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NOISY QUANTUM MEASUREMENT

Ben Saltzman

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Professor Sarada Rajeev
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1 Introduction

Not only does God play dice but... he sometimes throws them where they can't be seen.

Stephen Hawking

Despite Albert Einstein's insistence that the world is deterministic, quantum mechanics introduces a probabilistic world view. Measuring a quantum becomes a delicate task: measurements inherently affect the system in question. In this paper, based on a lecture given to fellow Kapitza members on December 3rd, 2017, I discuss the formulation of measurement in a noisy quantum system.

2 A New Description of Measurement

Measurement is commonly described by defining a set of projection operators that are complete, i.e.

$$\sum_j \Pi_j = I$$

Introducing a probe system, we can get information about the original system from the probe. Measurement can be described by the unitary interaction of the system with the probe, with operator U_{SP}.

Suppose we have a system S in state $|\Psi\rangle_S$ and a probe P with dimension d and orthonormal basis $\{|0\rangle_P, |1\rangle_P, ..., |d-1\rangle_P\}$. Defining the probe's initial state as $|0\rangle_P$, we can write the initial state of the whole system as

$$|\Psi\rangle_S \otimes |0\rangle_P$$

and measurement operators

$$\{|j\rangle\langle j|_P\}_{j\in\{0,1,...,d-l\}}$$

The operator can be written in the probe's basis as

$$U_{SP} = \sum_{j,k} M_S^{j,k} \otimes |j\rangle\langle k|_P \tag{1}$$

for operators $\{M_S^{j,k}\}$. The probability of outcome j from the probe is

$$p_J(j) = [\langle\Psi|_S \otimes \langle 0|_P U_{SP}^\dagger](I_S \otimes |j\rangle\langle j|_P)[U_{SP}|\Psi\rangle_S \otimes |0\rangle_P] \tag{2}$$

and the state of the system after measurement is

$$\frac{(I_S \otimes |j\rangle\langle j|_P)(U_{SP}\,|\Psi\rangle_S \otimes |0\rangle_P)}{\sqrt{p_J(j)}} \tag{3}$$

Note that $I_S \otimes |j\rangle\langle j|_P$ is the measurement operator on the full system, since only the probe is measured.

Since U_{SP} is unitary, we have $U_{SP}^\dagger U_{SP} = I_{SP} = I_S \otimes I_P$. Consider the case $k = 0$. For clarity, define $M^j \equiv M_S^{j,0}$.

$$\begin{aligned}
U_{SP}^\dagger U_{SP} &\xrightarrow{k=0} \left(\sum_{j'} M^{j'\dagger} \otimes |0\rangle\langle j'|_P\right)\left(\sum_j M^j \otimes |j\rangle\langle 0|_P\right) \\
&= \sum_j \sum j' M^{j'\dagger} M^j \otimes |0\rangle\,\langle j'|j\rangle\,\langle 0|_P \\
&= \sum_j M^{j\dagger} M^j \otimes |0\rangle\langle 0|_P
\end{aligned} \tag{4}$$

But $U_{SP}^\dagger U_{SP} \xrightarrow{k=0} I_S \otimes |0\rangle\langle 0|_P$. Using (4), we conclude that

$$I_S = \sum_j M^{j\dagger} M^j \tag{5}$$

Using the definition of U_{SP}, we can simplify (2) and (3). Substituting (1) into (2):

