Representations of the Rotation Groups $SO(N)$

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December 14, 2019

Abstract

This paper is based on my lecture for the Kapitza Society. In it, we discuss the representations of the Rotation Groups, $SO(N)$, and topics, like duals and adjoints, that naturally arise in this context. Necessary content from prior lectures will also be included. Zee's *Group Theory in a Nutshell for Physicists* is used as a reference for the bulk of the content, while Arovas's *Lecture Notes on Group Theory in Physics (A Work in Progress)* was used for supplemental information.

1 Introduction

The importance of Group Theory and Representation Theory in modern physics cannot be overstated. In physics, it is common for systems to be discussed in the context of their symmetries. These symmetries, in turn, are easily represented by the action of a group on another group. Some important symmetries in physics include, spatial translation in a homogeneous system, rotations in an isotropic system, time translation of a system in which energy is conserved, Lorentz transformations, parity, charge conjugation, time reversal, and permutation symmetry in many-body systems [2]. Many of these symmetries, and others, can be described by the Lie Algebras/Groups associated with the rotation (special orthogonal) groups $SO(N)$ or the special unitary groups $SU(N)$. Due to the importance of these groups, we will be focusing on the groups $SO(N)$ in this paper. We will begin with previous content that will be built from in the lecture.

2 Prerequisite Information

2.1 Rotation Groups

The rotation group in N-dimensional Euclidean space, $SO(N)$, is a continuous group, and can be defined as the set of N by N matrices satisfying the relations:

$$R^T R = I$$
$$\det\ R = 1$$

By our definition, we can see that the elements of $SO(N)$ can be represented very naturally by those N by N matrices acting on the N standard unit basis vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_N$ and transforming (permuting) them into one another. That is to say, this representation can be described by the vectors resulting from these matrices transforming the basis vectors in this manner. In this context, vectors are the objects that transform as:

$$V^i \xrightarrow{\text{Rotation}} V'^i = R^{ij} V^j \tag{1}$$

2.2 1-Dimensional Representations of $SO(2)$

It can be shown, by generalizing from representations of rotations of the finite group $Z_N = \mathbb{Z}/n\mathbb{Z}$ that has N 1-dimensional representations given by $D^{(k)}(e^{i2\pi j/N}) = e^{i2\pi kj/N}$ where $k = 0, 1, ..., N-1$, we can get a representation of $SO(2)$. We did this by taking $N \to \infty$, getting infinitely many irreducible 1-dimensional representations of $SO(2)$ given by $D^{(k)}(\theta) = e^{ik\theta}$ where $k = 0, 1, ..., \infty$, and our rotation is through angle θ. These representations work as expected, and behave as expected:

$$D^{(k)}(\theta)D^{(k)}(\theta') = e^{ik\theta}e^{ik\theta'} = e^{ik(\theta+\theta')} = D^{(k)}(\theta+\theta')$$
$$D^{(k)}(2\pi) = D^{(k)}(0) = 1$$

2.3 Use of Tensors for Irreducible Representations

In order to construct the larger irreducible representations of our rotation groups we need to generalize our concept of vectors to objects that transformed appropriately, but carry more indices. This is done by introducing tensors. In the case of a tensor T with 2 indices in N-dimensional space, we describe T by a collection of T^{ij} where $i, j = 1, 2, ..., N$ such that under a rotation:

$$T^{ij} \xrightarrow{\text{Rotation}} T'^{ij} = R^{ik}R^{jl}T^{kl} \tag{2}$$

This is very clearly a generalization of (1) to our idea of a tensor, defines how tensors transform under a rotation. It is worth noting that we will be using the convention of not explicitly writing our the sums over k and l from 1 to N on the right hand side of (2). Had we written the sums, the equation would read:

$$T^{ij} \xrightarrow{\text{Rotation}} T'^{ij} = \sum_{k=1}^{N}\sum_{l=1}^{N} R^{ik}R^{jl}T^{kl}$$

