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First Order Approximations in General Relativity

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1 Introduction

General relativity is a rich and complex field of physics, containing many important results coming from detailed, rigorous calculations. However, one can also gain a fair amount of knowledge and intuition about the subject by looking at first order approximations of various systems it describes. In this paper, I aim to present the results of first order of approximations in gravitational fields due to non-rotating and rotating objects, which can be used in the case of fairly weak gravitational fields and slow rotations, respectively.

2 Time Dilation due to Graviation

We begin this discussion by first looking at time dilation in general relativity. This will give us some insight as to how to formulate the weak field metric. To do this, we look at the case of a rocket ship accelerating through space. At one end of the rocket ship, a laser fires a pulse. At the other end, there is a detector which registers when the pulse has reached the other end. In this case, we assume that $\left(\frac{gh}{c^2}\right)^2$ and $\left(\frac{v}{c}\right)^2$ are negligible, where g is the acceleration of the rocket, h is the separation between the laser and the detector, and v is the velocity of the robot.This means that we can use Newtonian mechanics to describe the movement of the laser and the detector. We can write them as

$$z_d(t) = \frac{1}{2}gt^2, \quad z_l(t) = h + \frac{1}{2}gt^2, \tag{1}$$

where z_d is the position of the detector and z_l is the position of the laser. We consider the situation where we have two pulses of light fired from the laser. We define the time at which the first pulse is fired as $t = 0$, the time when the first pulse is received as t_1, the time the second pulse is fired as τ_a, and the time the second pulse is received as $t_1 + \tau_b$. Then, the distance traveled by the first pulse is ct_1, which means we can write the difference between the positions as

$$z_l(0) - z_d(t_1) = ct_1. \tag{2}$$

The distance travelled by the second pulse can be written as

$$z_l(\tau_a) - z_d(t_1 + \tau_b) = c(t_1 + \tau_b - \tau_a) \tag{3}$$

From these two expressions, we obtain a system of equations we can write as

$$h - \frac{1}{2}gt_1^2 = ct_1 \tag{4}$$

$$h - \frac{1}{2}gt_1^2 - gt_1\tau_b = c(t_1 + \tau_b - \tau_a) \tag{5}$$

Substituting 4 into 5, we get the expression

$$gt_1\tau_b = c\left(\tau_a - \tau_b\right) \tag{6}$$

$$\implies t_1 = c\frac{(\tau_a - \tau_b)}{g\tau_b} \tag{7}$$

However, we know from the quadratic formula and 4 that

$$t_1 = \frac{-c \pm \sqrt{c^2 + 2gh}}{g}, \tag{8}$$

where we take the positive solution, since we assume that time must be positive. Combining these two expressions and rearranging, we obtain

$$\tau_b = \frac{\tau_a}{\sqrt{1 + \frac{2gh}{c^2}}}. \tag{9}$$

Since we assume that $\left(\frac{gh}{c^2}\right)^2$ and smaller terms are negligible, we can take an expansion of this to obtain the time dilation of acceleration as

$$\tau_b = \tau_a \left(1 - \frac{gh}{c^2}\right). \tag{10}$$

Due to the equivalence principle, we know that this dilation due to acceleration is the same as due to a gravitational field. We can write gh as the potential difference between these two points, $\Phi_b - \Phi_a$. We can extend this to nonuniform fields by simply defining the potential difference between the two points as $\Phi(x_b) - \Phi(x_a)$. Then, our dilation can be written as

$$\tau_b = \tau_a \left(1 - \frac{\Phi(x_b) - \Phi(x_a)}{c^2}\right) \tag{11}$$

Therefore, when we derive our weak field metric, it must be consistent with this expression.

3 Geometric Newtonian Formulation of Gravity

3.1 Confirming Time Dilation

In general relativity, we know that the effect of gravity is to curve spacetime. In the Newtonian, first order approximation, we assume any infinitesimal line segment in this curved spacetime can be written as

$$ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right), \tag{12}$$

where $\Phi(x^i)$ is a gravitational potential and x^i represents a coordinate which only depends on the spatial position and not on time. We call this the static, weak metric, because it is time independent and valid for relatively small gravitational fields, such as the one produced by the sun. In this approximation, we assume that any terms of order greater than $\frac{1}{c^2}$ are negligible.

