Lie Algebra of $SO(3)$ and Ladder Operators

Fatima Zaidouni

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PHY 391- Prof. Rajeev - University of Rochester

Abstract

In this paper, we construct the irreducible representations of the Lie Algebra $SO(3)$ as a powerful example to be able to deal with more complex algebras. We introduce the ladder operators and address how to multiply two irreducible representations of $SO(3)$ together through the Clebsh-Gordan decomposition.

1 Representation of SO(3)

In the previous lecture, we constructed the irreducible representation of the group $SO(3)$. In this section, we will do so for the Lie algebra of $SO(3)$. Although the two are different, we will use the same notation $SO(3)$. We found that rotations are exponentials of linear combinations of the generators of the $SO(3)$ group which satisfy the following commutation relations, as seen in an earlier lecture,

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y \tag{1}$$

Recall that representing an algebra means finding the generators J_x, J_y, J_z such that equation 1 is satisfied. Note that we will done both the abstract operators and their matrix representation by J_i where $i = x, y, z$

The new generators do not commute per equation 1, therefore they can no be diagonalized simultaneously. Thus, we work in a basis where one of them is diagonal; J_z by convention. Let $J_\pm = J_x \pm iJ_y$ and let f be a test function, we have,

$$\begin{aligned}
[J_z, J_\pm]\, f &= J_z J_\pm f - J_\pm J_z f \\
&= J_z(J_x f \pm iJ_y f) - (J_x \pm iJ_y)(J_z f) \\
&= [J_z, J_x] f \pm i[J_z, J_y] f \\
&= iJ_y \pm i^2 J_x \\
&= \pm J_\pm
\end{aligned} \tag{2}$$

Similarly, we can show that

$$[J_+, J_-] = 2J_z \tag{3}$$

In what follows, we will use Dirac's notation instead of working with matrices. J_z is an eigenvector with eigenvalue m, we write

$$J_z \left|m\right\rangle = m \left|m\right\rangle$$

Since J_z is hermitian, its eigenvalues are real, thus m is real. Note that since $J_\pm$ is not commutative with J_z, $J_\pm$ is not diagonal in the basis where J_z is diagonal. Consider,

$$
\begin{aligned}
J_z J_+ |m\rangle &= (J_+ J_z + [J_z, J_+]) |m\rangle \\
&= (J_+ J_z + J_+) |m\rangle \\
&= (mJ_+ + J_+) |m\rangle \\
&= (m+1) J_+ |m\rangle
\end{aligned} \tag{4}
$$

Where we used both equation 1 and 3. Thus, $J_+ |m\rangle$ is an eigenstate of J_z with eigenvalue $m+1$. Therefore, the state $J_+ |m\rangle$ is equal to the state $|m+1\rangle$ multiplied by a normalization constant. We can write,

$$
J_+ |m\rangle = c_{m+1} |m+1\rangle \tag{5}
$$

Similarly,

$$
\begin{aligned}
J_z J_- |m\rangle &= (J_- J_z + [J_z, J_-]) |m\rangle \\
&= (J_- J_z + J_-) |m\rangle \\
&= (m-1) J_- |m\rangle
\end{aligned} \tag{6}
$$

By the same logic, we have,

$$
J_- |m\rangle = b_{m-1} |m-1\rangle
$$

We can think about $..., |m-1\rangle, |m\rangle, |m+1\rangle, ...$ as rungs of a ladder. J_+ acts as a raising operator that allows us to climb one rung of the ladder each time we use it. Similarly, J_- can be thought of as lowering operator. Thus, $J_\pm$ are the ladder operators.

Since J_x, J_y. J_z are hermitian operators, they are equal to their conjugate transpose, thus,

$$
\begin{aligned}
(J_+)^\dagger &= (J_x + iJ_y)^\dagger \\
&= J_x^\dagger - iJ_y^\dagger \\
&= J_x - iJ_y \\
&= J_-
\end{aligned} \tag{7}
$$