We could further generalize this equation to describe the way higher rank tensors transforms:

$$T^{ij...n} \to T'^{ij...n} = R^{ik}R^{jl}...R^{nm}T^{kl...m} \tag{3}$$

Previously in the course, we had seen that for $SO(3)$, the 9-dimensional representation given by rank 2 tensors decomposed into 5-, 3-, and 1-dimensional representations. As such, we say that in $SO(3)$, $9 = 5\oplus3\oplus1$. In [1], it is stated that for $SO(N)$, $N^2 = (\frac{1}{2}N(N+1)-1)\oplus\frac{1}{2}N(N-1)\oplus1$.

2.4 The Antisymmetric Symbol

The antisymmetric symbol $\varepsilon^{ijk...n}$ can be used to write the determinant of a matrix, and when generalized to N-dimensional space, the antisymmetric symbol carries N indices and satisfies the properties:

$$\varepsilon^{...l...m...} = -\varepsilon^{...m...l...}$$

$$\varepsilon^{12...N} = 1$$

The aforementioned calculation of the determinant of a matrix is given by:

$$\varepsilon^{ijk...n}R^{ip}R^{jq}R^{kr}...R^{ns} = \varepsilon^{pqr...s}\det\, R$$

Using the fact that for a rotation, R, det $R = 1$, we get:

$$\varepsilon^{ijk...n} R^{ip} R^{jq} R^{kr} ... R^{ns} = \varepsilon^{pqr...s}$$

Something else worth mentioning is that for a symmetric tensor U^{ij} and an antisymmetric tensor V^{ij}, the identity $U^{ij} V^{ij} = 0$ holds. We can choose $V^{ij} = \varepsilon^{ijk...n}$ and thus get that for the antisymmetric symbol contracted with any symmetric tensor U^{ij}, we have:

$$\varepsilon^{ijk...n} U^{ij} = 0$$

2.5 Dual Tensors and Contraction of Indices

In the case that we are contracting the antisymmetric symbol with an antisymmetric matrix, however, the result does not vanish, but instead, we get a tensor carrying $N-2$ indices:

$$\varepsilon^{ijk...n} A^{ij} = B^{k...n}$$

We can then proceed to show that this new object $B^{k...n}$ transforms in the manner necessary for it to be a tensor:

$$B^{k...n} \to \varepsilon^{ijk...n} R^{ip} R^{jq} A^{pq} = \varepsilon^{pqr...s} R^{kr} ... R^{ns} A^{pq} = R^{kr} ... R^{ns} B^{r...s}$$

These two tensors, A and B are called dual to each other. Moving on, we will look at another object which transforms like a tensor with 2 fewer indices: a tensor in which we have "contracted" two of the indices. Specifically, when contracting indices, we set two indices equal and sum over them. By taking a tensor $T^{ij...np}$ that transforms (in the usual way) as $T^{ij...np} \to T'^{ij...np} = R^{ik} R^{jl} ... R^{nm} R^{pq} T^{kl...mq}$, and choosing two indices, in this case we will use j and n, and make use of the fact that $R^{jl} R^{jm} = \delta^{lm}$, this process of contraction works as follows:

$$T^{ij...jp} \to T'^{ij...jp} = R^{ik} R^{jl} ... R^{jm} R^{pq} T^{kl...mq} = R^{ik} ... R^{pq} T^{kl...lq}$$

As such, our tensor $T^{ij...jp}$ with p indices, but contracted along 2 of them will transform like the tensor $T^{i...p}$ with $p-2$ indices.

2.6 SO(3)

Moving forward, we first show that as opposed to dealing with all possible tensors when looking for irreducible representations of $SO(3)$ of higher dimension, we only need to deal with the case of a totally symmetric, trace-less tensor. Any tensor of this type with j indices (where j is an arbitrary, positive integer), $S^{i_1 i_2 ... i_j}$, will, by definition, remain unchanged when a pair of indices is swapped, and when contracted along two indices, the components of the resultant sum will be identically zero. As opposed to redoing a proof of this result here, I will simply present an overview of the proof.

