University of Rochester

Cosmology

Professor Sarada Rajeev

The Basic Tools and Questions of Cosmology

David Mayrhofer

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1 Introduction

Cosmology represents an exciting field of physics, with much work being done computationally, theoretically, and experimentally. We aim to describe several of the tools and definitions used in cosmology, as well as introduce a few of the important mysteries in cosmology. The first tool we introduce is the Einstein Field Equation. This equation represents the cornerstone of general relativity, forming the equivalent of the Gaussian law of gravity in classical mechanics. Without it, fields such as cosmology would not be able to calculate the motion of objects in large gravitational fields accurately. We then go on to define different distances which are useful in cosmology, as well as how we can then apply the Einstein Field equation in a cosmology setting. From there, we discuss the distribution functions for different species of particles that make up the universe, and how we can use these functions to estimate the total energy density of these particles in the universe.

2 Einstein Field Equation

2.1 Basic Form of Equation

We would expect there to be some sort of relationship between the curvature of a certain point in spacetime and the stress-energy tensor, $\mathbf{T}$. We know that the curvature of spacetime can be described using the Riemann curvature tensor, often defined as

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(g_{\alpha\delta,\gamma\beta} - g_{\alpha\gamma,\delta\beta} - g_{\beta\delta,\gamma\alpha} + g_{\beta\gamma,\delta\alpha}\right). \tag{1}$$

where the indices after the comma represent a derivative with respect to that coordinate. We can define the Ricci tensor by taking the contraction of the Riemann tensor, $R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$. The Ricci scalar is then defined as a contraction of the Ricci scalar, $R = R^{\gamma}{}_{\gamma} = g^{\gamma\delta}R_{\gamma\delta}$. One can show mathematically that any tensor can be formed algebraically from the Ricci tensor [2], so we must have

$$R_{\alpha\beta} + \lambda g_{\alpha\beta} R = \kappa T_{\alpha\beta}. \tag{2}$$

To find what these constant λ and κ are, we make use of the conservation law for the stress-energy tensor. This means that we must have

$$\nabla^{\beta}\left(R_{\alpha\beta} + \lambda g_{\alpha\beta} R\right) = 0. \tag{3}$$

However, we know from the Bianchi identity that λ must be equal to $-\frac{1}{2}$. This does not give a value for κ, as this value will be found by taking the Newtonian limit of the Einstein field equation. However, we can still simplify this expression by writing $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$, where $G_{\alpha\beta}$ are the components of the Einstein curvature tensor. This leads to the field equation

$$\mathbf{G} = \kappa \mathbf{T}. \tag{4}$$

2.2 Newtonian Limit

In the Newtonian gravitation, we know that if we have $\nabla^2\phi = 0$, the metric for the system is

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2). \tag{5}$$

Our goal is to then determine how a system with a weak source behaves, using the same linear approximation and metric. In this system, we assume three things. The first is that for objects in this system, $v << c$. The second is that the rest energy density μ for the system is small. Lastly, we assume that the kinetic energy is proportional to μv^2, the potential is proportional to $\mu\phi$, and both are smaller than the rest mass density. With the three assumptions, the only nonzero term of the stress energy tensor is then

$$T^{00} = \mu + \mathcal{O}(\mu\phi, \mu v^2). \tag{6}$$

All other terms of the stress energy tensor are of order μv^2 or higher, so they may be neglected. If we calculate the components of the Einstein curvature tensor using the definition of the , we find that

$$G_{00} = 2\nabla^2\phi + \mathcal{O}(\phi^2); \quad G_{\alpha\beta} = \mathcal{O}(\phi^2);\ \alpha, \beta \neq 0. \tag{7}$$

Thus, from (4), we have

$$\begin{aligned} 2\nabla^2\phi &= \kappa\mu \\ \implies \nabla^2\phi &= \frac{\kappa}{2}\mu. \end{aligned}$$

To recover Newtonian gravity, we must then have $\kappa = 8\pi G$. Therefore, the Einstein field equation will take the form

$$\mathbf{G} = 8\pi G\mathbf{T}. \tag{8}$$

2.3 Additional Remarks

We show a brief example of calculating the curvature tensor from the stress energy tensor when $T_{\alpha\beta} = 0$. We then have

