Gravitational Lensing

Fatima Zaidouni

Thursday, December 20, 2018

PHY 391- Prof. Rajeev - University of Rochester

Abstract

In this paper, we explore how light bends under the effect of a gravitational field, a phenomenon referred to as Gravitational lensing. After introducing the topic, we dive into derivations of the angle by which light deflects as it passes by a massive body. Making use of the fact that Fermat's principle holds in general relativity, we derive an index of refraction that describes the effect of the graviational potential by a standard variational problem and make use of Euler-Lagrange equations to derive the angle of deflection for the weak gravity metric. We then generalize our derivation by considering the schwarzschild metric, deriving the orbital equations and integrating to obtain the deflection angle. We conclude by explaining the Saphiro delay.

1 Introduction

The subject of our paper is one of Newton's first queries: "Do not Bodies act upon Light at a distance, and by their action bend its Rays; and is not this action strongest at the least distance ?" (Newton, 1704).

This idea that mass can bend light comes to life in the context of **Gravitational lensing**. The gravitational field of a massive object will cause light rays passing by to bend. This is illustrated very well in Figure 1. In relatively rare observable occasions, the effects of gravitational lensing can manifest itself as a group of stretched out, lensed galaxies forming arcs around a cluster. One famous example is Abell 2218 as shown in Figure 2.

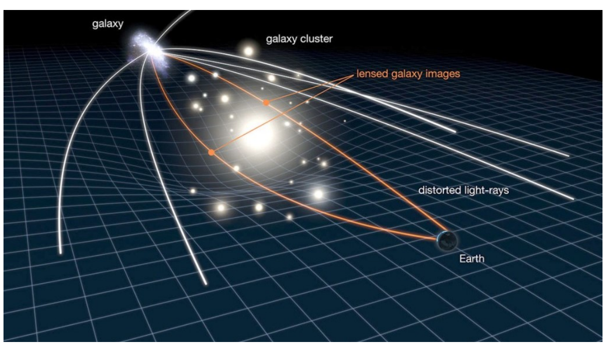


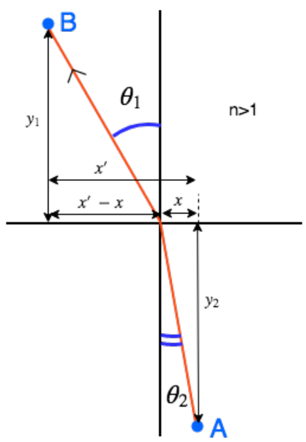
Figure 1: Gravitational lensing, exagerated scale (Calcada, 2011).

Figure 2: the galaxy cluster: Abell 2218 (Richard, 2008).

2 Derivations and Discussion

As explained in an online lecture by Prof. Massimo (Meneghetti); we will start off with Fermats principle, and then determine the index of refraction, after which we can derive the angle of deflection for the weak-gravity metric.

We will first briefly demonstrate a popular application of Fermat's principle as a starting point to transition to the notion of a continuously varying refractive index.



When light travels from point A to point B as in the figure on the left, it chooses the path that takes the least amount of time. Let the refractive index of the destination medium be n, hence, the speed of light in this medium is $\frac{c}{n}$ with c being the vacuum speed of light. Therefore, refering to the figure on the same figure, we have:

$$t = \frac{\sqrt{y_1^2 + x^2} + n\sqrt{y_2^2 + (x' - x)^2}}{c} \tag{1}$$

We can then obtain Snell's law by simplifying $\frac{dt}{dx} = 0$ which implies that:

$$\frac{x}{\sqrt{y_1^2 + x^2}} = n\frac{x' - x}{\sqrt{y_2^2 + (x' - x)^2}} \tag{2}$$

The L-H-S is equivalent to $sin(\theta_1)$ and the R-H-S is just $sin(\theta_2)$. Hence,

$$\sin\theta_1 = n\sin\theta_2 \tag{3}$$

What is important to realize is that Snells law allows us to describe the path of light passing through a boundary from one region of constant index of refraction to another region of constant index of refraction. In our context of gravitational lensing, we are dealing with a medium of continuously varying refractive index(Jacobsson, 1966). To clarify, the surroundings of a massive body are perturbed by its gravitational field, they can be treated as a medium whose refractive index varies continuously, this is the key motivation of the derivation outlined next.