The proof was done by induction over values of j, starting with the base cases of two and three. We had already shown that antisymmetric rank 2 tensors are equivalent to a rank 1 tensor, so we will begin from the case of $j = 3$. First, we start with our general tensor T^{ijk}, and decompose it, in first two indices, into an antisymmetric and a symmetric tensor:

$$\text{Symmetric} : T^{\{ij\}k} \equiv \left(T^{ijk} + T^{jik}\right)$$
$$\text{Antisymmetric} : T^{[ij]k} \equiv \left(T^{ijk} - T^{jik}\right)$$

We then simply note that the antisymmetric combination does not need to be dealt with, as we already treated the $j = 2$ case, and $\varepsilon^{ijl}T^{[ij]k} = B^{lk}$. Next, to deal with the symmetric tensors, we write our symmetric tensor in terms of a totally symmetric part, and two parts that can be treated as being rank 2:

$$3T^{\{ij\}k} = \left(T^{\{ij\}k} + T^{\{jk\}i} + T^{\{ki\}j}\right) + \left(T^{\{ij\}k} - T^{\{jk\}i}\right) + \left(T^{\{ij\}k} - T^{\{ki\}j}\right) \tag{4}$$

In (4), the fist term in parenthesis is the sum of the tensor with all permutations of the indices, and is thus completely symmetric, while the other two parenthetical terms are antisymmetric in ki and kj, and have thus, already been eliminated. We next proceeded to working with a totally symmetric tensor S^{ijk}, as these are the only ones that remained, and subtract its trace from it to get:

$$\tilde{S}^{ijk} = S^{ijk} - \frac{1}{N+2}\left(\delta^{ij}S^{hhk} + \delta^{ik}S^{hhj} + \delta^{jk}S^{hhi}\right)$$

As such, we only need to work with totally symmetric, trace-less tensors, like $\tilde{S}^{ijk}$, which satisfies that $\delta^{ij}S^{ijk} = 0$. This argument can then be increased to higher dimensions by showing that the antisymmetric tensors have already been dealt with and noting that symmetric tensors can be altered to be symmetric, trace-less tensors.

Having finally proven this result, we can put it to use in order to calculate the dimension of the irreducible representation that we can represent with $S^{i_1 i_2 \cdots i_j}$ (labeled by j, the number of indices). We note that if our indices can take on only values between 1 and 2, there are $j+1$ possibilities. When we then add the option for an index to take the value 3, we can now count the possibilities by summing $k+1$ options of how to make k indices 1 or 2 over all values of k from 0 to j, letting us get all possibilities:

$$\sum_{k=0}^{j}(k+1) = \frac{1}{2}(j+1)(j+2) \tag{5}$$

We then impose that our tensor must be trace-less: $\delta^{i_n i_m}S^{i_1 i_2 \cdots i_j} = 0$ for all $n, m \in \{i_1 i_2 ... i_j\}$. This, by the previous section, can be treated as a totally symmetric tensor with $j-2$ indices, which, by (5), has $\frac{1}{2}j(j-1)$ components which we are setting to zero. Because our general totally symmetric tensor has $\frac{1}{2}(j+1)(j+2)$ free parameters and we are now setting $\frac{1}{2}j(j-1)$ restrictions, which gives us $\frac{1}{2}(j+1)(j+2) - \frac{1}{2}j(j-1) = 2j+1$ free parameters in the tensors of the desired type. As such, our irreducible representations will be of dimension $2j+1$, giving that for $j = 0, 1, 2, 3, ...$ our dimensions will be $d = 1, 3, 5, 7, ...$.

