The Building Blocks for General Relativity in a Smooth, Expanding Universe

Kelly Aspinwall

13 May 2021

Kapitza Society Spring 2020

Introduction

In order to properly study cosmology, a set of mathematical tools are required for describing the universe. Without these, one cannot evaluate the mathematical aspect of spacetime. There would be immense amounts of hand-waving and jumping to conclusions, as well as blind trust in the author and conceptual intuition. This is not physics. So, in order to be able to rigorously study and understand the universe, one must begin with the "basics." In this case, basic meaning fundamental rather than simple. Here, a solid foundation is built from which more complex observations can be derived. The foundation in this case is the mathematical framework of general relativity.

Assumptions

A part of building the foundation is first evaluating the base case. Here, that means that it will be assumed that:

1. The universe is smooth, meaning no densities vary as a function of space.
2. The universe is in equilibrium.

Again, these things are not universally true, but the case explored here is one in which they are, as this is the simplest version of reality. Additionally, the convention that will be followed is this:

$$\hbar = c = k_B = 1$$

The Metric

Distance is a surprisingly complex concept. It seems pretty simple and explainable, but it turns out that it is not, especially when considering an expanding universe. Distance is usually thought of in terms of a coordinate system, but no single, basic coordinate system can account for the curvature of spacetime or the expansion of the universe. This is where the metric comes into play. The metric takes coordinate distance and turns it into physical distance, such that it no longer relies on the coordinate system of the measurement. This physical distance is invariant; it does not change based on coordinate transformations of different observers. Starting with the simplest case, the square of the invariant distance in two-dimensional space is as follows:

$$dl^2 = \sum_{i,j=1,2} g_{ij} dx^i dx^j$$

The metric here is g_{ij}, a 2x2 matrix. It is different depending on the coordinate being used. Here are the matrices for two common coordinate systems within this case:

Cartesian	Polar
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

A helpful aspect of the metric is that it accounts for the effects of gravity on spacetime. Rather than considering gravity as an external force that affects matter, the metric allows for gravity to be built into the geometry of space.

Spacetime, however, is not two-dimensional but rather four-dimensional. Along with convention, the zeroth dimension is time, and the first, second, and third are the three spatial dimensions. In this case, the square of the invariant distance is as follows:

$$ds^2 = \sum_{\mu,\nu}^{3} g_{\mu\nu} dx^{\mu} dx^{\nu}$$

In which $g_{\mu\nu}$ is a 4x4, symmetrical matrix. In Minkowski spacetime, the Minkowski metric is used. Thus, $g_{\mu\nu} = \eta_{\mu\nu}$, and:

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To adapt this for the expanding, flat universe, the Friedmann-Robertson-Walker, or FRW, metric is used. This takes the Minkowski metric and multiplies the spatial components by a scale factor of $a^2(t)$, resulting in:

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix}$$

The Geodesic Equation

In cartesian coordinates, the shortest distance from one point to another is a straight line between the two points. In more complicated geometries, this rule often fails to find the shortest possible path. The equation used to find this path is the geodesic, which is the path a particle follows in the absence of any external forces. To satisfy this, the acceleration of a particle needs to be set to zero.

In order to generalize this, the base case is once again a freely moving particle in two-dimensional Euclidian space, which yields:

$$\frac{d^2x^i}{dt^2} = 0$$

Applying a known coordinate system once again, how can this be generalized to polar coordinates? The basis vectors for polar coordinates are $\hat{r}$ and $\hat{\theta}$. These vectors vary over space, whereas the basis vectors for cartesian coordinates do not. Setting $x'^i = (r, \theta)$, the above condition does not imply the following condition:

$$\frac{d^2x'^i}{dt^2} = 0$$

Instead, the cartesian condition needs to be transformed. This is done using the transformation matrix as follows:

$$\frac{\partial x^i}{\partial x'^j} = \begin{bmatrix} \cos(x'^2) & -x'^1 \sin(x'^2) \\ \sin(x'^2) & x'^1 \cos(x'^2) \end{bmatrix}$$

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}$$

Thus transforming the velocity of the particle from cartesian coordinates to polar coordinates. The geodesic equation then becomes the following:

$$\frac{d}{dt}\left[\frac{dx^i}{dt}\right] = \frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = 0$$

Note that the time derivatives cannot simply cancel out, as the transformation being used is not linear. The following equality emerges from the above equations:

$$\frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\right] = \frac{\partial}{\partial x'^j}\left[\frac{dx^i}{dt}\right] = \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}$$

And thus, the following geodesic equation arises:

$$\frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = \frac{\partial x^i}{\partial x'^j}\frac{d^2x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0$$

The transformation is being used on the first term of the middle portion of the equality. Thus, this equality can be multiplied by the inverse of the transformation matrix to get a more palatable form.

$$\frac{d^2x'^l}{dt^2} + \left[\left(\left\{\frac{\partial x}{\partial x'}\right\}^{-1}\right)_i^l \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\right] \frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0$$

From this equation, the Christoffel symbol Γ_{jk}^l can be defined as the term in the square brackets. This symbol becomes useful in many calculations withing general relativity.

To generalize the derived geodesic equation to general relativity, the range of the indices changes from 1 and 2 to 0 to 3. Also, time can no longer be used as the standard evolution parameter, since it is now one of the coordinates. Thus, the parameter λ is introduced. This parameter increases along a particles path. Using this and the Christoffel symbol, the geodesic equation becomes the following:

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

Here, the Christoffel symbol was found through generalization of the geodesic over transformation of coordinates. However, it is generally simpler to use the following relation to find the Christoffel symbol.

$$\Gamma_{\alpha\beta}^\mu = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]$$

Einstein Equations

The Einstein equation for general relativity is as follows:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu}$$

There are many new terms in this equation, so here is a list of what they all are:

- $G_{\mu\nu}$: Einstein Tensor
- $R_{\mu\nu}$: Ricci Tensor
- $\mathcal{R}$: Ricci Scalar; contraction of the Ricci tensor, such that $\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu}$
- G: Newton's Constant
- $T_{\mu\nu}$: Energy-Momentum Tensor

The Einstein equations relate a function of the metric to a function of energy.

The Ricci Tensor can be defined as follows:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

In which the notation of ,i in the subscript denotes a partial derivative over x^i. Due to the nature of the Christoffel symbol, the Ricci Tensor only does not vanish for two cases: μ=ν=0 and μ=ν=i, with i being a spatial coordinate and 0 being the time coordinate.

$$R_{00} = -\Gamma^{i}_{0i,0} - \Gamma^{i}_{j0}\Gamma^{j}_{0i} = -\delta_{ii}\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 \delta_{ij}\delta_{ij} = -3\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = \delta_{ij}[2\dot{a}^2 + a\ddot{a}]$$

These can then be used to compute the Ricci Scalar.

$$\mathcal{R} = -R_{00} + \frac{1}{a^2}R_{ii} = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right] = 6\left[\frac{\ddot{a}}{a} + \frac{8\pi G}{3}\rho\right]$$

Reference

Dodelson, Scott. (2003). “2.1: The Smooth, Expanding Universe; General Relativity,” *Modern Cosmology* (pp.23-33). Academic Press.