$$\begin{aligned}
\sum_j p_J(j) &= \left[\langle\Psi|_S \otimes \langle 0|_P \left(\sum_{j',k} M_S^{j',k\dagger} \otimes |k\rangle\langle j'|_P\right)\right](I_S \otimes |j\rangle\langle j|_P) \\
&\quad \left[\left(\sum_{j',k} M_S^{j',k} \otimes |j'\rangle\langle k|_P\right)|\Psi\rangle_S \otimes |0\rangle_P\right] \\
&= \left[\sum_{j',k} \langle\Psi|_S\, M_S^{j',k\dagger} \otimes \langle 0|k\rangle\,\langle j'|_P\right](I_S \otimes |j\rangle\langle j|_P) \\
&\quad \left[\sum_{j',k} M_S^{j',k}\,|\Psi\rangle_S \otimes |j'\rangle\,\langle k|0\rangle_P\right] \\
&= \left[\sum_j \langle\Psi|_S\, M^{j\dagger} I_S \otimes \langle j|_P\right]\left[\sum_{j'} M^{j'}\,|\Psi\rangle_S \otimes |j'\rangle_P\right] \\
&= \sum_j\ \langle\Psi|M^{j\dagger} M^j|\Psi\rangle
\end{aligned}$$

and therefore

$$p_J(j) = \langle\Psi|M^{j\dagger}M^j|\Psi\rangle \tag{6}$$

Similarly, we can simplify the post-measurement state to

$$\frac{M^j\,|\Psi\rangle_S \otimes |j\rangle_P}{\sqrt{p_J(j)}} \tag{7}$$

The state of S can be read off from (7) easily, since S and P are in a pure product state. Measurement can therefore be described as a set of measurement operators $\{M_S^j\}$, instead of $\{\Pi_j\}$, that satisfy (5). See Appendix A for discussion on the application of this process to ensembles.

When transferring classical data over a quantum channel, the receiver doesn't need the post-measurement state to process the information in a quantum fashion. The relevant probability is the probability of error. For any such situation where the probability of an outcome matters and the post-measurement state does not, we can describe a positive operator-valued measure (POVM) with a set of operators $\{\Lambda_j\} = \{M_j^\dagger M_j\}$ that are non-negative and complete. Clearly, projection is a type of POVM. The probability of success of the POVM is

$$\sum_{x\in X} p_X(x)\,\mathrm{Tr}\,[\Lambda_x\rho_x]$$

where ρ_x is the density matrix for state $|\psi_x\rangle$.

3 Composite Systems

Suppose we have two indepedent ensembles, $\varepsilon_A = \{p_X(x), |\psi_x\rangle\}$ and $\varepsilon_B = \{p_Y(Y), |\phi_y\rangle\}$. The density matrix for the joint state $|\psi_x\rangle \otimes |\phi_y\rangle$ is

$$\begin{aligned}\mathrm{E}_{X,Y}[(|\psi_X\rangle \otimes |\phi_Y\rangle)(\langle\psi_X| \otimes \langle\phi_Y|)] &= \mathrm{E}_{X,Y}[|\psi_X\rangle\langle\psi_X| \otimes |\phi_Y\rangle\langle\phi_Y|] \\ &= \sum_{x,y} p_X(x)p_Y(y)\,|\psi_x\rangle\langle\psi_x| \otimes |\phi_y\rangle\langle\phi_y| \\ &= \sum_x p_X(x)\,|\psi_x\rangle\langle\psi_x| \otimes \sum_y |\phi_y\rangle\langle\phi_y| \\ &= \rho \otimes \sigma \end{aligned} \tag{8}$$

where ρ and σ are the density matrices for ε_A and ε_B respectively.

Now suppose we have a joint ensemble in which systems A and B are correlated classically. We'd like a formulation to express this ensemble similarly to the independent situation above. To do this, we introduce a new random variable Z that X and Y are conditioned on. The two ensembles are $\varepsilon_A = \{p_{X|Z}(x|z), |\psi_{x,z}\rangle\}$ (density matrix ρ_z) and $\varepsilon_B = \{p_{Y|Z}(y|z), |\phi_{y,z}\rangle\}$ (density matrix σ_z), and $X|Z$ and $Y|Z$ are independent. Using the same procedure as we did with (8), we obtain the density matrix of the total state:

$$\begin{aligned} &\mathrm{E}_{X,Y,Z}[(|\psi_{X,Z}\rangle \otimes |\phi_{Y,Z}\rangle)(\langle\psi_{X,Z}| \otimes \langle\phi_{Y,Z}|)] \\ &= \sum_{x,y,z} p_Z(z) p_{X|Z}(x|z) p_{Y|Z}(y|z)\, |\psi_{x,z}\rangle\langle\psi_{x,z}| \otimes |\phi_{y,z}\rangle\langle\phi_{y,z}| \end{aligned} \tag{9}$$