3 SO(2)

3.1 Tensors

Now that we have built up the necessary ideas, we will revisit $SO(2)$ to more thoroughly address its irreducible representations. In the same fashion as we found the possible dimensions of irreducible representations of $SO(3)$, we will simply deal with totally symmetric tensors and then impose the condition that they must be trace-less. From the preceding section, we know that there are $j+1$ totally symmetric tensor $S^{i_1 i_2 \cdots i_j}$ with each index ranging from 1 to 2. Arguing the exact same way as we did previously, we impose the restriction that $\delta^{i_n i_m}S^{i_1 i_2 \cdots i_j} = 0$, thus getting $(j-2)+1$ restrictions. As such, we will have $j+1-(j-2+1) = 2$ dimensions. This means that the dimension of all irreducible representations of $SO(2)$ is 2. Having some familiarity with rotation matrices, it is natural to think of the representation where our matrices will be:

$$D^{(j)}(\theta) = \begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ -\sin(j\theta) & \cos(j\theta) \end{pmatrix} \tag{6}$$

3.2 Decomposition

While (6) is a representation, it is not irreducible. We will show this by performing a unitary transformation $U^\dagger D^{(j)}(\theta)U$ where U is given by:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

This gives us:

$$\begin{aligned} U^\dagger D^{(j)}(\theta)U &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^\dagger \begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ -\sin(j\theta) & \cos(j\theta) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} e^{(ij\theta)} & e^{(-ij\theta)} \\ ie^{(ij\theta)} & -e^{(-ij\theta)} \end{pmatrix} \\ &= \begin{pmatrix} e^{(ij\theta)} & 0 \\ 0 & e^{(-ij\theta)} \end{pmatrix} \end{aligned}$$

As such, our 2-dimensional representation $D^{(j)}(\theta)$ decomposes into the two 1-dimensional representations $e^{(ij\theta)}$ and $e^{(-ij\theta)}$. From this, we also see that the columns, $\binom{1}{i}$ and $\binom{1}{-i}$, of our unitary matrix U are the eigenvectors of $D^{(j)}(\theta)$, where the eigenvalues are $e^{(ij\theta)}$ and $e^{(-ij\theta)}$ respectively. This is analogous to converting from Cartesian coordinates to complex coordinates. Another useful result of this computation is that we now know that $SO(2)$ and the unitary group $U(1)$ are isomorphic, where the map is $R(\theta) \leftrightarrow e^(i\theta)$.

4 Higher Dimensions

Now that we have discussed $SO(N)$ where $n = 2, 3$, we would like to move on to rotations of higher dimensional spaces, but unfortunately, with higher dimensional space, there are added complications.

To demonstrate one way in which we could run into issues, let us consider a tensor carrying 5 indices, T^{hijkl}, that is symmetric in three indices, hij, but antisymmetric in two indices, k and l. For $N = 3$, we were able to contract with ε^{klm} to reduce the number of indices from five to four, so we were able to consider only totally symmetric tensors. Unfortunately, once we go up to $N = 4$, if we contract the same tensor with the antisymmetric symbol ε^{klmn}, we would wind up with $\varepsilon^{klmn}T^{hijkl}$, a tensor that still has five indices. As such, in higher dimensions than 3, we will have to deal with additional complications.

4.1 SO(2N)

Due to these complications, it is valuable to take advantage of special properties of spaces of certain (higher) dimensions. One such example are the self-dual and antiself-dual representations when working with the rotations in an even-dimensional space. We will start, as we frequently do, with the antisymmetric tensor with n indices, $A^{i_1 i_2 \dots i_n}$, which has $(2n(2n-1)...(n+1))/n! =$

$(2n)!/(n!)^2$ components. Based on our previous work, one might expect this to be an irreducible representation, however, it is not. To see this, we construct a specific tensor:

$$B^{i_1 i_2 \dots i_n} \equiv \varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} A^{i_{n+1} \dots i_{2n}}$$

This B will be dual to A, and it can also be easily seen that A is dual to B:

$$A^{i_{n+1} \dots i_{2n}} \equiv \varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} B^{i_1 i_2 \dots i_n}$$