We first need to determine a varying index of refraction n that describes our gravitational field. We will make the assumption that our gravitational potential is very small compared to c^2 (Eq. 4), this is a valid assumption to make for most of the astrophysical objects of great interest but we will still generalize using a more "formal" method in the next section.

$$\frac{\phi}{c^2} << 1 \tag{4}$$

The line element of the weak gravity metric is:

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - (1 - \frac{2\phi}{c^2}) d\vec{x}^2 \tag{5}$$

with ϕ denoting the gravitational potential, $\vec{x}$ being the spatial dimension and ds being the line element. Light follows null geodesics so

$$ds = 0$$

We can therefore determine:

$$c' = \frac{|d\vec{x}|}{dt} = c\sqrt{\frac{1 + \frac{2\phi}{c^2}}{1 - \frac{2\phi}{c^2}}} \approx c(1 + \frac{2\phi}{c^2}) \tag{6}$$

Where c' is the speed of light within the gravitational field. The resulting index of refraction is therefore equal to

$$n = \frac{c}{c'} = \frac{1}{1 + \frac{2\phi}{c^2}} \approx 1 - \frac{2\phi}{c^2} \tag{7}$$

Finding the deflection angle is a variational problem that starts with looking for the path, $\vec{x}(l)$, for which:

$$\delta \int_a^b n\vec{x}(l) dl = 0 \tag{8}$$

Where a and b are fixed endpoints. We let:

$$dl = \left|\frac{d\vec{x}}{d\lambda}\right| d\lambda \tag{9}$$

Where λ is an arbitrary curve parameter. Eq 8 becomes:

$$\delta \int_{\lambda_a}^{\lambda_b} L(\dot{\vec{x}}, \vec{x}, \lambda) d\lambda = 0 \tag{10}$$

Where L is the lagrangian satisfying,

$$L(\dot{\vec{x}}, \vec{x}, \lambda) = n(\vec{x}(\lambda)) \left|\frac{d\vec{x}}{d\lambda}\right| \tag{11}$$

The corresponding Euler's equations can be written as:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 \tag{12}$$

Since $\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = \vec{\nabla} n|\dot{\vec{x}}|$ and $\frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$, we have:

$$\frac{d}{d\lambda}(n\dot{\vec{x}}) - \vec{\nabla} n = 0 \tag{13}$$

Which implies

$$n\ddot{\vec{x}} = \vec{\nabla} n - \dot{\vec{x}}(\vec{\nabla} n \cdot \dot{\vec{x}}) \tag{14}$$

The R-H-S is equal to the gradient of n perpendicular to the light path since the second term of the R-H-S is equivalent to the derivative along the light path. Therefore,

$$\ddot{\vec{x}} = \frac{1}{n}\vec{\nabla}_{\perp} n = \vec{\nabla}_{\perp}\ln(n) \tag{15}$$

By Eq 7 and Eq 4, we get

$$\ddot{\vec{x}} \approx \frac{-2}{c^2}\vec{\nabla}_{\perp}\phi \tag{16}$$

Since the new deflection angle satisfies $\vec{\theta}_{\text{new}} = \int_{\lambda_a}^{\lambda_b} \ddot{\vec{x}} d\lambda$ it can, therefore, be written as:

$$\vec{\theta}_{\text{new}} = \frac{2}{c^2}\int_{\lambda_a}^{\lambda_b} \vec{\nabla}_{\perp}\phi d\lambda \tag{17}$$

Since $\frac{\phi}{c^2} << 1$, the deflection angle will be small enough that we can use the Born approximation, as in scattering problems, if the closest approach of light to the body happens at $u = 0$ with an impact parameter b we obtain the following:

$$\vec{\theta}_{\text{new}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_{\perp}\phi du \tag{18}$$

In the next section, we will apply Eq 18 to a point mass to get the famous angle of deflection that Einstein derived in 1915.