To determine if this is consistent with our derivation of gravitational time dilation, we calculate the time difference between two events in this metric. Since we are only interested in the difference in time, we can set $dx = dy = dz = 0$. Therefore, the proper time $d\tau_a$ between two events at some point x_a can be written as

$$d\tau_a^2 = \frac{-ds^2}{c^2} = \left(1 + \frac{2\Phi(x_a)}{c^2}\right) dt^2 \tag{13}$$

$$\implies d\tau_a = \sqrt{1 + \frac{2\Phi(x_a)}{c^2}} dt \approx \left(1 + \frac{\Phi(x_a)}{c^2}\right) dt \tag{14}$$

Doing this for a separate point at x_b leads to the same expression for $d\tau_b$. If we solve for dt in the above equation and substitute it into the expression for $d\tau_b$, we can obtain

$$d\tau_b \approx \left(1 + \frac{\Phi(x_b) - \Phi(x_a)}{c^2}\right) d\tau_a, \tag{15}$$

again neglecting powers greater than $\frac{1}{c^2}$. This is the same expression for time dilation we derived earlier, but now using the weak field metric.

3.2 Equations of Motions

We can now define motion of particles in this metric. We do this the same way as we calculated the time difference between two particles within the metric, but now with some difference in the spatial coordinates between events a and b. We write

$$\tau_{ab} = \int_a^b d\tau = \int_a^b \left(-\frac{ds^2}{c^2}\right)^{1/2} \tag{16}$$

$$= \int_a^b \left[-\left(1+\frac{2\Phi}{c^2}\right)(cdt)^2 + \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right)\right]^{1/2} \tag{17}$$

This expression can be simplified somewhat if we assume that we can use the time coordinate as a parameter for the spatial coordinates. We can then extract a factor of dt from each term. Then, the spatial coordinate term can be written as $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$, where $v = ||\mathbf{v}||$ is the spatial velocity of the particle. The proper time can then be written as

$$\tau_{ab} = \int_a^b dt \left[\left(1+\frac{2\Phi}{c^2}\right) - \frac{v^2}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\right] \tag{18}$$

Since we neglect terms of higher order than $\frac{1}{c^2}$, we are left with

$$\tau_{ab} \approx \int_a^b dt \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 - \Phi\right)\right] \tag{19}$$

In general relativity, that the path taken by the particle is one which extremizes the proper time. Therefore, to solve for the equations of motion, we need to extremize the integral

$$\int_a^b dt \left(\frac{1}{2}v^2 - \Phi\right) \tag{20}$$

We use the calculus of variations to do this, with a Lagrangian equal to the argument of the integral. Using the Lagrangian equations of motion, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \tag{21}$$

$$\implies \frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi, \tag{22}$$

where $\mathbf{x}$ is a vector of the spatial coordinates. From this, we see that this first order approximation in our metric results in the Newtonian equations of motion.

4 The Covariant Derivative

4.1 Derivatives of Scalars

Before we can discuss a first order approximation of rotating bodies, we first must develop our notion of the derivative. which will help us with this problem. We first look at taking the derivative of a scalar function. We define the derivative of a scalar function f along a curve with can be parameterized with respect to some σ as

$$\frac{df}{d\sigma} = \lim_{\epsilon\to 0}\frac{f\left(x^\alpha(\sigma+\epsilon)\right) - f\left(x^\alpha(\sigma)\right)}{\epsilon} = \frac{dx^\alpha}{d\sigma}\frac{\partial f}{\partial x^\alpha}. \tag{23}$$