From this pair of tensors that are dual to each other, we can take combinations of them to get two tensors, one that is self-dual and one that is antiself-dual:

$$T_{\pm}^{i_1 i_2 \dots i_n} = \left(A^{i_1 i_2 \dots i_n} \pm B^{i_1 i_2 \dots i_n}\right)$$

To see that these T_+ and T_- are self- and antiself- dual respectively, we simply use multiply them by the antisymmetric symbol and manipulate the tensors algebraically:

$$\begin{aligned}
\varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} T_{\pm}^{i_1 i_2 \dots i_n} &= \varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} \left(A^{i_1 i_2 \dots i_n} \pm B^{i_1 i_2 \dots i_n}\right) \\
&= \varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} A^{i_1 i_2 \dots i_n} \pm \varepsilon^{i_1 i_2 \dots i_n i_{n+1} \dots i_{2n}} B^{i_1 i_2 \dots i_n} \\
&= B^{i_{n+1} \dots i_{2n}} \pm A^{i_{n+1} \dots i_{2n}} \\
&= \pm \left(A^{i_{n+1} \dots i_{2n}} \pm B^{i_{n+1} \dots i_{2n}}\right) \\
&= \pm T_{\pm}^{i_{n+1} \dots i_{2n}}
\end{aligned} \tag{7}$$

(7) shows us that T_+ is dual to itself and that T_- is dual to minus itself.

As such, under a transformation that is an element of $SO(2N)$, T_+ transforms into a linear combination of T_+, and T_- also transforms into a linear combination of T_-. These two tensors are elements of irreducible representations with dimension half of the representation with antisymmetric tensors that we initially had. Specifically, the self-dual and antiself-dual representations will be of dimension $(2n)!/2(n!)^2$. For example, in the case of $SO(4), SO(6), SO(8)$, $d = 3, 10, 35$ respectively.

4.2 Subgroups

Suppose, now, that we already have an irreducible representation of some group G. By restricting our analysis to some subgroup $H \subset G$, it will be common to find that the irreducible representation of G breaks up into several irreducible representations of H.

We will illustrate this by looking at how irreducible representations of $G = SO(4)$ break up into irreducible representations of $H = SO(3)$. In order to examine this, we will be working with the representation of $SO(4)$ by a 4-vector $V^i, i = 1, 2, 3, 4$, and our version of $SO(3)$ will be the rotations that only affect $\{V^1, V^2, V^3\}$. This splits our V^i into two sets $\{V^1, V^2, V^3\}$ and $\{V^4\}$. This can be written as $4 \to 3 \oplus 1$, which is to say that the 4-dimensional representation of $SO(4)$ breaks up into 1- and 3-dimensional representations of $SO(3)$.

Next is the case of the 6-dimensional irreducible representation of $SO(4)$ given by the antisymmetric tensors A^{ij}. These tensors are defined by the six values $A^{14}, A^{24}, A^{34}, A^{12}, A^{23}, A^{31}$. It is clear to see that exactly three of these are affected by elements of our H, and three will not be. As such, in this case, $6 \to 3 \oplus 3$.

Our last example will be the 9-dimensional irreducible representation of $SO(4)$ given by the trace-less symmetric tensors S^{ij}. We begin by picking out that the S^{44}s give the 1-dimensional representation and that our S^{14}, S^{24}, S^{34} give our 3-dimensional representation of $SO(3)$. Finally, our 6 remaining terms in our symmetric tensor, when accounting for the additional restriction that $S^{11}+S^{22}+S^{33}+S^{44}=0$ and that S^{44} is already accounted for, gives us five independent parameters, and thus a 5-dimensional irreducible representation of $SO(3)$. So, for the 9-dimensional irreducible representation of $SO(4)$, we have that $9 \rightarrow 5 \oplus 3 \oplus 1$.

4.3 Adjoint Representation

We now take a short detour from working exclusively with our rotation groups to develop a concept that we will then apply to our $SO(N)$: the adjoint representation. In this type of representation, we are using the structure constants of our Lie Algebra as our representation.