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R &= 0 \\ \implies g_{\alpha\beta}R_{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta}R &= 0 \\ \implies R - \frac{1}{2}R = 0 \implies R &= 0. \end{aligned}$$

As we would expect, when the stress energy tensor is 0, there is no curvature. Due to the symmetry of the Ricci tensor, we obtain a set of ten nonlinear partial differential equations from this expression. However, by using the Bianchi identity, we can note that these ten equations are not independent. Instead, we have only six independent equations to solve. This is consistent with the metric tensor, as this tensor has ten free components, four of which can be fixed by our choice of coordinate system.

We note that the justification for (8) is not unique. Specifically, we could add a constant term to the left hand side of the equation, and the expression would still be divergenceless. This additional term is called the cosmological constant, and is usually denoted as Λ. However, this terms is usually associated with the stress energy tensor, as a sort of energy density associated with the vacuum. Specifically, we can write

$$T^{\text{vac}}_{\alpha\beta} = -\frac{\Lambda}{8\pi G}g_{\alpha\beta}. \tag{9}$$

3 Cosmology

Now armed with the Einstein Field equation, we move on to define a few basic quantities which are important when discussing cosmology. First, we define a redshift factor, namely z. It is convenient to define a scaling factor, which is related to the redshift factor by

$$a = \frac{1}{1+z} \tag{10}$$

As the name implies, this scaling factor provides a measure of the expansion of the universe. These, in combination with the derivations from the previous section, will allow for in depth study into many of the phenomena within the field.

3.1 Distances

There are two ways to measure distance in an expanding universe. The first is what is known as a comoving distance, which is constant with respect to time. This is the distance between two objects as if they were measured on a grid that was expanding at the same rate as the universe. The second is the physical distance, which is defined in the usual way and is not constant as a result of the expanding universe.

One useful comoving distant is the comoving horizon. It is defined as distance light could have traveled since the beginning of the universe at $t = 0$. We can write this mathematically as

$$\eta = \int_0^t \frac{dt'}{a(t')}. \tag{11}$$

We know that no information can have propagated farther than η, so this puts an upper bound on the distance that can be between two connected events. In addition, we can estimate η in the cases of certain universes. For instance, in a matter dominated universe, we have $\eta \propto \sqrt{a}$. In a radiation dominated universe, $\eta \propto a$.

Another useful comoving distance is the distance between Earth and some distance emitter, which we can write

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} \tag{12}$$

If we know that the universe is flat (i.e. k=0) and matter dominated, then $H \propto a^{\frac{3}{2}}$ and we have

$$\frac{2}{H_0}\left(1 - a^{\frac{1}{2}}\right) = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{13}$$

We can also measure distances by measuring the angle θ subtended in the sky by some object of known size l. Then, in a non-expanding universe, we have that

$$d_A = \frac{l}{\theta} \tag{14}$$

We call the d_A the angular diameter distance. In a flat, expanding universe, we must adjust by adding the effects of the scale factor. The angle subtended by the object is now $(l/a)/\chi(a)$, so the angular diameter distance is now

$$d_A^{\text{flat}} = a\chi = \frac{\chi}{1+z} \tag{15}$$

If we are in a non-flat universe, then we have

$$d_A = \frac{a}{H_0\sqrt{|\Omega_k|}} \begin{cases} \sinh\left(\sqrt{\Omega_k} H_0 \chi\right), \ \Omega_k > 0 \\ \sin\left(\sqrt{-\Omega_k} H_0 \chi\right), \ \Omega_k < 0 \end{cases} \tag{16}$$

Yet another way to measure distance is via the luminosity of an object. We know that the flux of a luminous object through the surface of some sphere a distance d away from the object is

$$F = \frac{L}{4\pi d^2}. \tag{17}$$

To adjust for an expanding universe, we take

$$F = \frac{L(\chi)}{4\pi\chi^2(a)}. \tag{18}$$

If we assume the object is emitting photons of only one energy, then $L(\chi)$ is the number of photons moving through the sphere times the energy of the photons. If we look at a fixed time interval, the photons will travel a larger comoving distance earlier in the time interval than they will later in the time interval, due to the expanding universe. Therefore, the number of photons crossing the shell will be smaller now than when they were first emitted by a factor of a. The energy of the photons will also be smaller by a factor of a, due to the redshift. Therefore, we get for the flux

$$F = \frac{L}{4\pi d_L^2}, \tag{19}$$

where d_L is the luminosity distance and is defined as $d_L = \frac{\chi}{a}$.

