Double Covers and Some Applications to Spinor Representations

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1 Introduction

The algebraic structure of groups lends itself to multitudes of connections between groups with different definitions. Naturally, some groups are easier to visualize, understand, or represent. Thus, finding relations between known groups and lesser known groups can help bridge the gap of understanding to these lesser understood groups. In this paper, we will discuss a few double coverage relations between groups (or direct products of groups) and how it can be applied to spinor representations. The following is based on my second lecture for Kapitza Spring 2020.

2 The Lorentz Group

As it is important in later sections, we will briefly review the Minkowski Metric and Lorentz transformations.

Utilizing Einstein summation notation, we can express The Minkowski Metric as follows:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad \eta_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu > 0 \\ 0 & \mu \neq \nu \end{cases} \qquad ds^2 = dt^2 - d\vec{x}^2 \tag{1}$$

By convention, the time component of the four vector is indexed by 0. Thus, the matrix indices for η are one lower than usual matrix indices. Regardless of rotations and boosts of the frame, the length of ds^2 remains constant, for all frames. Thus, if we find a group that shows this invariance, we can conclude that it is part of the Lorentz Group.

The algebra of the Lorentz Group can be summarized by two operators:

$$J_{\pm n} = \frac{1}{2}(J_n \pm iK_n) \qquad J_{\pm n} = (J_{\pm n})^{\dagger} \tag{2}$$

The J_n component represents a rotation, while the K_n component represents a Lorentz boost. We can arrange these compound operators in three ways to get the commutation relations for the algebra, which are the following, derived in a previous lecture:

$$[J_{+n}, J_{+m}] = i\epsilon_{mnp}J_{+p} \qquad [J_{-n}, J_{-m}] = i\epsilon_{mnp}J_{-p} \qquad [J_{+n}, J_{-m}] = 0 \tag{3}$$

The last relation is the most important as it suggests a near independence between the two operators, creating two distinction algebras.

3 SU(2) & SO(4)

We will start by determining the relation between $SU(2)$ and $SO(4)$.

Recall the 2 by 2 Pauli Matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

Let us define the following matrix. The dot symbol in this case represents scalar multiplication of each Pauli matrix by the respective component of $\vec{x}$, and not right matrix multiplication with a vector. $(x^4, \vec{x})$ is a four vector, with $\vec{x}$ having three components.

$$X_E = x^4 I + i\vec{x} \cdot \vec{\sigma} \qquad X_E = \begin{bmatrix} x^4 + ix^3 & ix^1 + x^2 \\ ix^1 - x^2 & x^4 - ix^3 \end{bmatrix} \tag{5}$$

Now we impose the following restriction:

$$(x^4)^2 + \vec{x}^2 = 1 \tag{6}$$

Applying the above restriction, the determinant is 1.

$$\det X_E = (x^4 + ix^3)(x^4 - ix^3) - (ix^1 + x^2)(ix^1 - x^2) \tag{7}$$
$$(x^4)^2 + (x^3)^2 + (x^1)^2 + (x^2)^2 = (x^4)^2 + \vec{x}^2 = 1$$

Invoking rotational invariance, let $\vec{x} = x^3$ with $x^1 = x^2 = 0$

$$X_E^* X_E = \begin{bmatrix} x^4 - ix^3 & 0 \\ 0 & x^4 + ix^3 \end{bmatrix} \begin{bmatrix} x^4 + ix^3 & 0 \\ 0 & x^4 - ix^3 \end{bmatrix} \tag{8}$$
$$X_E^* X_E = \begin{bmatrix} (x^4 - ix^3)(x^4 + ix^3) & 0 \\ 0 & (x^4 + ix^3)(x^4 - ix^3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The above utilizes the fact that there is only an x^3 component of $\vec{x}$ and thus after multiplying the binomials out, $(x^4)^2 + (x^3)^2 = (x^4)^2 + \vec{x}^2 = 1$

Since the determinant of X_E is 1 and the matrix is unitary, we can conclude that X_E defines an element in $SU(2)$. Under closure, a new transformed version of X_E can be created by simultaneous right and left multiplication of matrices in $SU(2)$. Thus X_E can be transformed into some X_E'. This transformation is $SU(2) \otimes SU(2)$.

$$A, B \in SU(2) \implies A^* X_E B = X_E' \in SU(2) \tag{9}$$
$$(A, B) \in SU(2) \otimes SU(2)$$

However, there is something else happening at the same time. Within the matrix, X_E there exists the four vector, $(x^4, \vec{x})$. Similarly, within X_E' there exists another four vector, $(x^{4\prime}, \vec{x}')$ of transformed coordinates. Thus when we simultaneously right and left multiply by matrices in $SU(2)$ we are transforming a unit four vector into another unit four vector. This is 4 dimensional rotation and the same as an $SO(4)$ transformation.