To develop this concept, we start by utiizing the fact that for the Lie Algebra of $SO(3)$, we had three matrix generators:

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, J_y = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, J_z = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By commuting these generators, we find the commutation relation:

$$[J_a, J_b] = i\varepsilon_{abc} J_c; \text{ where } a, b, c = x, y, z$$

An interesting observation that one could make at this point is that the entries of our generating matrices are directly related to the the structure constants (the antisymmetric symbol in this case) as follows:

$$(J_a)_{bc} = -i\varepsilon_{abc}$$

We will now show that we can always have a representation given by our structure constants and this is not just a nice coincidence for $SO(3)$.

In order to do this, we will start with the Jacobi Identity. This identity will hold for any three matrices or operators $A, B,$ and C:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

This can be seen to be valid by simply expanding all of the terms:

$$\begin{aligned}
&[[A, B], C] + [[B, C], A] + [[C, A], B] \\
&= [A, B] C - C [A, B] + [B, C] A - A [B, C] + [C, A] B - B [C, A] \\
&= (AB - BA)C - C(AB - BA) + (BC - CB)A - A(BC - CB) + (CA - AC)B - B(CA - AC) \\
&= ABC + CBA + BCA + ACB + CAB + BAC - BAC - CAB - CBA - ABC - ACB - BCA \\
&= 0
\end{aligned}$$

Now, taking this identity, and using it with the fact that Lie Algebras are defined by their commutation relation:

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{8}$$

with indices $a, b, c, ...$ ranging over n values, where n is the number of generators of the algebra. Now we simply apply the Jacobi identity to T^a, T^b, T^c:

$$\left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^b, T^c\right], T^a\right] + \left[\left[T^c, T^a\right], T^b\right] = 0 \tag{9}$$

By working with (9), we can find that all three of the terms in it are linear combinations of other generators. For example, the first term is equal to $if^{abd}\left[T^d, T^c\right] = (if^{abd})(if^{dcg})T^g$. If we use these equations and then set the coefficients of the T^g's to be zero, we get that:

$$f^{abd}f^{dcg} + f^{bcd}f^{dag} + f^{cad}f^{dbg} = 0 \tag{10}$$

or equivalently, we can define our matrix generator T^b as follows:

$$(T^b)^{cd} = -if^{dbg} \tag{11}$$

In order to prove (11), we manipulate (10) to get back to (8):

$$\begin{aligned}
f^{abd}f^{dcg} &= if^{abd}(T^d)^{cg} \\
f^{bcd}f^{dag} &= -(T^b)^{cd}(-T^a)^{dg} \\
f^{cad}f^{dbg} &= -(-T^a)^{cd}(-T^b)^{dg}
\end{aligned}$$

$$\begin{aligned}
f^{abd}f^{dcg} + f^{bcd}f^{dag} + f^{cad}f^{dbg} &= if^{abd}(T^d)^{cg} - (T^b)^{cd}(-T^a)^{dg} - (-T^a)^{cd}(-T^b)^{dg} \\
&= if^{abd}(T^d)^{cg} + (T^bT^a)^{cd} - (T^aT^b)^{cd} \\
&= if^{abd}(T^d)^{cg} - \left[T^a, T^b\right]^{cg} = 0
\end{aligned}$$

$$\left[T^a, T^b\right]^{cg} = if^{abd}(T^d)^{cg}$$

As such, we can always create a representation of any Lie Algebra by using the structure constants of the algebra. We call this representation the adjoint representation (this representation is of dimension n where n is the number of generators).

