Field Theory in Curved Spacetime and the Stress-Energy Tensor

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Introduction

When introducing a new area of physics, various assumptions are generally made. Early assumptions include assuming that drag forces can be ignored, that mass is always conserved, and that objects can be taken as point particles. After that, the assumptions get a little more advanced: assume the effects of general relativity on a system are negligible, assume an inertial reference frame, and assume collisions are elastic and energy is conserved. The farther along one gets in the study of physics, the less they are allowed to make these assumptions. This is because in simple systems the assumptions hold true. However, at extremes of energy and size, they do not. Quantum field theory works within these extremes, so fewer assumptions are allowed. Still, as with everything, it needs to start somewhere, so it starts at the simplest possibility: flat spacetime. Spacetime is not flat; it is curved. In general cases, spacetime can be approximated as flat. In order to form a more accurate theory, the effects of the curvature of spacetime are taken into account. A developed outline of quantum field theory can be combined with basic general relativity to formulate a version of quantum field theory more consistent with reality.

Coordinate Transformations

First, a coordinate transformation of spacetime must be performed. To do this, we start with the invariant Minkowskian spacetime interval:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dt)^2 - (d\vec{x})^2$$

Using Einstein's theory of gravity, this interval is replaced by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Where $g_{\mu\nu}$ is a metric tensor defined as a function of the spacetime coordinate x. To explore why this transformation is valid, the action *S* is explored. The principle of general covariance states that *S* must be invariant under an arbitrary coordinate transformation from x to x'(x), like the one proposed. In this instance, this principle allows the effect of a gravitational field can be viewed instead as a coordinate transformation. We can look at the transformation in the following way:

$$ds^2 = g'_{\lambda\sigma} dx'^\lambda dx'^\sigma = g'_{\lambda\sigma} \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$$

From this, we can see that the metric tensor $g_{\mu\nu}$ transforms as follows:

$$g'_{\lambda\sigma}(x') \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = g_{\mu\nu}(x)$$

Transformation over a scalar field has no effect, such that, on the scalar field φ, φ(x) = φ'(x'). However, the gradient of said scalar field will transform in the following way:

$$\partial_\mu \varphi(x) = \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial \varphi'(x')}{\partial x'^\lambda} = \frac{\partial x'^\lambda}{\partial x^\mu} \partial'_\lambda \varphi'(x')$$

Similarly, a covariant vector field *A* transforms as follows:

$$A_\mu(x) = \frac{\partial x'^\lambda}{\partial x^\mu} A'_\lambda(x')$$

Thus, ∂φ takes on the same form as a vector field transformation and can therefore be considered a vector field.

Now, consider two vector fields: $A_\mu(x)$ and $B_\nu(x)$. Contracting them with the metric tensor $g^{\mu\nu}(x)$ forms the scalar $g^{\mu\nu}(x)\, A_\mu(x)\, B_\nu(x)$. The gradient of the scalar field φ is such a vector field, so $g^{\mu\nu}(x)\, \partial_\mu\varphi(x)\, \partial_\nu\varphi(x)$ is also a scalar. We now consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right)$$

Replacing the Minkowski metric $\eta^{\mu\nu}$ with the Einstein metric $g^{\mu\nu}$ allows the Lagrangian to be invariant under coordinate transformations. Integrating the Lagrangian over spacetime leads to the action *S*. To do this, the following coordinate transformation is used:

$$d^4x' = d^4x\left[\det\left(\frac{\partial x'}{\partial x}\right)\right]$$

To do this, we define the determinant of the Einstein metric as follows:

$$g \equiv \det(g_{\mu\nu}) = \det\left(g'_{\lambda\sigma}(x')\frac{\partial x'^{\lambda}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\right) = g'\left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^2$$

Noting that $d^4x\sqrt{-g} = d^4x'\sqrt{-g'}$, which is also invariant under coordinate transformation.

