Inflationary Period: The Horizon Problem

Kai J. Reed

University of Rochester

Spring 2021 Kapitza Society

Abstract

Perhaps one of the most famous problems in Cosmology, the Horizon problem asks how particles outside the limits of causal physics could share many similar attributes, the most worrisome being Temperature and Energy. These limits are known as the "Comoving Horizon" and the "Comoving Hubble Radius." The explanation of the problem, as well as a possible solution known as "Inflation Theory," will be presented in the following paper.

Theory

Before a solution to the Horizon problem can be presented, the problem must be explained. Back in the 1960's, the CMB (Cosmic Microwave Background) allowed physicists to make a much needed conclusion that our Universe started with a Big Bang, rather than being a Steady State Universe.[1] However, it also raised other questions due to its uniformity and isotropic nature. If one thinks of the CMB as a free-stream of photons, most of which are at about 3K on the Kelvin temperature scale, then the question arises how photons coming from opposite reaches of space can share such uniformity in energy. **Figure-1** below illustrates this conundrum.

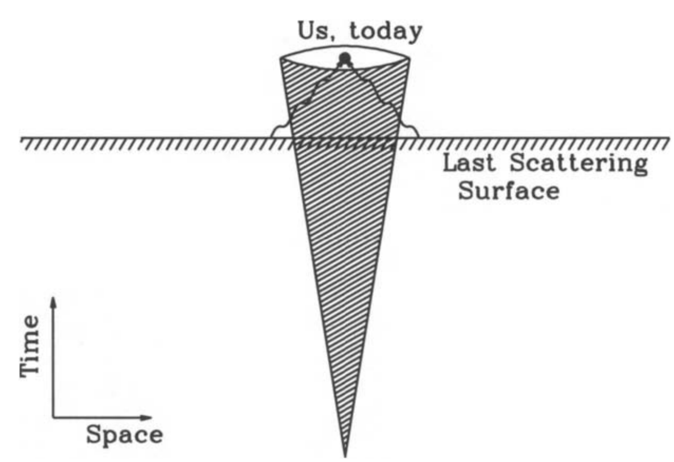


Figure-1, Dodelson pg. 145

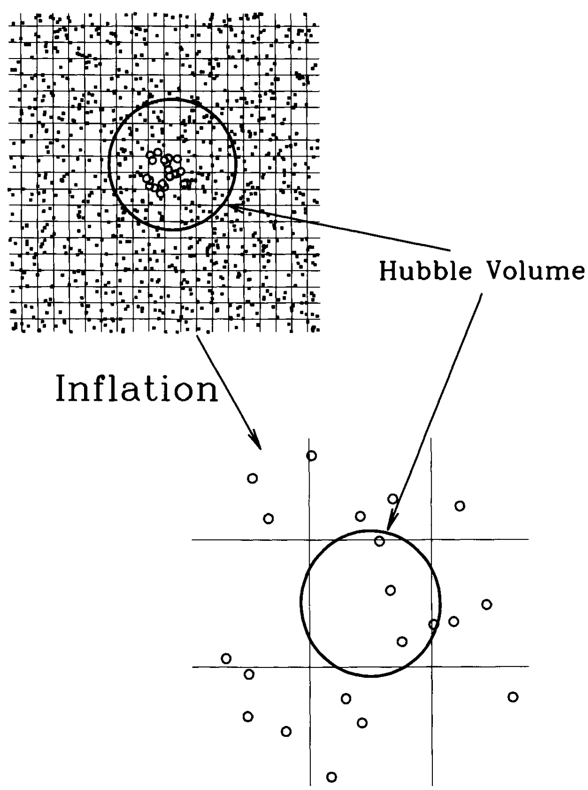
Figure-1 graphs a light cone (shaded region) in which we can view causal physics and measure what happens, whereas anything outside the light cone is unable to interact with us or other objects of the Universe on opposite sides of the light cone. The two squiggly-lines represent photons from the CMB, each beginning outside the realm of causality with us and one another. However, once they've traversed into the light cone, we are able to measure and view how they interact with other cosmological objects. It turns out that, despite being absurdly out of touch with one another, both photons will carry similar energies and thus obtain similar temperatures. This is the Horizon Problem: these photons could not possibly communicate to one another before they reached the light cone, or crossed within the Comoving Horizon, yet if they share similar temperatures Compton Scattering tells us they probably scattered from the same surface.