3.2 Evolution of Energy

We can use the Robertson-Walker metric and the Einstein field equation to deduce information about how energy evolves in universe. We know the Robertson-Walker can be in spherical coordinates as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{20}$$

We take the case of a perfect fluid, that is, where $T_{\alpha\beta} = \text{diag}\,(\rho, P, P, P)$, where ρ is the energy density of the fluid and P is its pressure. We assume that there are no gravitational forces, and velocities of the system are negligible. We take a set of basis vectors $\{e_{\hat{\alpha}}\}$ with $e_{\hat{0}} = u$ and $e_{\hat{i}}$ pointing along (r, θ, ϕ). From (1) and (8), we can calculate that we have

$$G_{00} = \frac{3}{a^2}(k + \dot{a}^2) = 8\pi\rho \tag{21}$$

$$G_{\hat{i}\hat{i}} = -\left[2\frac{\ddot{a}}{a} + \frac{1}{a^2}(k + \dot{a}^2)\right] = 8\pi P. \tag{22}$$

The first equation is known as the Friedman equation. We can derive the first law of thermodynamics from these relations. We do this by first multiplying (21) by a^3 and rearranging to get to get

$$\rho a^3 = \frac{3a}{8\pi}\left(k + \dot{a}^2\right). \tag{23}$$

Differentiating both sides, we obtain

$$\frac{d}{dt}\left(\rho a^3\right) = \frac{3}{8\pi}\left(k\dot{a} + \dot{a}^3 + 2a\dot{a}\ddot{a}\right). \tag{24}$$

We then manipulate (22) to find that

$$\begin{aligned} -3\dot{a}a^2 P &= \frac{3}{8\pi}\left(k\dot{a} + \dot{a}^3 + 2a\dot{a}\ddot{a}\right) \\ \implies -P\frac{da^3}{dt} &= \frac{3}{8\pi}\left(k\dot{a} + \dot{a}^3 + 2a\dot{a}\ddot{a}\right). \end{aligned} \tag{25}$$

Therefore, we get

$$\frac{d}{dt}\left(\rho a^3\right) = -P\frac{da^3}{dt}. \tag{26}$$

To obtain the first law of thermodynamics from this expression, we first consider how some differential volume ΔV changes over time. In an expanding universe, this volume is related to the volume defined in some coordinate system by $\Delta V = a^3 \Delta V_{\text{coord}}$. The ΔV_{coord} is constant with respect to time, so we multiply both sides by this quantity and bring it into the derivatives to find

$$d\left(\rho a^3 \Delta V_{\text{coord}}\right) = d\left(\Delta E\right) = -P\ d\left(\Delta V\right). \tag{27}$$

This is the first law of thermodynamics for a fluid expanding adiabatically.

We can also use (26) to determine how energy density scales in different situations. For instance, in a matter dominated universe, we know that $p = 0$, so we find that $\rho_m \propto a^{-3}$. In a radiation dominated universe, $p = \frac{rho}{3}$. In this situation, we have

$$\frac{d\rho_r a^3}{dt} + \frac{\rho_r}{3}\frac{da^3}{dt} = a^{-4}\frac{d\rho_r a^4}{dt} = 0. \tag{28}$$

Therefore, we find that $\rho_r \propto a^{-4}$.

3.3 Distribution Functions

We want to express energy and pressure in terms of common temperature, as we often have particles in equilibrium with each other. To do this, we introduce a distribution function which counts the number of particles in a particular region of phase space. We can then determine the total energy of the system described by the distribution function by summing over the entire phase space. That is, we take $\sum f\left(\vec{x}, \vec{p}\right) E(p)$, where f is our distribution function and $E(p) = \sqrt{p^2 + m^2}$.

