Kerr Geometry and Rotating Black Holes

PHY391

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December 13, 2018

Abstract

The Schwarzschild metric is not the most generalizable metric since it assumes spherical symmetry. If spherical symmetry is not assumed, a metric known as the Kerr metric becomes relevant. The Kerr metric describes rotating black holes, which are parameterized by angular momentum and mass. This paper will cover the basics of the Kerr metric, orbits about Kerr black holes, and ergospheres.

1 Introduction

Prior to covering this topic, a metric known as the Schwarzschild metric was used for black holes since it was a (relatively) simple solution to the Einstein equations. The main way that the Schwarzschild metric was simple is that it assumed black holes were spherical symmetric and hence were only parameterized by their mass. When dropping the assumption of spherical symmetry, a new metric called the Kerr metric is obtained. Black holes described in the Kerr metric are parameterized by both their angular momentum and mass.

2 Kerr Geometry

Roy Kerr discovered a family of geometries in 1963 now referred to as the Kerr Geometry. This metric is as follows,

$$\boldsymbol{ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \left(\frac{4Marsin^2\theta}{\rho^2}\right)d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2sin^2\theta}{\rho^2}\right)sin^2\theta d\phi^2} \quad (1)$$

where

M is the mass of the black holes, J is the angular momentum of the black hole, $a \equiv J/M$, a is known as the Kerr parameter $\rho^2 \equiv r^2 + a^2cos^2\theta$, $\Delta \equiv r^2 - 2Mr + a^2$.

The coordinates used in this geometry are of the form (t, r, θ, ϕ) and are called the "Boyer-Lindquist coordinates" after the two scientists, Robert Boyer and

Richard Lindquist, who derived them in a 1967 paper. The Kerr metric has two symmetries. One in time (stationary) and one along a certain axis (axisymmetric). The killing vectors that correspond to each symmetry are as follows:

$$\text{Time symmetry: } \xi^{\alpha} = (1, 0, 0, 0)$$

(2)

$$\text{Axis symmetry: } \eta^{\alpha} = (0, 0, 0, 1)$$

(3)

We can also see that the absence of θ killing vectors in Kerr geometry correspond to the lack of spherical symmetry. Another notable feature of Kerr geometry is that there are singularities when $\rho = 0$, which occurs only when $r = 0$ and $\theta = 0$, and when $\Delta = 0$. In the first situation, space time would have an infinite curvature so it is useful to think of this as a "real" singularity. However, Δ vanishes at two radii, r_- and r_+.

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \qquad (4)$$

It turns out that r_+ is the radius of the Kerr black hole's horizon. Seemingly, this assumes that $a \leq M$. However, this instead indicates that the angular momentums of Kerr black holes are limited by the square of their mass. Kerr black holes with the largest possible angular momentum ($J = M^2$) are called "extremal black holes". It tends to be that matter, principally from an accretion disk around the black holes, spirals into the black hole and thus adds its angular momentum to it. This in turn means that many black holes naturally tend towards becoming an extremal black hole. Despite this, black holes of an extremal nature tend to have a Kerr parameter of only around 0.998M. This is because as the angular momentum of the black hole increases, the amount of matter orbiting it that falls into it decreases, which in turn is a result of more matter going through a scattering orbit. In theory, no matter with a positive angular momentum can fall into an extremal black hole. This means that r_+ will always exist for a Kerr black hole and in turns means that the real singularities of a Kerr black hole are always behind their horizon. A notable conjecture in black hole physics that this conclusion supports is that of cosmic censorship, which is that all real singularities are not visible from infinity i.e. all real singularities are hidden behind a horizon from which light cannot escape.