How is this possible? For years physicists tried to explain the phenomena to no avail, until some time in the 1980's the idea of "inflation" began working its way into the theory. Cosmologists today are not entirely sure the Inflation Model is truly the answer to the Horizon Problem, but another theory that solves many of the questions in the Horizon Problem has yet to be developed as much as Inflation has been, thus it is currently the "best guess."

The theory states that for a short period of time at the very genesis of the Universe, physical distances of microscopic scale blew-up to cosmic proportions. For instance, an Angstrom of space (about 10^{-10} meters) grew to Megaparsec scale (about $3*10^{6}$ lightyears) in the span of perhaps a few fractions of a second. This exponential growth is why the term "inflation" is used in descriptions of the theory, but just "how big" was this growth?

To attempt a derivation, the usage of Modern Cosmology by Scott Dodelson will be necessary. Most of the math is taken from Chapter 6, but other ideas such as the CMB and Boltzmann Equations come from earlier chapters. To start, remember that Inflation took place before anything else happened after the Big Bang. While the Universe was still a hot soup of photons, Inflation occurred and basically stretched the physical space where these photons existed. It expanded so quickly that the photons were unable to continue interacting with each other, or anything else for that matter, thus whatever they scattered off before Inflation decided their energies. **Figure 2** exemplifies this expansion.

Figure 2, Dodelson pg. 148



The photons didn't stop moving, however, thus once they traveled unabated through empty space to reach us now, we can measure them to have the exact energies they had back before Inflation occurred. These photons are the particle make-up of the CMB we see today, but we do not see them until they converge back into the Hubble Volume on **Figure 2**, if we were to put ourselves at its center. To define the Hubble Volume, a few distinctions must be made. The Comoving Horizon has been mentioned, and a simple definition is that it represents the distance causal physics is allowed to occur across space for all of time. No matter what point in time in the Universe, matter dominated era or radiation dominated era or something else, the Comoving Horizon defines the total distance light can travel since the very genesis of the Universe at time $t = 0$. (This definition comes from page 51 of Dodelson.)

Along with the Comoving Horizon, another cosmic distance of great importance is the Hubble Radius. This is what is meant by "Hubble Volume," in that the volume is a sphere of physical space with radius equal to the Hubble Radius. Now, the Hubble Radius is dependent on the Hubble Rate, which in turn depends on the scale factor of rapid expansion of the Universe. The scale factor is denoted $a(t)$, a function of time. However the time-dependence changes as the Universe evolves. For instance, $a(t) \propto t^{1/2}$ during radiation-dominated era, whereas matter-dominated eras have scale factor $a(t) \propto t^{2/3}$(Dodelson pg. 3). This leads us to the definition of the Hubble Rate, denoted $H(a)$, and is given below as a mathematical expression.

Hubble Rate

$$H(a) = \frac{da/dt}{a(t)}$$

The Hubble Rate is a ratio of how fast the Universe expands divided by the total distance it has expanded, giving it units of $(seconds)^{-1}$. Due to these units, inverting the Hubble Rate gives a measure of just how long the Universe has existed, with current estimates around 14 billion years. With these definitions, we can now define the **Comoving Hubble Radius**, the expression of the same name below. This defines the distance particles can travel over one expansion time, or over the time it takes for the scale factor, $a(t)$, to double (Dodelson pg. 146).

Comoving Hubble Radius

$$R = (a(t) \cdot H(a))^{-1}$$

To connect this back to the theory of Inflation, remember that it is the idea that physical distances were much, much smaller than they are now, thus it makes sense to look at how the Comoving Hubble Radius needed to change during this period of Inflation to get the cosmological scale we see today. Referring back to **Figure 2**, before inflation the Comoving Horizon resided within the Hubble Volume, but after Inflation it seems the Hubble Volume "shrunk," thus allowing particles that were in causal contact before are no longer in contact afterwards. To put numbers to this shrinkage, let's define the **Comoving Horizon** in terms of the Hubble Radius, denoted by the greek letter eta, and discuss what it means for the Radius to shrink.