We can define the $\frac{dx^\alpha}{d\sigma}$ as a vector t^α, which will be tangent to the curve along which we are taking the derivative at all points. We can therefore define the total derivative as

$$\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha}. \tag{24}$$

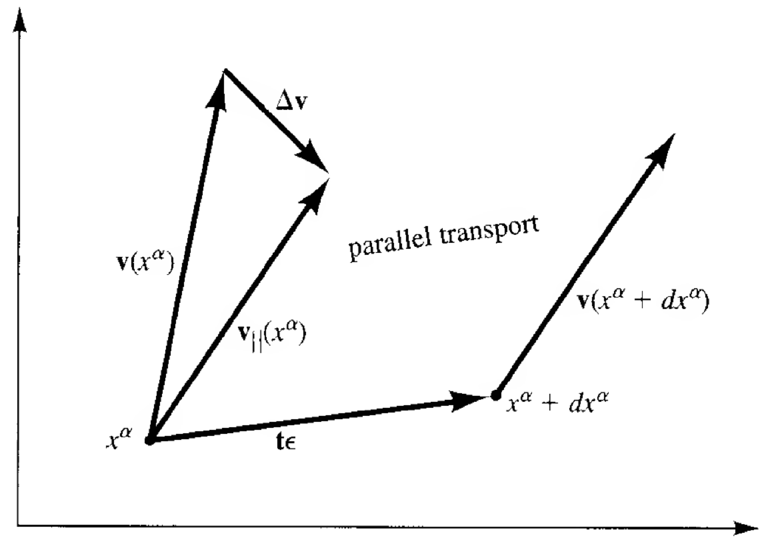

Figure 1: Demonstration of parallel transport, which allows us to take the derivative of a vector.

4.2 Derivatives of Vectors

The next step to take is to define the derivative of a vector. The issue with derivatives of vectors is that the derivative involves looking at how a vector changes at different points in spacetime, but the basic properties of vectors, such as addition and subtraction, are only defined at a single point. Any changes in the coordinate where the vector is defined will also lead to changes in the spacetime for non-flat geometries. To remedy this issue, we start in the flat, Cartesian case. To take care of the fact the vectors are defined at different points, we can move the translated vectors back to the point of the original vector along a path parallel to the translation vector. This process is called parallel transport, and it shown in figure 1. This assists us in non-flat spacetime as we can treat any point in any non-flat geometry as being locally equivalent to a flat spacetime. We can then define the covariant derivative in some direction $\mathbf{t}$ as

$$\nabla_{\mathbf{t}} \mathbf{v} \left(x^{\alpha}\right) = \lim_{n \to \infty} \frac{\mathbf{v} \left(x^{\alpha} + t^{\alpha} \epsilon\right)_{\|} - \mathbf{v} \left(x^{\alpha}\right)}{\epsilon}, \tag{25}$$

where the $\|$ represents that the vector has been moved back to the original point by parallel transport. In flat spacetime time (or a local inertial frame of curved spacetime), the parallel transport does not change the coordinates of the vector, and we can write the derivative in the same way we did for scalars, as

$$\left(\nabla_{\mathbf{t}} \mathbf{v}\right)^{\alpha} = t^{\beta} \frac{\partial v^{\alpha}}{\partial x^{\beta}}, \tag{26}$$

which can also be written as

$$\nabla_{\beta} v^{\alpha} = \frac{\partial v^{\alpha}}{\partial x^{\beta}}. \tag{27}$$

This equation only works in flat spacetime with Cartesian coordinates, as the coordinates of the vector may change when undergoing parallel transport even when in flat, non-Cartesian coordinates, such as spherical coordinates. We expect the parallel transported vector to be the sum of the original vector and the changes in the vector due to the change in the coordinates. Generally, we can write this as

$$v_{\parallel}^{\alpha}\left(x^{\alpha}\right)=v^{\alpha}\left(x^{\delta}+\epsilon t^{\delta}\right)+\Gamma_{\beta\gamma}^{\alpha}x^{\delta}v^{\gamma}(x^{\delta})\left(\epsilon t^{\beta}\right), \tag{28}$$

where $\Gamma_{\beta\gamma}^{\alpha}$ can be shown to be the Christoffel symbols. Taking the covariant derivative of this expression, we obtain

$$\nabla_{\beta}v^{\alpha}=\frac{\partial v^{\alpha}}{\partial x^{\beta}}+\Gamma_{\beta\gamma}^{\alpha}v^{\gamma}. \tag{29}$$