3 Horizons of Kerr Black Holes in Depth

The horizon of a black hole is defined as the interior boundary surface of the region in spacetime where light can escape to infinity from any point. This means that any light inside of the horizon cannot escape to infinity and that any light exactly along the boundary neither go out of or into the horizon. This means that we can think of the horizon as the 3-surface generated by null vectors and the generalized form of the vectors tangent to this surface are as follows:

$$t^{\alpha} = (t^{t}, 0, t^{\theta}, t^{\phi}) \tag{5}$$

where evidentially, since we've already established that the horizon is at a constant r_+, the r component of tangential vectors must be zero. Since the surface is made up of null vectors, at any point on the surface a null vector ℓ in the form of (5) will fulfill the following:

$$\ell \cdot \ell = g_{tt}(\ell^{t})^2 + 2g_{t\phi}\ell^{t}\ell^{\phi} + g_{\phi\phi}\left(\ell^{\phi}\right)^2 + g_{\theta\theta}\left(\ell^{\theta}\right)^2 = 0 \tag{6}$$

Then, by evaluating (6) at r_+, we get,

$$\left(\frac{2Mr_+ sin\theta}{\rho_+}\right)^2 \left(\ell^{\phi} - \frac{a}{2Mr_+}\ell^{t}\right)^2 + \rho_+^2\left(\ell^{\theta}\right)^2 = 0 \tag{7}$$

where ρ_+ is simply ρ with $r = r_+$. Evidentially, $\rho_+, r_+ \neq 0$, so the only solution to (7) is when $\ell^\theta = 0$ and $\ell^\phi = \frac{a}{2Mr_+}\ell^t$. So we can write the generalized solution for the null vectors as,

$$\ell^\alpha = c(1, 0, 0, \Omega_H) \quad (8)$$

where c is some scalar and Ω_H is

$$\Omega_H = \frac{a}{2Mr_+} \quad (9)$$

One may mistakenly assume that since r_+ is constant the horizon and thus the black hole are spherically symmetric. However, it is important to remember that r_+ is constant only in Boyer-Lindquist coordinates. If we apply $r = r_+$ and t = constant to the Kerr metric, a surface with the following line element is created:

$$d\Sigma^2 = \rho_+^2(\theta)d\theta^2 + \left(\frac{2Mr_+}{\rho_+(\theta)}\right)^2 sin^2\theta d\phi^2 \quad (10)$$

The surface/horizon ends up looking like a Boston cream-filled donut, with a wide middle that greatly decreases as you near the poles. The total surface area of the horizon, calculated using (10) is as follows:

$$Surface\ Area\ (A) = 8\pi Mr_+ \quad (11)$$

4 Equatorial Orbits

Since the geometry of Kerr black holes is only axisymmetric, the orbital paths of objects about these black holes are often complex. A simple case, useful for calculating the binding energy of Kerr black holes, is the orbits within the equatorial plane of the black hole (i.e. $\theta = \pi/2$). The Kerr metric in this case looks like this:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4aM}{r}dtd\phi + \frac{r^2}{\Delta}dr^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)d\phi^2 \qquad (12)$$

evidentially, since this new metric lacks a $d\theta$ component (or rather since θ is constant) all four-velocities **u** *have zero θ components. The time independence means energy per unit mass (e) is conserved and so too does the ϕ-independence mean for angular momentum per unit mass along the axis of symmetry (ℓ). Both ℓ and e can be expressed as linear combinations of u^t and u^ϕ in the following way:*

$$e = -\left(g_{tt}u^t + g_{t\phi}u^\phi\right) \qquad (13)$$

$$\ell = g_{\phi t}u^t + g_{\phi\phi}u^\phi \qquad (14)$$

solving for u^t and u^ϕ in (13) and (14) yields:

$$u^t = \frac{dt}{d\tau} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)e - \frac{2Ma}{r}\ell\right] \qquad (15)$$

$$u^\phi = \frac{d\phi}{d\tau} = \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)\ell + \frac{2Ma}{r}e\right] \qquad (16)$$

u *is timelike and hence* **u • u** *= -1. Recalling this, an equation for dr/dτ can be reached since we know* u^t, u^ϕ, *and* u^θ. *This can be written as,*

$$\frac{dr}{d\tau} = \sqrt{e^2 - 1 + \frac{M}{r} - \frac{\ell^2 - a^2(e^2-1)}{2r^2} + M\frac{(\ell - \mathrm{a}e)^2}{r^3}} \qquad (17)$$

which can then be written in a similar form to the Schwarzschild metric:

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r, e, \ell) \qquad (18)$$

where

$$V_{eff}(r, e, \ell) = -\frac{M}{r} + \frac{\ell^2 - a^2(e^2-1)}{2r^2} - M\frac{(\ell - \mathrm{a}e)^2}{r^3} \qquad (19)$$