4.4 Adjoint Representation of SO(N)

Now that we have developed the idea of an adjoint representation in general, we will finish our discussion of $SO(N)$ by describing the adjoint representation of $SO(N)$. Earlier in the course, we learned two things that we will now put to use: 1.) the antisymmetric tensors T^{ij} in $SO(N)$ provide us with an $\frac{1}{2}N(N-1)$-dimensional irreducible representation of $SO(N)$ and 2.) The number of generators in $SO(N)$ is $\frac{1}{2}N(N-1)$, where the generators represent the defining N-dimensional representation as follows:

$$\mathcal{J}^{ij}_{(mn)} = (\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}) \tag{12}$$

we also define the hermitean matrices $J_{(mn)} = -i\mathcal{J}_{(mn)}$. In (12) the indices m and n are used to identify which of the $\frac{1}{2}N(N-1)$ matrices it is, while i and j are used to determine which of the $N \times N$ entries in the matrix we are working with. If we write (mn) as a instead, and have it range from 1 to $\frac{1}{2}N(N-1)$, we have $\frac{1}{2}N(N-1)$ matrices $\mathcal{J}^{ij}_a$.

Next, by looking at our antisymmetric tensor T^{ij} as an $N \times N$ matrix and express it as a linear combination of our $\mathcal{J}^{ij}_a$'s with some scalars A_a:

$$T^{ij} = \sum_{a=1}^{\frac{1}{2}N(N-1)} A_a\mathcal{J}^{ij}_a$$

Because there are $\frac{1}{2}N(N-1)$ of the T^{ij} and A_a, we can regard them as linear combinations of each other, but this initially seems odd. We address this "weirdness" by looking at how our

A_a transform. We are well-aware of the fact that $T^{ij} \to T'^{ij} = R^{ik}R^{jl}T^{kl} = R^{ik}T^{kl}(R^T)^{lj}$, so we know that:

$$T \to T' = RTR^T = RTR^{-1}$$

We now look at an infinitesimal analogue of this rotation:

$$R \simeq I + \theta_a \mathcal{J}_a$$

$$T' \simeq (I + \theta_a \mathcal{J}_a)T(I - \theta_a \mathcal{J}_a) \simeq T + \theta_a [\mathcal{J}_a, T] \tag{13}$$

From (13), we can then say that the variance of our tensor T under the infinitesimal rotation is given by:

$$\delta T = \theta_a [\mathcal{J}_a, T] = \theta_a [\mathcal{J}_a, T_b \mathcal{J}_b] = \theta_a T_b [\mathcal{J}_a, \mathcal{J}_b]$$

By taking $[J_a, J_b] = if^{abc}J_c$ and modifying it slightly, we can get that $[\mathcal{J}_a, \mathcal{J}_b] = f_{abc}\mathcal{J}_c$. We can now see that:

$$\delta T = \delta A_c \mathcal{J}_c = \theta_a A_b [\mathcal{J}_a, \mathcal{J}_b] = \theta_a A_b f_{abc} \mathcal{J}_c$$

$$\delta A_c = f_{abc}\theta_a A_b \tag{14}$$

From (14), we can finally see that our A_a can also be see as giving us the adjoint representation.

For $SO(3)$, we had the same result, except, as was already acknowledges, in that case, the structure constants are simply the antisymmetric symbol ε_{abc}. Furthermore, because for $SO(3)$, the indices take on three values, so we can recognize A_a as a typical 3-vector, and write that $\delta\vec{A} = \vec{\theta} \otimes \vec{A}$. If we then choose to let $\vec{\theta}$ point in the z-direction, we get the familiar result that $\delta A_x = -\theta A_y$ and $\delta A_y = \theta A_x$. In much the same way, we can write that $\delta\vec{J} = \vec{\theta} \otimes \vec{J}$.

The adjoint representation is so useful because it is a characterization of how the generators of the Lie Algebra transform, and can be constructed for any Lie Algebra.

5 Conclusion

In our exploration of the useful, yet specific, case of the special orthogonal groups, we were able to develop some key ideas that are applicable in more general circumstances by beginning with a very familiar setting. An added bonus of beginning with $SO(N)$ is the usefulness of the rotation groups in Classical Mechanics, Non-relativistic Mechanics, and Electromagnetism, fields of physics that students are very familiar with as well. This concludes our discussion of $SO(N)$ in the course.

References

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