2.1 Application on a point mass body

For a point mass, the gravitational potential is:

$$\phi = \frac{-GM}{r} \tag{19}$$

In cartesian coordinates, the impact parameter is

$$b = \sqrt{x^2 + y^2}$$

and

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2}$$

The gravitational potential perpendicular to the light path is:

$$\vec{\nabla}_{\perp}\phi = \begin{cases} \text{x-axis: } \partial_x\phi = \frac{GMx}{r^3} \\ \text{y-axis: } \partial_y\phi = \frac{GMy}{r^3} \end{cases} \tag{20}$$

Using Eq 18, we can write

$$\vec{\theta}_{\text{new}}(b) = \begin{cases} \text{x-axis: } \vec{\theta}_{\text{new},x}(b) = \frac{2GMx}{c^2} \int_{-\infty}^{+\infty} \frac{dz}{(b^2+z^2)^{\frac{3}{2}}} \\ \text{y-axis: } \vec{\theta}_{\text{new},y}(b) = \frac{2GMy}{c^2} \int_{-\infty}^{+\infty} \frac{dz}{(b^2+z^2)^{\frac{3}{2}}} \end{cases} \tag{21}$$

The integrals simplify to:

$$\vec{\theta}_{\text{new}}(b) = \begin{cases} \text{x-axis: } \vec{\theta}_{\text{new},x}(b) = \frac{4GM}{c^2 b} \cos\phi \\ \text{y-axis: } \vec{\theta}_{\text{new},y}(b) = \frac{4GM}{c^2 b} \sin\phi \end{cases} \tag{22}$$

Since $\begin{cases} x = b\cos\phi \\ y = b\sin\phi \end{cases}$, Eq 22 becomes

$$|\vec{\theta}_{\text{new}}| = \frac{4GM}{c^2 b} \tag{23}$$

For the deflection of light as it is passing by the surface of the sun, we obtain:

$$\theta_\odot = \frac{4GM_\odot}{c^2 b_\odot} \simeq \frac{4 \times 6.674 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \times 1.989 \times 10^{30}\text{kg}}{(2.998 \times 10^8 \text{ms}^{-1})^2 \times 6.957 \times 10^8 \text{m}} \simeq 8.492 \times 10^{-6}\text{rad} \tag{24}$$

2.2 Formal strong-deflection bending angle in the Schwarzshild metric

As mentioned in (Iyer,2007), a "Schwarzschild black hole is the unique static, spherically symmetric, asymptotically flat vacuum solution of the Einstein equation" defined by the following metric (in $c = 1$ units):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{25}$$

The explicit metric is written as:

$$g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{26}$$

We follow the derivation presented in Chapter 9 of Hartle. We first notice that the metric is independent of both t and ϕ, we can therefore write the following constant quantities as function of the corresponding Killing vectors ξ and η :

$$e \equiv -\xi \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \text{and} \qquad \ell \equiv \boldsymbol{\eta} \cdot \mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\lambda} \tag{27}$$

Where $u^\alpha = \frac{dx^\alpha}{d\lambda}$ and λ is an affine parameter.
We then make use of the requirement that the tangent vector must be null:

$$\mathbf{u} \cdot \mathbf{u} = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

and consider that the orbit of the light ray is in the equatorial plane (ie. $\theta = \frac{\pi}{2}$), we get:

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{28}$$

Substituting Eq. 27 in Eq. 28 and multiplying by $\frac{(1-\frac{2M}{R})}{l^2}$ yields:

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \tag{29}$$

Where $b^2 = \frac{l^2}{e^2}$.
Eq. 29 has the form of an energy intergral for the radial motion with $\frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$ representing the effective potential and b^{-2} representing the energy. By freedom of normalization of the affine parameter, we can make our equations rely solely on the ratio of $\frac{l}{e}$ as we will see next.