We know from Heisenberg's uncertainty principle that the three dimensional phase space cannot be divided up into smaller volume elements that $(2\pi\hbar)^3$. Therefore, in any region $d^3x d^3p$ of phase space, there are $\frac{d^3x d^3p}{(2\pi\hbar)^3}$ volume elements. The energy density for a particular species of particle i is

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i\left(\vec{x}, \vec{p}\right) E(p), \tag{29}$$

where we take g_i as the degeneracy of the particle, and $\hbar = 1$. For bosons and fermions, we have

$$f_{\text{BE}} = \frac{1}{e^{(E-\mu)/T} - 1}, \quad f_{\text{FD}} = \frac{1}{e^{(E-\mu)/T} + 1}, \tag{30}$$

where T is the temperature of the system, and μ is the chemical potential. It is important to note that these functions depend only on momentum, and not on position. However, they are only accurate to the 0^{th} order approximation of a smooth universe, and they change as we add inhomogeneities and anisotropies.

The pressure for the system can also be expressed in a similar manner,

$$P_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i\left(\vec{x}, \vec{p}\right) \frac{p^2}{3E(p)}. \tag{31}$$

Typically, we have that $\mu << T$, so we can approximate $\frac{E-\mu}{T}$ as just $\frac{E}{T}$. Then, we find that

$$\frac{\partial P_i}{\partial T} = \frac{\rho_i + P_i}{T}. \tag{32}$$

By expanding the $\frac{da^3}{dt}$ term in (26), we can find

$$\begin{aligned}
&0 = a^{-3}\frac{\partial \rho a^3}{\partial t} + 3\frac{\dot{a}}{a}P \\
\implies &0 = a^{-3}\frac{\partial \rho a^3}{\partial t} + 3\frac{\dot{a}}{a}P + \frac{\partial P}{\partial t} - \frac{\partial P}{\partial t} \\
&= a^{-3}\frac{\partial \rho a^3}{\partial t} + a^{-3}\frac{\partial P a^3}{\partial t} - \frac{\partial P}{\partial t} \\
&= a^{-3}\frac{\partial (\rho + P) a^3}{\partial t} - \frac{\partial P}{\partial t} \\
&= a^{-3}\frac{\partial (\rho + P) a^3}{\partial t} - \frac{dT}{dt}\frac{\rho + P}{T} \\
&= a^{-3} T \frac{\partial}{\partial t}\left(\frac{(\rho + P)a^3}{T}\right)
\end{aligned}$$

We can define the entropy density of a system as $s = \frac{\rho+P}{T}$. Therefore, by the above derivation, the entropy density must scale with a^{-3}.

4 Balancing the Universal Mass-Energy Budget

4.1 Photons

Experimentally, we know that the temperature of cosmic microwave background photons to be $T = 2.725 \pm 0.002\text{K}$. We can then calculate the energy density of these photons as

$$\rho_\gamma = 2\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T} - 1}. \tag{33}$$

We take the chemical potential μ to be 0, since photons can be formed from electrons and positrons without any loss of energy to the system. There is also no angular dependence in this integral, since the magnitude of the momentum is the only variable in this expression. We can then calculate this integral as

$$\rho_\gamma = \frac{8\pi T^4}{(2\pi)^3}\int_0^\infty \frac{dx\, x^3}{e^x - 1} = \frac{8\pi T^4}{(2\pi)^3}(6\zeta(4)) = \frac{\pi^2}{15}T^4, \tag{34}$$

where ζ is the Riemann ζ function. We find a ratio of this energy density with the critical energy density today, ρ_{cr}.

$$\begin{aligned}
\frac{\rho_\gamma}{\rho_{\text{cr}}} &= \frac{\pi^2}{15}\left(\frac{2.725\text{K}}{a}\right)^4 \frac{1}{8.098 \times 10^{-11} h^2 \text{eV}^4} \\
&= \frac{2.47 \times 10^{-5}}{h^2 a^4},
\end{aligned} \tag{35}$$

where h is the dimensionless Hubble constant defined as $h = H_0/100$ s Mpc km^{-1}.