Define a new random variable $W = X \wedge Y \wedge Z$. We can write the density matrix in (9) as

$$\sum_w p_W(w)\, |\phi_w\rangle\langle\phi_w| \otimes |\phi_w\rangle\langle\phi_w| \tag{10}$$

So, we can write any state with the properties discussed in this paragraph as a product of pure states. This type of state is termed *separable*, and contains no entanglement. In other words, a separable state can always be prepared classically. See Appendix B for an application involving separable states.

4 Local Density Operators

Suppose systems A and B are in an entangled Bell state $|\Phi^+\rangle_{AB}$. Take a POVM $\Lambda^j{}_j$ on A. The measurement operators for the system are $\Lambda^j_A \otimes I_B{}_j$. The probability of outcome j is

$$\begin{aligned} p_J(j) &= \left\langle \Phi^+ \middle| \Lambda^j_A \otimes I_B \middle| \Phi^+ \right\rangle_{AB} \\ &= \frac{1}{2} \sum_{k,l=0}^{1} \langle kk|\Lambda^j_A \otimes I_B|ll\rangle_{AB} \\ &= \frac{1}{2} \sum_{k,l=0}^{1} \langle k|\Lambda^j_A|l\rangle_A \, \langle k|I_B|l\rangle_B \\ &= \mathrm{Tr}\left[\Lambda^j_A \, \frac{1}{2} \sum_{k=0}^{1} \Big(|k\rangle\langle k|_A \Big) \right] \\ &= \mathrm{Tr}\left[\Lambda^j_A \pi_A \right] \end{aligned} \tag{11}$$

where the "local density operator" for A is the maximally mixed state $\pi_A = \frac{1}{2}\sum_{k=0}^{1} |k\rangle\langle k|_A$. This process goes similarly for B. Thus, the following global state gives the same predictions in local measurements as $|\Phi^+\rangle_{AB}$:

$$\pi_A \otimes \pi_B$$

We'd like to define what we mean by a local density operator in order to describe the results of local measurements. To do this, we need to define the *partial trace* operation.

Suppose $\{|k\rangle_A\}$ and $\{|l\rangle_B\}$ are orthonormal bases for the Hilbert spaces of A and B. Then $\{|k\rangle_A \otimes |l\rangle_B\}$ is an orthonormal basis for the product of the Hilbert spaces. For density operator ρ_{AB}, the probability of outcome j is

$$\begin{aligned} p_J(j) &= \mathrm{Tr}\big[(\Lambda_A^j \otimes I_B)\rho_{AB}\big] \\ &= \sum_{k,l} [\langle k|_A \otimes \langle l|_B][(\Lambda_A^j \otimes I_B)\rho_{AB}][|k\rangle_A \otimes |l\rangle_B] \\ &= \sum_{k,l} \langle k|_A \, [I_A \otimes \langle l|_B][(\Lambda_A^j \otimes I_B)\rho_{AB}][I_A \otimes |l\rangle_B] \, |k\rangle_A \\ &= \sum_k \langle k|_A \, \Lambda_A^j \sum_l \left[(I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B)\right] |k\rangle_A \\ &= \mathrm{Tr}\left[\Lambda_A^j \sum_l \left[(I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B)\right]\right] \end{aligned} \tag{12}$$

So, we define the partial trace for B as

$$\mathrm{Tr}_B[X_{AB}] = \sum_l \left[(I_A \otimes \langle l|_B)X_{AB}(I_A \otimes |l\rangle_B)\right] \tag{13}$$

and the local density operator for A as

$$\rho_A = \mathrm{Tr}_B[\rho_{AB}] \tag{14}$$

Therefore, (12) becomes

$$p_J(j) = \mathrm{Tr}\big[\Lambda_A^j \rho_A\big] \tag{15}$$

Alice can predict the outcome of local measurements with (15).