We can then use this equation to obtain a concise definition of a geodesic. To do this, we note that in a local inertial frame, any geodesic can be defined as a straight line. A straight line can be defined as a curve whose tangent vector at any point along the line will be propagated along the line itself. Take $\mathbf{u}$ to be such a tangent vector. If we take the derivative of $\mathbf{u}$ along itself, by the previous condition, we have

$$\left(\nabla_{\mathbf{u}}\mathbf{u}\right)^{\alpha}=u^{\beta}\left(\frac{\partial u^{\alpha}}{\partial x^{\beta}}+\Gamma_{\beta\gamma}^{\alpha}u^{\gamma}\right)=0, \tag{30}$$

since we know $u^{\alpha}=\frac{dx^{\alpha}}{d\tau}$. Therefore, in order for a curve to be a geodesic, we must have that

$$\nabla_{\mathbf{u}}\mathbf{u}=0. \tag{31}$$

5 Slow Rotations

Now that we have developed a rigorous way to determine the geodesic equation, we can investigate the effect rotations have on gravity. We would expect there to be some effect, as we know the rotations of a spherical object causes it to compress, making it no longer spherical. For instance, the sun rotates with a period of 27 days, causing it to be slightly oblong. However, we can find that even when we neglect the higher order terms in angular momentum which cause the deformation of spherical objects, we still find a change in the gravitational field. This is because the gravitational fields are affected not only by the mass of the object, but also by its motion.

5.1 A Gyroscope Orbiting A Nonrotating Body

We can think of the affect of this rotation as "dragging" the inertial frames along with it. That is, the frames of reference with respect to the rotations may be thought of as inertial frames of reference with a small induced angular moment associated with them. To observe this dragging, we can imagine a test gyroscope in one of these frames of reference. A gyroscope can be thought of as an object which has not only a four-velocity $\mathbf{u}(\tau)$, but also a four-spin $\mathbf{s}(\tau)$. If we choose a local inertial frame where the gyroscope is at rest, the spin takes on the form $s^{\alpha}(\tau)=(0,\vec{s})$, whereas the velocity takes on the form $u^{\alpha}(\tau)=(1,\vec{0})$. Therefore, we obtain the condition

$$\mathbf{s}\cdot\mathbf{u}=0, \tag{32}$$

which can be shown to be valid for any frame of reference. We already know that the velocity of the gyroscope must satisfy $\nabla_{\mathbf{u}}\mathbf{u}=0$. However, we have the same condition for the spin as well. That is, we have $\nabla_{\mathbf{u}}\mathbf{s}=0$, which is known as the gyroscope equation.

As an example, consider a gyroscope in a circular orbit around a nonrotating sphere. In the reference frame of the gyroscope, the magnitude of the spin remains constant, and if it starts in the equatorial plane, it will remain in the equatorial plane. As the gyroscope orbits around the sphere, the spin will begin to precess. We can measure the angle which the gyroscope precesses as some $\Delta\phi$. We derive this expression for the $\Delta\phi$ by using the metric for the Schwarzschild geometry, since we have assumed the sphere is nonrotating initially. We have for the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{33}$$

We assume we are at the Schwarzschild radius R, and that $\theta = \frac{\pi}{2}$, since we are in the equatorial plane. Then the only spatial, nonzero component of the velocity is u^ϕ, which we can write as

$$u^\phi = \frac{d\phi}{d\tau} = \frac{d\phi}{\underset{\cdot}{t}}\frac{dt}{d\tau} = \Omega u^t, \tag{34}$$

where we have defined $\Omega = \frac{d\phi}{\underset{\cdot}{t}}$ as the angular velocity. We can then define $\mathbf{u}$ as $\mathbf{u} = u^t(1, 0, 0, \Omega)$. To solve for $\mathbf{s}$, we first note that $s^\theta = 0$, since the precession only occurs in the equatorial plane. We solve for s^t by using the condition specified by 32.