Upon further inspection, cancellation of negatives will cause (A, B) and $(-A, -B)$ to correspond to the same four dimensional rotation, meaning that there is only a local isomorphism between these two groups, and twice as many transformations exists than are needed to cover all of $SO(4)$. Because of this, we can conclude the following statement of double coverage:

$$\frac{SU(2) \otimes SU(2)}{\mathbb{Z}_2} = SO(4) \tag{10}$$

4 Lorentz Group & Special Linear Group over $\mathbb{C}$

Of course double coverage applies between other groups as well. We can now repeat essentially the same steps, but this time finding the relation for $SO(3,1)$.

For distinction, x^4 becomes x^0.

$$X_M = x^0 I + \vec{x} \cdot \vec{\sigma} \qquad X_M = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \tag{11}$$

Taking the determinant of this matrix, we find the following:

$$\det X_M = (x^0 + x^3)(x^0 - x^3) - (x^1 - ix^2)(x^1 + ix^2) = (x^0)^2 - \vec{x}^2 \tag{12}$$

Once again, the scalar component is expressed on its own, but this time the squared vector is subtracted instead of added.

Let us now consider another 2 by 2 matrix with a unit determinant, M. Simultaneous matrix multiplication by M and its adjoint transform X_M into some new matrix X'_M. This transformation corresponds to $SL(2,\mathbb{C})$.

$$M \in SL(2,\mathbb{C}) \implies M^\dagger X_M M = X'_M \tag{13}$$

Once again under the guise of the matrices, while $X_M \to X'_M$, at the same time the four vectors from which these matrices are devised are transforming as well. $(x^0, \vec{x}) \to (x^{0\prime}, \vec{x}')$. If we take the determinant of the transformation:

$$\det(M^\dagger X_M M) = \det X'_M \tag{14}$$

$$\det(M^\dagger M) det X_M = \det X'_M$$

$$\det X_M = \det X'_M$$

With the determinants of X_M and X'_M being the same and both matrices taking the same general form, albeit built from different four vectors, we can express the following:

$$(x^0)^2 - \vec{x}^2 = (x^{0\prime})^2 - \vec{x}'^2 \tag{15}$$

The above expression is a satisfaction of the Minkowski Metric and because of this we know that $X_M \in SO(3,1)$. This matrix is part of the Lorentz Group.

Once again, this matrix is connected to other groups as well. By applying an $SL(2,\mathbb{C})$ transformation to X_M we transform one element of $SO(3,1)$ into another element of $SO(3,1)$. However, just as before, by virtue of the how the transformation M is applied to X_M, the matrix M corresponds to the same transformation as $-M$. Thus, we can make the following statement of double coverage:

$$\frac{SL(2,\mathbb{C})}{\mathbb{Z}_2} = SO(3,1) \tag{16}$$

A well known instance of double coverage is $SU(2)$ and $SO(3)$. Using the above relation, we can actually derive this as a specific instance. If we restrict $SL(2,\mathbb{C})$ to its $SU(2)$ subgroup then $M \in SU(2)$ and the transformation will not longer effect the x^0 component of the four vector. The remaining components will change in the same fashion as previous. With no transformation occurring to the x^0 component, $SO(3,1)$ becomes restricted to $SO(3)$. Alternatively, $M \in SU(2)$ means that $M^\dagger M = I$ and the transformation is just a 3 dimensional rotation. Thus we get the familiar double coverage relation:

$$\frac{SU(2)}{\mathbb{Z}_2} = SO(3) \tag{17}$$

5 Using Double Coverage on Lorentz Algebra

The existence of double coverage allows us to extract representations of new groups from the representations of groups we are already familiar with. In this specific case, if we already know how to express the irreducible representations of $SU(2)$, we can find the irreducible representations of $SO(3,1)$.

