Decoherence induced by non-unitary evolution in a qubit subsystem

Jose M. Torres López. Kapitza Society, University of Rochester. 14th December 2021.

0. Abstract.

This paper illustrates the relation between non-unitary time evolution of one part of a bipartite system and the onset of entanglement between the two parts. From the point of view of one of the system's parts, this process of entanglement with the other part can be interpreted as an increase of decoherence due to interaction with the environment. This correspondence is particularly simple in the context of one qubit which is considered as part of a two-qubit system undergoing unitary evolution; and can be understood through the concepts of quantum channel and Schmidt number.

1. Operator-sum representation of quantum channels

Quantum channels are linear transformations on the set of density matrices corresponding to a given Hilbert space of quantum states. Since the density matrices are themselves operators on the Hilbert space, i.e. they are linear transformations mapping quantum states to quantum states, quantum channels can be thought of as maps acting on operators instead of states; for this reason, they are also known as "superoperators". In this section, we introduce the operator-sum representation of quantum channels by considering how unitary evolution of a bipartite system generally transforms a single one of the component subsystems. This presentation is based on Section 3.2.1 of *Lecture Notes for Physics 229, Quantum Information and Computation*, by John Preskill (Ref. [1]).

To begin with, let our bipartite system be $A \otimes B$, and imagine that we as observers only have access to the information contained in A alone. Also assume that we know the initial state of subsystem B to be a pure state, which we label $|0\rangle_B$; then, the initial density matrix on the bipartite system has the following tensor-product form:

$$\boldsymbol{\rho}_A \otimes |0\rangle_{B\,B}\langle 0| \qquad [1]$$

where $\boldsymbol{\rho}_A$ represents an arbitrary density matrix on the subsystem A. Next, we allow the combined system to evolve for some time under the action of a unitary time evolution operator U_{AB}:

$$\boldsymbol{\rho}_A \otimes |0\rangle_{B\,B}\langle 0| \rightarrow \boldsymbol{U}_{AB}(\boldsymbol{\rho}_A \otimes |0\rangle_{B\,B}\langle 0|)\boldsymbol{U}_{AB}^{\dagger}. \qquad [2]$$

After performing the partial trace over the Hilbert space H_B of subsystem B, this expression reduces to the density matrix giving the final state of system A:

$$\begin{aligned}\boldsymbol{\rho}'_A &= tr_B\left(\boldsymbol{U}_{AB}(\boldsymbol{\rho}_{\boldsymbol{A}} \otimes |0\rangle_{B\,B}\langle 0|)\boldsymbol{U}_{AB}^{\dagger}\right)\\ &= \textstyle\sum_{\mu}{}_B\langle \mu|\boldsymbol{U}_{AB}|0\rangle_B\, \boldsymbol{\rho}_{\boldsymbol{A}}\, {}_B\langle 0|\boldsymbol{U}_{AB}^{\dagger}|\mu\rangle_B \end{aligned} \qquad [3]$$

expressed in terms of a given orthonormal basis $|\mu\rangle_B$ of H_B. Note that ${}_B\langle\mu|\boldsymbol{U}_{AB}|0\rangle_B$ is an operator on H_A which does not necessarily inherit the unitary property from U_{AB}. Now, label this operator $\boldsymbol{M}_\mu = {}_B\langle\mu|\boldsymbol{U}_{AB}|0\rangle_B$, in order to rewrite Eq. [3] as

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\rho}_A) = \boldsymbol{\rho}'_A = \sum_\mu \boldsymbol{M}_\mu\, \boldsymbol{\rho}_A\, \boldsymbol{M}_\mu^\dagger. \qquad [4]$$