$$\mathbf{s} \cdot \mathbf{u} = -\left(1 - \frac{2M}{R}\right) s^t u^t + R^2 s^\phi u^\phi = 0 \tag{35}$$

$$\implies s^t = R^2 \Omega \left(1 - \frac{2M}{R}\right)^{-1} s^\phi \tag{36}$$

To solve for the s^r and the s^ϕ components, we use the gyroscope equation, substituting in the appropriate Christoffel symbols.

$$\nabla_{\mathbf{u}} \mathbf{s} = 0$$

$$\implies \frac{ds^t}{dt} - (R - 3M)\,\Omega s^\phi = 0 \tag{37}$$

$$\frac{ds^\phi}{dt} + \frac{\Omega}{R} s^r = 0 \tag{38}$$

Solving for s^ϕ first, we get

$$\frac{d^2 s^\phi}{dt^2} + \left(1 - \frac{3M}{R}\right) \Omega^2 s^\phi = 0 \tag{39}$$

$$\implies s^\phi(t) = -s\left(1 - \frac{2M}{R}\right)^{1/2} \left(\frac{\Omega}{\omega R}\right) \sin(\omega t), \tag{40}$$

where we have used the initial condition that $s^\phi(0) = 0$, defined s to be the magnitude of $\mathbf{s}$, and defined

$$\omega = \left(1 - \frac{3M}{R}\right)^{1/2} \Omega. \tag{41}$$

We can then define the determine the radial component of $\mathbf{s}$

$$s^r(t) = s\left(1 - \frac{2M}{R}\right)^{1/2} \cos(\omega t). \tag{42}$$

We choose this normalization so that $(\mathbf{s} \cdot \mathbf{s})^{1/2} = s$. To determine the angle of the precession, $\Delta\phi$, we project the direction of the spin after one period ($t = 2\pi\Omega$) onto the direction of the initial spin. According to the equations we derived, the initial spin is in the $\hat{r}$ direction. Therefore, we have

$$\left(\frac{\mathbf{s}}{s}\right)\cdot\hat{r} = \cos\left(2\pi\frac{\omega}{\Omega}\right) = \cos\left[2\pi\left(1-\frac{3M}{R}\right)^{1/2}\right]. \tag{43}$$

The change in the angle per orbit can therefore be written as

$$\Delta\phi = 2\pi\left[1-\left(1-\frac{3M}{R}\right)^{1/2}\right] \tag{44}$$

5.2 The Effect of Slow Rotations Rotation on Spacetime

With the precession of a gyroscope orbiting a nonrotating body determined, we can begin to investigate the effects of a slowly rotating body. In this scenario, we assume that angular momentum (J) terms of $\mathcal{O}(J^2)$ or higher are negligible. Therefore, our metric becomes

$$ds^2 = ds^2_{\text{Schwarzschild}} - \frac{4GJ}{c^3 r^2}\sin^2\theta(r\,d\phi)(c\,dt) + \mathcal{O}(J^2) \tag{45}$$

To study this metric, we look at the case of a gyroscope freely falling towards a slowly rotating spherical body along the line of its rotational axis. In the case of the nonrotating object, the spin would not precess as the gyroscope fell, as we have a symmetry in the ϕ coordinate in the Schwarzschild metric when we take $\phi \to -\phi$. However, in the metric of the slowly rotating object, this is no longer the case. To determine this precession, we need to again use the gyroscope equation. To do this, we first convert the metric from spherical coordinates into Cartesian, as this will ease the difficulty of calculations.

$$ds^2 = (ds^2)_{\text{Schwarzschild}} - \frac{4GJ}{c^3 r^2}(c\,dt)\left(\frac{x\,dy - y\,dx}{r}\right) \tag{46}$$

To solve the gyroscope equation, we also assume that terms of the order $1/c^4$ are negligible. This means any terms in the Schwarzschild geometry of order $1/c^2$ cannot contribute, as we get terms of $1/c^5$ when they are multiplied by the rotational correction term become of order $1/c^5$ or higher. This means we can use a flat spacetime metric instead of the Schwarzschild metric for our calculations. In addition to this, we assume that the gyroscope is moving purely along the z-axis. Therefore, we can write the velocity and spin of the gyroscope as

$$u^\alpha = (u^t, 0, 0, u^z) \tag{47}$$
$$s^\alpha = (0, s^x, s^y, 0) \tag{48}$$

