial t} + 3\frac{da/dt}{a} n^{(0)}_{\rm dm} = 0 \tag{19}$$

where $n^{(0)}_{\rm dm}$ is the zeroth-order part of $n_{\rm dm}$. Notice that

$$\begin{aligned}\frac{\partial (n^{(0)}_{\rm dm} a^3)}{\partial t} &= \frac{\partial n^{(0)}_{\rm dm}}{\partial t} a^3 + 3a^2 \frac{\mathrm{d}a}{\mathrm{d}t} n^{(0)}_{\rm dm} \\ &= a^3\left(\frac{\partial n^{(0)}_{\rm dm}}{\partial t} + 3\frac{da/dt}{a} n^{(0)}_{\rm dm}\right) = 0\end{aligned}$$

So we can write (19) equivalently as

$$\frac{\mathrm{d}(n_{\mathrm{dm}}^{(0)}a^3)}{\mathrm{d}t} = 0 \quad \rightarrow \quad n_{\mathrm{dm}}^{(0)} \propto a^{-3} \tag{20}$$

This means that, to zero order, the density of dark matter goes as the inverse cube of the temporal scale factor a.

Now we will deal with the first order part. Note that we can change all n_{dm} factors multiplying first-order quantities to $n_{\mathrm{dm}}^{(0)}$, since the higher-order part drops off. Elsewhere, we need to expand n_{dm} into a first-order perturbation:

$$n_{\mathrm{dm}} = n_{\mathrm{dm}}^{(0)}(1 + \delta(t, \vec{x})) \tag{21}$$

Plugging this in, we get

$$\begin{aligned} n_{\mathrm{dm}}^{(0)}\frac{\partial \delta}{\partial t} + \frac{1}{a}n_{\mathrm{dm}}^{(0)}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t}n_{\mathrm{dm}}^{(0)} = 0 \\ \rightarrow \quad \frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \end{aligned} \tag{22}$$

We have two variables to work with in this equation, δ and v. To fix this, we'll take the *first moment* of (1), by multiplying by $d^3p(p/E)\hat{p}^j/(2\pi)^3$ and integrating, getting

$$\begin{aligned} 0 = &\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}} \frac{p\hat{p}^j}{E} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_{\mathrm{dm}} \frac{p^2\hat{p}^i\hat{p}^j}{E^2} \\ &- \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right]\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{p^3\hat{p}^j}{E^2} - \frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{\hat{p}^i\hat{p}^j p^2}{E} \end{aligned}$$

The first term is just the time derivative of $n_{\mathrm{dm}}v^i$, as before, and the second is of order $(p/E)^2$, so we can drop it.

For the third term, we again must integrate by parts. Note first that $(p/E)(\partial/\partial E) = (\partial E/\partial p)(\partial/\partial E) = \partial/\partial p$. So we can write the third term as

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial p}\frac{p^2\hat{p}^j}{E} &= \int \frac{d\Omega\hat{p}^j}{(2\pi)^3}\int_0^\infty dp\frac{p^4}{E}\frac{\partial f_{\mathrm{dm}}}{\partial p} \\ &= -\int \frac{d\Omega\hat{p}^j}{(2\pi)^3}\int_0^\infty f_{\mathrm{dm}}\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right) \end{aligned}$$

The p^5/E^3 term is negligible, so we can drop it. The integral of the term that's left is $-4n_{\mathrm{dm}}v^j$. We can perform the same steps on the last term, using the fact that

$$\int d\Omega\hat{p}^i\hat{p}^j = \delta^{ij}\frac{4\pi}{3}$$

So we have the first moment of the Boltzmann equation:

$$\frac{\partial(n_{\text{dm}} v^j)}{\partial t} + 4\frac{da/dt}{a} n_{\text{dm}} v^j + \frac{n_{\text{dm}}}{a}\frac{\partial \Psi}{\partial x^j} = 0 \tag{23}$$

Since the velocity is first-order, this equation has no zeroth-order parts. So we can set $n_{\text{dm}} = n_{\text{dm}}^{(0)}$. Using the fact we found earlier, that $n_{\text{dm}}^{(0)} \propto a^{-3}$, we get

$$\begin{aligned} n_{\text{dm}}^{(0)} \frac{\partial v^j}{\partial t} + n_{\text{dm}}^{(0)} \frac{da/dt}{a} v^j + \frac{n_{\text{dm}}^{(0)}}{a}\frac{\partial \Psi}{\partial x^j} = 0 \\ \rightarrow \quad \frac{\partial v^j}{\partial t} + \frac{da/dt}{a} v^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0 \end{aligned} \tag{24}$$

We can rewrite (22) and (24) by taking their Fourier transforms and writing in terms of derivatives with respect to conformal time η:

$$\dot{\widetilde{\delta}} + ik\widetilde{v} + 3\dot{\widetilde{\Phi}} = 0 \tag{25}$$

$$\dot{\widetilde{v}} + \frac{\dot{a}}{a}\widetilde{v} + ik\widetilde{\Psi} = 0 \tag{26}$$

4 Conclusion

In this paper, we have derived the Boltzmann equations governing perturbations in the photon and dark matter distribution functions. The use of this is not immediately obvious; we have these differential equations, so what?

Combining these with the Boltzmann equation for Baryons, this makes up basically all matter in the early universe. Using these equations, we can derive perturbed versions of the Einstein field equations, which will allow us to solve for the perturbations themselves. This can give us an idea of how the metric, and the distributions, actually evolve over time.

This is rather complicated, but I was able to find a powerpoint presentation that I've uploaded to the drive that goes through the process. The result is that at early times, the fluctuations are larger than the horizon, so the gravitational potential doesn't immediately evolve. At intermediate times, however, we have a radiation-dominated universe which has fluctuation modes inside the horizon. Radiation pressure ends up dominating, and we have a decaying potential. At later times, the universe is matter-dominated, and the potential is constant.

We also find that in the radiation-dominated epoch, the growth of dark matter is slowed by decaying potentials, and thus its density only grows logarithmically. At later times, however, the universe is matter-dominated, allowing dark matter to grow linearly.