Where $\dagger$ denotes the conjugate transpose. if we multiply equation 5 by $\langle m+1|$, we obtain,

$$
\langle m+1|J_+|m\rangle = c_{m+1} \langle m+1|m+1\rangle = c_{m+1}
$$

Taking the conjugate,

$$
\begin{aligned}
(c_{m+1})^\star &= (\langle m+1|J_+|m\rangle)^\star \\
&= \langle m|J_+^\dagger|m+1\rangle \\
&= \langle m|J_-|m+1\rangle \quad \text{using (7)} \\
&= \langle m|b_m|m\rangle \\
&= b_m \langle m|m\rangle \\
&= b_m
\end{aligned} \tag{8}
$$

Therefore, we proved that $b_{m-1} = (c_m)^\star$. Thus,

$$J_- |m\rangle = b_{m-1} |m-1\rangle = (c_m)^\star |m-1\rangle \tag{9}$$

If we act on equation 9 by J_+ we obtain,

$$J_+ J_- |m\rangle = (c_m)^\star J_+ |m-1\rangle = \mid c_m \mid^2 |m\rangle$$

. Similarly, if we act on $J_+ |m\rangle$ by J_-, we obtain,

$$J_- J_+ |m\rangle = \mid c_{m+1} \mid^2 |m\rangle$$

2 Ladder termination

Since the representation of $SO(3)$ is finite dimensional, the ladder must terminate. Let $\max(m) = j$ thus, there is a state $|j\rangle$ where $J_+ |j\rangle = 0$ that represents the top of the ladder. This implies that

$$J_+ |j\rangle = c_{m+1} |m+1\rangle = 0$$

Thus,

$$\begin{aligned}
\langle j | J_- J_+ | j \rangle &= \langle j | J_+ J_- - [J_+, J_-] | j \rangle \\
&= |j\rangle \left(J_+ J_- |j\rangle - 2J_z |j\rangle \right) \\
&= |j\rangle \left(\mid c_j \mid^2 |j\rangle - 2j |j\rangle \right) \\
&= \mid c_j \mid^2 - 2j \\
&= 0
\end{aligned} \tag{10}$$

Thus $\mid c_j \mid^2 = 2j$. Additionally,

$$\langle m | [J_+, J_-] | m \rangle = \langle m | (J_+ J_- - J_- J_+) | m \rangle = |c_m|^2 - |c_{m+1}|^2 = \langle m | 2J_z | m \rangle = 2m \tag{11}$$

Thus, we have both $\mid c_m \mid^2 = \mid c_{m+1} \mid^2 + 2m$ and $\mid c_j \mid^2 = 2j$. These two equations allows to determine $\mid c_m \mid$ as follows,

$$|c_{j-1}|^2 = |c_j|^2 + 2(j-1) = 2(2j-1)$$

Let $m = j - 2$:

$$|c_{j-2}|^2 = |c_{j-1}|^2 + 2(j-2) = 2(3j-1-2)$$

In general, we can write,

$$|c_{j-s}|^2 = 2[(s+1)j - \sum_{i=1}^{s} i]$$

By Gauss's law $\sum_{i=1}^{s} i = \frac{1}{2} s(s+1)$ so,

$$|c_{j-s}|^2 = 2\left((s+1)j - \frac{1}{2} s(s+1) \right) = (s+1)(2j-s) \tag{12}$$

Substituting $s = 2j$ yields $|c_{j-s}|^2 = 0$ thus,

$$J_-|-j\rangle = c^*_{-j}|-j-1\rangle = 0$$

This implies the the $\min(m) = -j$. s counts the number of rungs climbed down, it is therefore an integer. Since the ladder terminates at $s = 2j$, j is either an integer or half integer depending on whether s is even or odd. Since our states are $|-j\rangle$,....,$|j\rangle$, the total number of states is $2j+1$. To emphasize the j dependence we denote a state $|n\rangle$ by $|j,m\rangle$.

We have shown that the representation of $SO(3)$ are $2j+1$ when j is an integer. Therefore, the methods of tensors(previous lecture) and Lie algebra agree. What about the case when j is a half integer?

When $j = \frac{1}{2}$, $2j+1 = 2$ thus we have a $2-D$ representation consisting of $|-\frac{1}{2}\rangle$, $\frac{1}{2}$ states. This will be solved in an upcoming lecture when discussing $SU(2)$ especially in the context of electron spin.