This now leads to converting a given quantum field theory into a quantum field theory of curved spacetime. This is done by replacing the Minkowski metric in the Lagrangian with the Einstein metric and including a factor of $\sqrt{-g}$. Ordinary partial derivatives also need to be replaced by covariant derivatives, as used in general relativity. However, since the derivatives are acting on the scalar field φ, the ordinary partial derivatives and covariant derivatives are equivalent here. Now, we can derive the action *S* by integrating the Lagrangian over spacetime:

$$S = \int \sqrt{-g}\,\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right)d^4x$$

Which is invariant under coordinate transformation, as desired.

From this, it is shown that $g_{\mu\nu}(x)$ is a classical field. Integrating over this field would lead to a quantization of gravity. However, this has not been an easy task. Attempting to do so leads to more confusion, requiring entirely new theories to reconcile the obstacles it produces. One such theory that allows this integration is string theory.

The action S can be defined as the sum of two terms: S_g and S_M. S_g is the gravitational action. It describes the dynamics of the gravitational field $g_{\mu\nu}(x)$. S_M is the matter action. It describes the dynamics caused by all other known fields, also known as matter fields. One such field is the scalar field φ discussed prior.

The graviton and the stress-energy tensor

To better understand the field $g_{\mu\nu}(x)$, it is assigned the particle called the graviton. While the existence of this particle has not been experimentally verified, it is widely accepted as the particle associated with the gravitational field. According to Einstein's theory of gravity, the graviton is affected by energy and momentum.

Next, the stress-energy tensor is defined as follows:

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}(x)}$$

It is described as the variation of the matter action S_M over the metric $g_{\mu\nu}$. Energy E and momentum P^i are defined as follows:

$$E = P^0 = \int \sqrt{-g}T^{00}(x)d^3x$$

$$P^i = \int \sqrt{-g}T^{0i}(x)d^3x$$

Returning to flat spacetime, we turn our focus to the stress-energy tensor. To return to flat spacetime, the following relationship is defined:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Here, $h_{\mu\nu}(x)$ is a symmetric tensor analogous to the graviton field. With that, S_M can be expanded by h to the first order:

$$S_M(h) = S_M(h=0) - \int \frac{1}{2}(h_{\mu\nu}T^{\mu\nu} + O(h^2))d^4x$$

The graviton field thus couples to the stress-energy tensor. For comparison, the electromagnetic current J^{μ}(x) couples to the photon field in the same way. Choose a generic S_M as follows:

$$S_M = \int d^4x\sqrt{-g}(A + g^{\mu\nu}B_{\mu\nu} + g^{\mu\nu}g^{\lambda\rho}\, C_{\mu\nu\lambda\rho} + \cdots)$$

Then note:

$$-g = 1 + \eta^{\mu\nu}h_{\mu\nu} + O(h^2)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$$

This then leads to the following stress-energy tensor:

$$T_{\mu\nu} = 2\left(B_{\mu\nu} + 2C_{\mu\nu\lambda\rho}\eta^{\lambda\rho} + \cdots\right) - \eta_{\mu\nu}\boldsymbol{\mathcal{L}}$$

This result for the stress-energy tensor can also lead to familiar and easier to understand results for the electromagnetic field. The Lagrangian for the electromagnetic field is defined in the following way:

$$\mathcal{L} = -\frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} + \frac{1}{2}m^2g^{\mu\nu}A_{\mu}A_{\nu}$$

Where F_{ij} is the field tensor for the electromagnetic field. This leads to the following outcome for the stress-energy tensor of this field:

$$T_{\mu\nu} = -F_{\mu\lambda}F_{\nu}^{\lambda} + m^2A_{\mu}A_{\nu} - \eta_{\mu\nu}\boldsymbol{\mathcal{L}}$$

For the electromagnetic field, the associated particle is the photon, which is massless; thus, m=0. The Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(-2F_{0i}^2 + F_{ij}^2\right) = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)$$

Then, the stress energy tensor of this field becomes

$$T_{00} = -F_{0\lambda}F_0^{\lambda} - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

Which is consistent with electromagnetism.

Reference

Zee, A. (2010). I.11 Field Theory in Curved Spacetime, *Quantum Field Theory in a Nutshell* (2nd ed.) (pp. 81-87). Princeton University Press.