We define $b = |\frac{l}{e}|$ and consider a light ray moving parallel to the x-axis a distance of d away from it, b represents the impact parameter since for far enough away from the source of curvature, the light ray is moving in a straight line and therefore, for $r >> 2M$, the quantity b just d.
Solving Eq. 27 for $\frac{d\phi}{d\lambda}$ and $\frac{dr}{d\lambda}$, dividing the results, and then simplifying using $b^2 = \frac{l^2}{e^2}$, Eq.29 yields:

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{30}$$

We integrate to obtain the shape of the orbit, notice that the total angle swept out as the light ray goes from infinity and back out again is just twice the angle swept out from the turning point $r = r_1$ to infinity as indicated in the following figure:

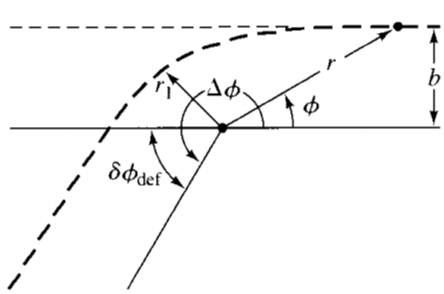


Figure 3: Quantities of the deflection of light (Hartle,2003)

The integration yields:

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{31}$$

We replace $r = \frac{b}{w}$ and expand both inverse factors of $1 - (\frac{2M}{b})w$ in powers of $\frac{2M}{b}$ and keep only the linear terms, we obtain our final result:

$$\Delta\phi = 2\int_{0}^{w_1} dw \frac{1 + (M/b)w}{[1 + (2M/b)w - w^2]^{1/2}} \tag{32}$$

As a comparison to our result in the last section, we find the solution to this integral for small $\frac{M}{b}$ to be:

$$\Delta\phi \approx \pi + \frac{4M}{b} \tag{33}$$

As indicated in the last figure, the angle of deflection is $\delta\phi_{\text{def}}$ which is related to $\Delta\phi$ by $\delta\phi_{\text{def}} = \Delta\phi - \pi$, reinserting G and c yields:

$$\delta\phi_{\text{def}} = \frac{4GM}{c^2 b} \tag{34}$$

which is the same result obtained in Eq. 23.

2.3 Shapiro delay

As explained in (Shapiro, 1964), since the speed of light is less when it is traveling through a gravitational field, we can determine the additional time it needs to travel relatively to the time needed to cover the same distance in a vacuum.

We outline an valuable experiment explained in Hartle. If a radar signal is sent from Earth to pass close to the Sun and reflect off another planet or a spacecraft. The time interval between the emission of the first pulse and the reception of the reflected pulse can be measured.

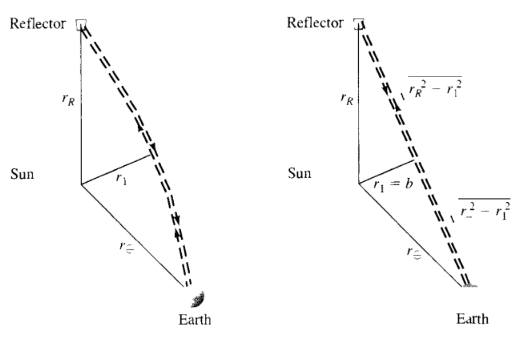


Figure 4: On the left, radar waves are sent from earth to a distant reflector, their path is deflected. On the right, it shows their path if they were propagating in flat spacetime(straight lines) (Hartle , 2003)

As shown in the figure, we let $r_\oplus$ be the radius of the orbit around the earth, r_R be the radius of the orbit around the reflector, and r_1 be the Schwarzschild radius of closest approach to the sun.