We can return back to the final commutation relation of Equation 3, and the fact that these operators form two distinct $SU(2)$ operators. Two distinct group algebras of this group can be expressed as $SU(2) \otimes SU(2)$ which has already been referenced in Section 2.

Assuming we already know how to create irreducible representations in $SU(2)$, the double coverage relation in Equation 10, allows a quick way of discovering the irreducible representation of $SO(4)$ and by extension the Lorentz Group.

Representations of $SU(2)$ are defined by the half integers starting from 0. Each representation can be indexed by values ranging from the negative value of the number that defines its representation up, to the same number in the positives. Thus the dimension of the n^{th} representation has dimension $2n + 1$. The 1 is added because of the inclusion of 0 in the range. By extension, a direct product representation can be expressed by a set of two of these numbers. Iterating over all possible values, representations of the Lorentz Group can be expressed as the following sequence:

$$(0,0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (1,0), (0,1), \left(\frac{3}{2}, 0\right), \left(0, \frac{3}{2}\right) \dots \tag{18}$$

Similarly, the dimension each of these representations will just be the product of the dimensions of the two sub representations that comprise it. In essence, the dimension is $(2n + 1)(2m + 1)$. With n and m standing for the representations in the sets described

6 Weyl Spinors

Plucking out the second and third representations from the above equation, we can restrict this $SO(3, 1)$ representation to the $SO(3)$ rotation group. We can decompose these two representations by computing $m \otimes n$. This can be accomplishes by taking the direct sum of a sequence of representations, each with a value one less than the previous, spanning from the complete sum to the complete difference of the two values:

$$(n + m) \oplus (n + m - 1) \oplus (n + m - 2) + \dots + (n - m) \tag{19}$$

Using the above, both $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ decompose into $\frac{1}{2}$. This implies that the double set of values used to describe a representation of $SO(3,1)$ decomposes into a 2 dimensional spinor.

We now describe two two-indexed objects, u and v respective to $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. Both u and v have two components.

Starting with $(\frac{1}{2},0)$ we can return to Equation 2. J_+ corresponds to the first entry $\frac{1}{2}$ while J_- corresponds to the second entry 0. Similiarly, $(0,\frac{1}{2})$ means J_+ corresponds to the first entry 0 while J_- corresponds to the second entry $\frac{1}{2}$. Adding in the Pauli matrix, we can summarize the results with the following:

$$\left(\frac{1}{2},0\right) \rightarrow \qquad J_{+n} = \frac{1}{2}(J_n + iK_n) = \frac{1}{2}\sigma_n \qquad J_{-n} = \frac{1}{2}(J_n - iK_n) = 0 \tag{20}$$

$$\left(0,\frac{1}{2}\right) \rightarrow \qquad J_{+n} = \frac{1}{2}(J_n + iK_n) = 0 \qquad J_{-n} = \frac{1}{2}(J_n - iK_n) = \frac{1}{2}\sigma_n \tag{21}$$

To isolate the J_n and K_n operators, we can add the two relations together, and then subtract them apart.

$$\left(\frac{1}{2},0\right) \rightarrow \qquad J_{+n} + J_{-n} = J_n = \frac{1}{2}\sigma_n \qquad J_{+n} - J_{-n} = iK_n = \frac{1}{2}\sigma_n \tag{22}$$

$$\left(0,\frac{1}{2}\right) \rightarrow \qquad J_{+n} + J_{-n} = J_n = \frac{1}{2}\sigma_n \qquad J_{+n} - J_{-n} = -iK_n = \frac{1}{2}\sigma_n \tag{23}$$

This allows us to see some of the similar, and more importantly, contrasting behaviour of the denotations u and v. We see that in both cases, the J_n operator comes out to be defined the same way, however, K_n varies between the two denotations by a negative sign. This implies that the spinors u and v transform exactly the same under rotation, but oppositely over Lorentz boost.

The spinors, u and v both supply representations for the Lorentz group and are known as the Weyl Spinors.

7 Conclusion

Due to the relations between differing groups, representations of more complex groups can be expressed through the understanding of simpler, more well understood groups. Double covers are one such relation that arises between these group structures and can be utilized inn the case of the Lorentz Group. The Lorentz Group can be expressed into the direct product of $SU(2)$ with itself, as decomposing this further leads to expressing a representation of the Lorentz Group with rotational algebra as two indexed spinors known as the Weyl Spinors.