A radial equation then depends on the impact parameter $b \equiv |\ell / e|$ *and the direction of the orbit relative to the rotation of the black hole (parameter* $\sigma \equiv$ *sign(*ℓ*)), and can be written as*

$$\frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - W_{eff}(r, b, \sigma) \qquad (20)$$

where $W_{eff}(r, b, \sigma)$ *is the effective potential of a photon and can be written as*

$$W_{eff}(r, b, \sigma) = \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2M}{r}\left(1 - \sigma\frac{a}{b}\right)^2\right] \qquad (21)$$

Earlier it was asserted that particles would largely enter a scattering orbit as the Kerr parameter of a Kerr black hole approached 0.998M. While that fact may not be immediately obvious from (21), it is at least apparent that particles falling from infinity will have a different effect potential depending on whether their spiral into the black hole is with or against its rotation.

To prove the assertion made in part 2, consider an extremal Kerr black hole (a = M). Suppose we try dropping a particle with a positive angular momentum. Then we'd expect that, if the particle could enter the black hole, it would increase the angular momentum of it to the degree where r_+ would cease to exist and, presumably, the horizon would disappear. If we, however, consider (18) then it is apparent that a particle with a maximum effective potential greater than $\frac{e^2-1}{2}$ will not execute a bound orbit (since $\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \geq 0$) i.e. it must undergo a scattering orbit where it travels off to infinity. If we consider (19) when a = M, then the minimal effective potential maximum is reached when the particle has $\ell = 2\mathrm{M}e$. Yet at r = M, the maximum of this case, the effective potential is exactly $\frac{e^2-1}{2}$. Consider, then that when $\ell = 2\mathrm{M}e$, the Kerr parameter remains constant since the increase in mass exactly off puts the increase in *angular momentum. It is then apparent that any particle with an* $\ell > 2\mathrm{M}e$ *cannot under the extremal Kerr black hole because it will enter a scattering orbit.*

5 The Ergosphere

For a non-rotating black hole an observer can, assuming they can produce any amount of lift, get arbitrarily close to the horizon of said black hole and remain stationary with respect to infinity. However, this fact changes when the black hole has some rotation. Evidentially, the four-velocity of the observer takes the form:

$$u^{\alpha}_{obs} = (\frac{dt}{d\tau}, 0, 0, 0) \tag{22}$$

and must be timelike, so

$$\boldsymbol{u}_{obs} \cdot \boldsymbol{u}_{obs} = g_{tt}(u^{t}_{obs})^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)(u^{t}_{obs})^2 = -1 \tag{23}$$

yet if g_{tt} *goes to zero, this cannot hold no matter what* u^{t}_{obs} *is (i.e. an observer with a velocity of the form in (22) is not possible).* g_{tt} *goes to zero at r =* $r_e(\theta)$*, which is:*

$$r_e(\theta) = M + \sqrt{M^2 - a^2 cos^2\theta} \tag{24}$$

The surface at $r_e(\theta)$ is the boundary of the space in which stationary observers cannot only have time components of their four-velocity. Since $0 \leq cos^2\theta \leq 1$, this surface will always encompass the horizon of a rotating black hole. The space between the horizon and this surface is known as the ergosphere. Evidentially, when a = 0, there is no difference between $r_e(\theta)$ and r_+, so non-rotating black holes do not have ergospheres (hence why an observer can get arbitrarily close to their horizons).

Even though it is not possible to remain stationary with respect to infinity inside the ergosphere, it is still possible to remain at a fixed r and θ. This is done by rotating with the black hole. The four-velocities of such observers are in the following form:

$$u_{obs}^{\alpha} = u_{obs}^{t}(1, 0, 0, \Omega_{obs}) \tag{25}$$

which can also be written as

$$\boldsymbol{u}_{obs} = u_{obs}^{t}(\boldsymbol{\xi} + \Omega_{obs}\boldsymbol{\eta}) \tag{26}$$

References

[1] James B. Hartle, *Gravity an Introduction to Einstein's General Relativity*. Pearson Education Inc., 2003.

[2] C. MacLauren. "Boyer-Lindquist coordinates." http://colinscosmos.com/wiki/boyer-lindquist-coordinates/. Accessed on 2018-12-14.