4.2 Baryons

In order to determine the energy density of baryons, we must measure the particles directly. This is because, unlike photons, they cannot be modeled as a gas. We use the approximation that $\rho_m \propto a^{-3}$

to state that $\frac{\rho_b}{\rho_{\text{cr}}} = \Omega_b a^{-3}$, where Ω_b is is the ratio of the total matter density today to the critical density.

There are four different ways to measure this baryon energy density. We can look at the gas in different galaxies, as this gas forms the greatest contribution to the energy density of baryons in the universe. By measuring this quantity, we find that $\Omega_b \approx 0.02$. We can also look at the spectra of emitted light of distant quasars, as this light is a measure of the hydrogen near these quasars. Doing this, we find that $\Omega_b h^{1.5} \approx 0.02$, but with a significant uncertainty. The third way to measure the anisotropies of the universe, which gives $\Omega_b h^2 = 0.024^{+0.004}_{-0.003}$. Lastly, the proportions of the lighter elements depend sensitively on the baryon density. By studying these abundances, cosmologists have estimated an $\Omega_b h^2 = 0.0205 \pm 0.0018$. All four techniques roughly agree, and have $\Omega_b \approx 0.02$. However, there is in fact more matter in the universe than this suggests, meaning there must be more non-baryonic matter out there.

4.3 Other Matter

The above arguments all depend on the interaction between matter and light. However, we can estimate the mass of matter without directly relying on light. The rotation of a galaxy about its center, for instance, does not depend on light-matter interactions. Again, we can use the relation $\frac{\rho_m}{\rho_{\text{cr}}} = \Omega_m a^{-3}$. Experimental data seems to show that $\Omega_m \approx 0.3$. In addition, we can also find experimental evidence for the ratio $\frac{\Omega_b}{\Omega_m}$. We can determine this ratio by looking at the ratio of the hot gas clusters at the center of the galaxy to the total mass of the galaxy by examining X-ray emission or by analyzing the CMB which comes from the direction of the cluster. From this, we find that $\frac{\Omega_b}{\Omega_m} = 0.15 \pm 0.07$. This means that the total matter density is potentially five times larger than the mass density of baryons, which means there must be another type of matter. We call this new type of matter dark matter.

4.4 Matter-Radiation Equality

When we consider the case of equal energy density between matter and radiation, we can gain additional insight into how the universe is evolving over time. To find the total energy density of radiation, we use the energy density photons found in (35) and add to the energy density of neutrinos, which was not derived in this paper. From these, we have

$$\frac{\rho_r}{\rho_{\text{cr}}} = \frac{4.15 \times 10^{-5}}{h^2 a^4} = \frac{\Omega_r}{a^4}. \tag{36}$$

To determine at what age the universe had equal mass and radiation energy densities, we set the scaling factor to

$$a_{\text{eq}} = \frac{4.15 \times t0^{-5}}{\Omega_m h^2}. \tag{37}$$

Another way to interpret this result is in terms of the redshift of the universe. In this case, we have

$$1 + z_{\text{eq}} = 2.4 \times 10^4 \Omega_m h^2. \tag{38}$$

Interestingly, this is much larger than when it is known that photons have decoupled from matter, which occurred around $z \approx 10^3$. Therefore, this even occurred well into the matter dominated portion of the universe's history.

5 Conclusion

With this, we have investigated some of the basic questions currently plaguing cosmologists, as well as provided some of the tools being used to answer these questions. The Einstein field equation forms the bedrock for any calculations involving large masses, and can even be used to derive the first law of thermodynamics. We can define distribution functions for many of the different types of particles, and then use those distribution functions to determine the total energy of that particular species of particle. For light, this is straightforward, and gives us a concrete answer. However, for baryons and other massive particles, we must rely on experimental measurements. From these measurements, we find that the amount of baryonic matter falls short of the predicted total energy density of the universe, and that dark matter must make up this missing mass.

6 References

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