By equation 12, for $s = j - m = 1$, we have $j - s = j - j + m = m$ thus,

$$|c_m|^2 = (j+m)(j+1-m)$$

which implies,

$$J_+|m\rangle = c_{m+1}|m+1\rangle = \sqrt{(j+1+m)(j-m)}|m+1\rangle \tag{13}$$

Similarly,

$$J_-|m\rangle = c^*_m|m-1\rangle = \sqrt{(j+1-m)(j+m)}|m-1\rangle \tag{14}$$

3 Multiplying two $SO(3)$ representations

Using the tensor approach introduced in the previous lecture, suppose we have two $SO(3)$ tensors; a symmetric traceless tensor S^{ij} and a vector T^k. They furnish the 5-dimensional and 3-dimensional irreducible representations, respectively. Thus, the product $P^{ijk} = S^{ij}T^k$ is a 3-indexed tensor with 15 components. Note that P^{ijk} is not necessarily symmetric and traceless. However since the irreducible representations of $SO(3)$ are furnished by symmetric traceless tensors, we can write P^{ijk} as a linear combination of symmetric traceless tensors.

We can construct the symmetric tensor:

$$U^{ijk} = S^{ij}T^k + S^{jk}T^i + S^{ki}T^j$$

The trace is $U^k = \delta^{ij}U^{ijk} = 2S^{ik}T^i$. To make it traceless, we define:

$$\tilde{U}^{ijk} = S^{ij}T^k + S^{jk}T^i + S^{ki}T^j$$

which furnishes a 7-dimensional irreducible representation.
To extract the antisymmetric part of $S^{ij}T^k$, we contract it with the antisymmetric symbol

$V^{il} = S^{ij}T^k\varepsilon^{jkl}$. The symmetric and antisymmetric parts of V^{il} are $W^{il} = V^{il} + V^{li}$ and $X^{il} = V^{il} - V^l$ respectively.
We can write,

$$\begin{aligned} X^{il} &= \frac{1}{2}X^{il}\varepsilon^{mil} \\ &= S^{ij}T^k\varepsilon^{jkl}\varepsilon^{mil} \\ &= S^{ij}T^k\left(\delta^{jm}\delta^{ki} - \delta^{ji}\delta^{km}\right) \\ &= S^{im}T^i \\ &= \frac{1}{2}U^m \end{aligned} \tag{15}$$

This furnishes a 3-dimensional irreducible representation.
We can see that the symmetric part is traceless by setting $i = l$ in

$$W^{il} = S^{ij}T^k\varepsilon^{jkl} + S^{lj}T^k\varepsilon^{jki}$$

This has $\frac{1}{2}(3.4) - 1 = 5$ components. We have, therefore, showed that:

$$5 \otimes 3 = 7 \oplus 5 \oplus 3 \tag{16}$$

In the general case, the product of $S^{i_1\cdots i_j}$ and $T^{k_1\cdots k_{j'}}$ is a tensor with $j + j'$ indices. If we symmetrize and take out its trace as shown above, we get the irreducible representation labeled by $j + j'$.

If we contract it with ε^{ikl},, we trade two indices, i and k, for one index l, which results in a tensor with $j + j' - 1$ indices. We get the irreducible representation labeled by $j + j' - 1$. We can keep repeating this process. If we let $j \geq j'$. , without loss of generality, we have shown that $j \otimes j'$ contains the irreducible representations $(j + j') \oplus (j + j' - 1) \oplus (j + j' - 2) \oplus \cdots \oplus (j - j' + 1) \oplus (j - j')$. Using the absolute value, we can write:

$$j \otimes j' = \left(j + j'\right) \oplus \left(j + j' - 1\right) \oplus \left(j + j' - 2\right) \oplus \cdots \oplus \left(\left|j - j'\right| + 1\right) \oplus \left|j - j'\right| \tag{17}$$

The number of components in (17) is:

$$\sum_{|j-j'|}^{j+j'} (2k + 1) = \left(j + j' + 1\right)^2 - \left(j - j'\right)^2 = (2j + 1)\left(2j' + 1\right) \tag{18}$$

4 Conclusion

We have used our definition of the algebra $SO(3)$ to construct the irreducible representations of the Lie Algebra $SO(3)$. We introduced the Ladder operators and proved their most important properties. Finally, we showed how to multiply together two irreducible representations of $SO(3)$.

References

Zee, A. (2016). Group theory in a nutshell for physicists. Princeton and Oxford: Princeton University Press.