We consider the Earth to be stationary over the travel time of the pulse. The total time interval between the emission and return of a pulse as measured by a clock on Earth is the Schwarzschild coordinate time interval $(\Delta t)_{\text{total}}$ between these events corrected for the influence of the Earth on spacetime and other effects.
We need to fint t as a function of r along the path of the pulse. This is like finding the shape of the orbit.
We solve Eq. 27 for $\frac{dt}{dx}$ and for $\frac{dr}{dx}$, and divide them to obtain:

$$\frac{dt}{dr} = \pm\frac{1}{b}\left(1-\frac{2M}{r}\right)^{-1}\left[\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-1/2} \tag{35}$$

The total elapsed time is:

$$(\Delta t)_{\text{total}} = 2t\left(r_\oplus, r_1\right) + 2t\left(r_R, r_1\right) \tag{36}$$

where $t(r, r_1)$ is the travel time from the turning point r_1 to a radius r given by (first order approximation):

$$t\left(r, r_1\right) = \sqrt{r^2 - r_1^2} + 2M\log\left[\frac{r+\sqrt{r^2-r_1^2}}{r_1}\right] + M\left(\frac{r-r_1}{r+r_1}\right)^{1/2} \tag{37}$$

The first term is the Newtonian expression for the propagation time as in the right part of Figure 4, the next terms are the relativistic corrections.
The excess delay over the Newtonian prediction, for $r_1/r_R \ll 1$ and $r_1/r_\oplus \ll 1$ is (reinserting G and c):

$$(\Delta t)_{\text{excess}} \approx \frac{4GM}{c^3}\left[\log\left(\frac{4r_R r_\oplus}{r_1^2}\right)+1\right] \tag{38}$$

For instance, when the Earth, Sun, and Venus are aligned, Shapiro calculated that a radar traveling from earth to Venus would be delayed by $200ms$ by the gravitational field of the sun.

3 Conclusion

In this paper, we reproduced the derivation of the deflection angle of light passing by a massive object and concluded with Eq. 23 , 34, and 32.
These derivations are only the starting point of understanding gravitational lensing and its effect on light. There are many other variables such as the Einstein radius, the effective convergence of the lens, the weight function, etc., that further describe this phenomenon to help us locate it in the sky, detect it with our instruments, or use it to infer other quantities.

One important example of such quantities is the study of the distribution of dark matter. This latter can be understood as performing the reverse calculation outlined in our application in section 2.1.
An over-simplification of such procedure would work as follows, as understood from a lecture by Prof. Eric Blackman:

- We first observe luminous objects with highly similar spectra in a ring-shaped area of the sky, we determine that they must represent an object behind being gravitationally lensed by a second object in front of it.
- We then calculate the observable mass and conclude that it is not sufficient to induce the bending we observe.
- Through calculating this difference, we, therefore, can determine how much dark matter is needed to exist in that region of the sky for our observations to make sense.

In reality, this process can prove to be much more cumbersome, for instance, in the event when scientists are able to observe a seemingly gravitationally lensed object, there are still debates about whether multiple observed objects are truly similar objects with similar spectra or if it is just a single gravitationally lensed object.

References

Calada, L. (2011). Gravitational lensing in action. NASA, ESA. Retrieved from: https://www.spacetelescope.org/images/heic1106c

Hartle, J. (2003). Gravity.

Iyer, S. (2007). Lights Bending Angle due to Black Holes: From the Photon. Retrieved December 10, 2018, from the arXiv database.

Jacobsson, R. (1966). V Light Reflection from Films of Continuously Varying Refractive Index. Progress in optics, 5, 247-286.

Meneghetti, M. (n.d.). Retrieved from Ruprecht-Karls-Universitt Heidelberg: http://www.ita.uni-heidelberg.de/

Newton, I. (1704). Opticks: or, A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light.

Shapiro, Irwin I. (1964). Fourth Test of General Relativity. Phys. Rev. Lett., 13(26), 789-791.