Comoving Horizon (Dodelson pg. 146)

$$\eta = \int_0^a \frac{da'}{a'} \frac{1}{a'H(a')}$$

The primes in the equation are used to follow Integration rules, the variable integrated over cannot also be a limit to the integral, but what we see is that the Comoving Horizon is the logarithmic integral of the Hubble Radius. This implies then that shrinking the Hubble Radius corresponds to exponential growth. However, what would cause the Hubble Radius to shrink? During eras of radiation dominant or matter dominant epochs, the Hubble Radius will increase, so if it is to decrease, there must be some other dominant form of energy density. Some believe this to be "dark energy," with representation in the Einstein Equations as the Cosmological Constant Λ, but the mechanism isn't the focus of this paper.

Since we cannot base the evolution of the Hubble Radius on what type of energy dominates the Universe during Inflation, we can instead define it by how the scale factor, $a(t)$, evolves during this time period. The common way to do this is to look at how we defined the **Hubble Rate**, $H(a)$, and hold the value H itself to be constant. Rearranging **Hubble Rate** then yields the following expressions:

$$\frac{da}{a} = Hdt$$

$$ln(a(t)) - ln(a_e(t)) = H(t - t_e)$$

$$\Rightarrow a(t) = a_e e^{H(t-t_e)}$$

Here all subscript-e variables represent values at the end of Inflation, and the derivation is based on Dodelson page 147. Since we chose to hold H constant, this restriction will also apply to the Comoving Hubble Rate and therefore also to the Comoving Horizon. This means any decrease in the Hubble Radius, $(a(t)\cdot H)^{-1}$, is exclusively from an exponential increase in the scale factor $a(t)$.

To fully understand the implications, however, we look to the Einstein Equations of General Relativity. Noting that $a(t)$ is always positive, the natural number e can never be less than or equal to zero and a_e represents physical expansion which we know cannot be negative, we can see that it increases during Inflation. Now, knowing that a decrease in $(a(t)\cdot H)^{-1}$ implies an increase over time for $(a(t)\cdot H)$, thus:

$$\frac{d}{dt}[a\cdot\frac{da/dt}{a}] = \frac{d^2a}{dt^2} > 0$$

This shows that the increase in the scale factor actually *accelerates* during Inflation, and we can now be presented with the following **Einstein Equations** from page 151 of Dodelson.

Einstein Equations

$$H^2 = (\frac{da/dt}{a})^2 = \frac{8\pi G}{3}\rho$$

$$\frac{d^2a/dt^2}{a} + \frac{1}{2}H^2 =- 4\pi G\mathcal{P}$$

Here rho is the energy density and the script P is the pressure of the Universe, and G is Newton's Gravitational Constant. Combining them in the way described on page 151 of Dodelson is presented below, where the first is multiplied by ½ then subtracted from the second:

$$\frac{1}{2}H^2 = \frac{4\pi G}{3}\rho$$

$$\Rightarrow \frac{d^2a/dt^2}{a} + \frac{1}{2}H^2 - \frac{1}{2}H^2 =- 4\pi G\mathcal{P} - \frac{4\pi G}{3}\rho$$

$$\frac{d^2a/dt^2}{a} =- \frac{4\pi G}{3}(3\mathcal{P} + \rho)$$

It has been shown that expansion accelerates during Inflation, thus the term in parentheses must also be negative:

$$3\mathcal{P} + \rho < 0$$

$$\Rightarrow \mathcal{P} < -\frac{\rho}{3}$$

This shows that for Inflation to occur, there must be a negative pressure in the Universe causing the accelerated expansion. This solidifies the idea that this era of the Universe's history is neither radiation or matter dominant, but what else could possibly have a non-zero energy density to cause this negative pressure? According to the official NASA website, link given in **Sources** at the end of this paper, it is a "cosmological constant type of vacuum energy density," but beyond this notion is a world of physics in need of deeper exploration.

Conclusion

We have discussed the Theory of Inflation and shown that it is a possible answer to the Horizon Problem developed by Cosmologists back in the 1980's. It aims to explain why the CMB is so isotropic despite photons being separated by distances much greater than the Comoving Horizon with the idea of a shrinking Hubble Radius. Due to this idea the concept of "negative pressure" was found to be the mechanism by which Inflation occurred. It is here though that more experimentation is needed before any conjectures can be made as to the physical nature of this mechanism.

Sources

Dodelson, S. (2002). *Modern cosmology: anisotropies and inhomogeneities in the universe*. Academic.

NASA/WMAP Science Team. (2010, April 16). *WMAP Inflation Theory*. NASA. https://wmap.gsfc.nasa.gov/universe/bb_cosmo_infl.html.