To solve the gyroscope equation, we use the appropriate Christoffel symbols. The only nonvanishing Christoffel symbols of order less than $1/c^4$ are

$$\Gamma^x_{ty} = \frac{2GJ}{c^2 z^3} \tag{49}$$
$$\Gamma^y_{tx} = -\frac{2GJ}{c^2 z^3}. \tag{50}$$

With these, we obtain the following expressions for the gyroscope equation

$$\frac{ds^x}{dt} = -\frac{2GJ}{c^2 z^3}s^y \tag{51}$$
$$\frac{ds^y}{dt} = +\frac{2GJ}{c^2 z^3}s^x \tag{52}$$

This is the equation for two coupled simple harmonic oscillators. We can see that the precession of the gyroscope is in the same direction as the rotation of the body the gyroscope is falling towards. We can also note that the precession is the same whether it is measured from the frame of the gyroscope or from a frame in which the center of the body it is falling towards is motionless. This is because the precession is transverse to the direction of motion, and is therefore not affected by time dilation.

The rate of precession at some height above the body is

$$\Omega_{LT} = \frac{2GJ}{c^2 z^3}, \tag{53}$$

which is called the Lense-Thirring precession. We can measure the Lense-Thirring precession along an axis other than the axis which the body is rotating around by using

$$\vec{\Omega}_{LT} = \frac{G}{c^2 r^3} \left[3 \left(\vec{J} \cdot \hat{r} \right) \hat{r} - \vec{J} \right], \tag{54}$$

where arrows have been used to denote three-vectors. Interestingly enough, this is similar to the dipole term in a multipole expansion for a charge distribution, with $\vec{J}$ acting as the dipole moment. We can see the effect of the Lense-Thirring precession in conjunction with the geodetic precession caused by orbital motion in figure 2.

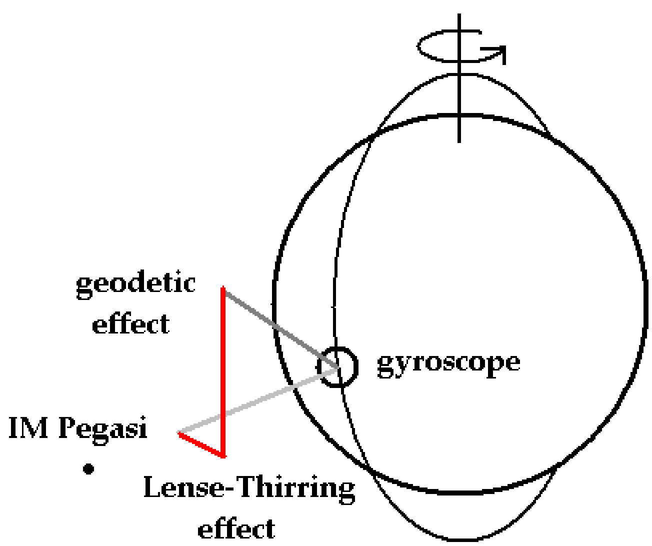

Figure 2: The precession of the spin of a gyroscope due to both Lense-Thirring and geodetic precession [2].

6 Conclusion

We see that we can gain a wealth of information about various complicated systems by simply looking at first order approximations. By taking the first order approximations to our metric, we can obtain a geometric version of the Newtonian equations of motion. If we have derived the equations for a geodesic, we also fairly simply look at the effects the rotations of an object to the first order. This gives us basic ideas about how this rotation will affect objects in its orbit, including predictions of the precession. With these calculations, we are fully prepared to look into higher order terms, as we have a good idea of the limiting behavior of these physical phenomenon.

7 References

[1] Hartle, James, *Gravity*, Addison Wesley, 2003.
[2] Bechtol, Keith "Gravity Probe B and the Search for the Lense-Thirring Effect," Stanford, 2007.