5 Classical-Quantum Ensembles

Suppose Alice prepares a quantum system with density matrix ρ_A^x and probability distribution $p_X(x)$. She passes this ensemble to Bob, who must learn about it. There is a loss of information in X after preparation which is minimized if the state is pure. $\rho_A = \sum_x p_x(x) \, |x\rangle\langle x|_A$ for an orthonormal basis $\{|x\rangle\}_{x\in X}$. For a mixed state, $\rho_A = \sum_x p_X(x)\rho_A^x$ is more difficult to extract information from.

One solution is for Alice is to prepare a *classical-quantum ensemble*:

$$\{p_X(x), |x\rangle\langle x|_X \otimes \rho_A^x\}_{x\in X}$$

This ensemble is so-called because system X is classical, while system A is quantum. The density operator for the entire system is

$$\rho_{XA} = \sum_x p_X(x)\, |x\rangle\langle x|_X \otimes \rho_A^x \tag{16}$$

Suppose Bob makes a measurement of the system with $\{I_X \otimes \Lambda_A^j\}$. This is akin to Bob measuring an isolated system A with $\{\Lambda_A^j\}$. Why?

$$\begin{aligned}
\operatorname{Tr}\left[\rho_{XA}(I_X \otimes \Lambda_A^j)\right] &= \operatorname{Tr}\left[(\sum_x p_X(x)\, |x\rangle\langle x|_X \otimes \rho_A^x)(I_X \otimes \Lambda_A^j)\right] \\
&= \operatorname{Tr}\left[\sum_x p_X(x)(|x\rangle\langle x|_X\, I_X \otimes \rho_A^x \Lambda_A^j)\right] \\
&= \sum_x \operatorname{Tr}[|x\rangle\langle x|_X\, I_X]\operatorname{Tr}\left[p_X(x)\rho_A^x \Lambda_A^j\right] \\
&= \sum_x \operatorname{Tr}\left[p_X(x)\rho_A^x \Lambda_A^j\right] \\
&= \operatorname{Tr}\left[\rho_A \Lambda_A^j\right]
\end{aligned}$$

So Bob can extract information about A from the whole system with a local measurement on A.

6 Conclusion

> *At the heart of quantum mechanics is a rule that sometimes governs politicians or CEOs - as long as no one is watching, anything goes.*
>
> Lawrence M. Krauss

Measurement of quantum systems is tricky, and matters only get more complicated when you consider composite quantum systems. With judicious choices of measurement operators and careful preparation of a system or composite system, one can make quantum measurement seem more akin to the classical case.

Appendix A: Redefining Measurement for Ensembles

Suppose we have an ensemble

$$\{p_X(x), |\psi_x\rangle\}_{x\in X}$$

with density operator

$$\rho = \sum_{x\in X} p_X(x)\, |\psi_x\rangle\langle\psi_x|$$

Using the same procedure as outlined in section 2:

$$\begin{aligned}
\sum_j p_J(j) &= \sum_{x\in X} p_X(x) \left[\langle\psi_x|_S \otimes \langle 0|_P \left(\sum_{j',k} M_S^{j',k\dagger} \otimes |k\rangle\langle j'|_P \right) \right] (I_S \otimes |j\rangle\langle j|_P) \\
&\quad \left[\left(\sum_{j',k} M_S^{j',k} \otimes |j'\rangle\langle k|_P \right) |\psi_x\rangle_S \otimes |0\rangle_P \right] \\
&= \sum_{x\in X} p_X(x) \left[\sum_{j',k} \langle\psi_x|_S\, M_S^{j',k\dagger} \otimes \langle 0|k\rangle\, \langle j'|_P \right] (I_S \otimes |j\rangle\langle j|_P) \\
&\quad \left[\sum_{j',k} M_S^{j',k}\, |\psi_x\rangle_S \otimes |j'\rangle\, \langle k|0\rangle_P \right] \\
&= \sum_{x\in X} p_X(x) \left[\sum_j \langle\psi_x|_S\, M^{j\dagger} I_S \otimes \langle j|_P \right] \left[\sum_{j'} M^{j'}\, |\psi_x\rangle_S \otimes |j'\rangle_P \right] \\
&= \sum_{x\in X} p_X(x) \sum_j \langle\psi_x|M^{j\dagger}M^j|\psi_x\rangle \\
p_J(j) &= \sum_{x\in X} p_X(x)\, \langle\psi_x|M^{j\dagger}M^j|\psi_x\rangle \\
&= \left\langle M^{j\dagger}M^j \right\rangle \\
&= \mathrm{Tr}\left[M^{j\dagger}M^j\rho \right]
\end{aligned}$$

This is a reformulation of the Born Rule. The post-measurement state is

$$\frac{M^j\rho M^{j\dagger}}{p_J(j)}$$

Appendix B: The CHSH Game

Consider a game with two players, Alice (A) and Bob (B). Alice and Bob cannot communicate with each other, but they can communicate with a referee. The referee sends out two bits: x to Alice and y to Bob. Each player sends a bit back to the referee (a and b respectively). The win condition is

$$x \wedge y = a \oplus b$$

It is known that the maximum probability of success for any strategy that Alice and Bob agree to use beforehand is 0.75 for classical strategies and $\cos^2(\pi/8) \approx 0.85$ for quantum strategies. The form of a classical strategy is

$$p_{AB|XY}(a, b|x, y) = \int \mathrm{d}\lambda p_{\Lambda}(\lambda) p_{A|\Lambda X}(a|\lambda, x) p_{B|\Lambda Y}(b|\lambda, y)$$

where Λ is a continuous index for their strategies.

The density operator for the state of the system is

$$\rho_{AB} = \int \mathrm{d}\lambda p_{\Lambda}(\lambda) \, |\psi_\lambda\rangle\langle\psi_\lambda| \otimes |\phi_\lambda\rangle\langle\phi_\lambda|$$

In this case, the classical strategy is

$$\begin{aligned} p_{AB|XY}(a, b|x, y) &= \mathrm{Tr}\Big[(\Pi_a^{(x)} \otimes \Pi_b^{(y)})\rho_{AB}\Big] \\ &= \mathrm{Tr}\Big[\int \mathrm{d}\lambda p_{\Lambda}(\lambda) \; \Pi_a^{(x)} \, |\psi_\lambda\rangle\langle\psi_\lambda|_A \otimes \Pi_b^{(y)} \, |\phi_\lambda\rangle\langle\phi_\lambda|_B\Big] \\ &= \int \mathrm{d}\lambda p_{\Lambda}(\lambda) \, \langle\phi_\lambda|\Pi_a^{(x)}|\phi_\lambda\rangle \; \langle\phi_\lambda|\Pi_b^{(y)}|\phi_\lambda\rangle \end{aligned}$$

Therefore, we have

$$p_{A|\Lambda X}(a|\lambda, x) = \langle\phi_\lambda|\Pi_a^{(x)}|\phi_\lambda\rangle \quad \text{and} \quad p_{B|\Lambda Y}(b|\lambda, y) = \langle\phi_\lambda|\Pi_b^{(y)}|\phi_\lambda\rangle$$

The above are classical strategies that simulate quantum strategies, provided that the initial system is in a separable state. This implies that the upper limit on the probability of winning for a separable states is 0.75, not $\cos^2(\pi/8)$.

References

[1] Wilde, M. M. (2013). *Quantum information theory.* Cambridge University Press.

[2] Sándor, I. and Ferenc, B. *Quantum Computing and Communications - An Engineering Approach, Chapter 3: Measurements.* Accessed December 5 2017.