This gives the state obtained from $\boldsymbol{\rho}_A$ after the unitary evolution of the combined system, defining the linear map $\boldsymbol{\mathcal{L}}$ that takes linear operators to linear operators. Besides, this map has the important property that $\sum_\mu \boldsymbol{M}_\mu\, \boldsymbol{M}_\mu^\dagger = \boldsymbol{I}_A$, which follows directly from the unitarity of $\boldsymbol{U}_{AB}$:

$$\sum_\mu \boldsymbol{M}_\mu\, \boldsymbol{M}_\mu^\dagger = \sum_\mu\ {}_B\langle 0|\boldsymbol{U}_{AB}^\dagger|\mu\rangle_B\ {}_B\langle\mu|\boldsymbol{U}_{AB}|0\rangle_B\ = {}_B\langle 0|\boldsymbol{U}_{AB}\boldsymbol{U}_{AB}^\dagger|0\rangle_B \qquad [5]$$

since the $|\mu\rangle_B$ form a basis (so $\sum_\mu\ |\mu\rangle_{BB}\langle\mu| = \boldsymbol{I}_{B\otimes B}$). A map on linear operators that satisfies this property is known as *quantum channel* or *superoperator*. Similarly, Eq. [4] is called the operator-sum representation or *Kraus* representation of the quantum channel $\boldsymbol{\mathcal{L}}$ (and the $\boldsymbol{M}_\mu$ are the *Kraus operators*). Of course, we have to check that the outputs of $\boldsymbol{\mathcal{L}}$ always satisfy the properties required of density matrices: hermiticity, unit trace, and positivity. This check is straightforward and available in Preskill's notes. Similarly, Preskill shows that the converse direction is also true: for any given superoperator in operator-sum there is a corresponding unitary representation; this is the essence of the *Kraus representation theorem*. We omit the proof because we will not need to use this fact explicitly here. Instead, the next section focuses on the relation between the non-unitary evolution of system A, mediated by a quantum channel map, and the increase of decoherence in system A (this is, increase of its entanglement with system B).

2. Evolution of two-qubit system

In this section, we focus on a two-qubit system as an illustrative example of the general dynamics introduced in the previous section. More concretely, we will show how unitary evolution of the two-qubit system can transform a pure state in A into a mixed state, which means that decoherence increases in the process. Thus, let both A and B represent single qubits, and consider a pure, uncorrelated initial state of the form:

$$(\alpha|0\rangle + \beta|0\rangle)_A \otimes |0\rangle_B \qquad [6]$$

where $|\alpha|^2 + |\beta|^2 = 1$. This state is uncorrelated because of its simple tensor-product structure and, of course, follows from the initial state in the previous section simply by choosing $\boldsymbol{\rho}_A$ to be a general pure state for a qubit. Now, an interaction between the two qubits is represented by some unitary transformation of the combined system, and example of which is:

$$U\{(\alpha|0\rangle + \beta|1\rangle)_A \otimes |0\rangle_B\} = \alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B. \qquad [7]$$

This transformation is clearly unitary, because the length of the state $|\alpha|^2 + |\beta|^2$ is conserved, so it is related to a particular quantum channel as derived in the previous section. In addition, we can see directly that the final state is in fact entangled, by noting that the outcome of a measurement in the $\{|0\rangle_B, |1\rangle_B\}$ basis will also project the final state $|\psi'\rangle_A$ of subsystem A to a definite post-measurement result. This statement will be made quantitative in the next section, by calculating the increase in Schmidt number from the initial to the final density matrix. For now, note that the measurement induced in A would be orthonormal in this scenario. But we can pick a different basis in B for the expansion above, such that the measurement induced in A is given by a non-orthogonal POVM; this alternative basis expansion will make it easier to rewrite the final state in terms of the Kraus operators, for consistency with the previous section. Clearly, an adequate choice of basis for these purposes is $\left\{\left|\pm\right\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_B \pm |1\rangle_B)\right\}$. In fact, a measurement of B in this basis is related to the following possible outcomes in A:

$$\alpha|0\rangle_A \pm \beta|1\rangle_A$$

which are only orthogonal to each other if $|\alpha| = |\beta|$. We can rewrite Eq. [7] in the new basis as:

$$|\psi'\rangle_{AB} = \boldsymbol{M}_+|\psi\rangle_A \otimes |+\rangle_B + \boldsymbol{M}_-|\psi\rangle_A \otimes |-\rangle_B \qquad [8]$$

for the Kraus operators $\boldsymbol{M}_+ = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix} = \frac{1}{\sqrt{2}}\boldsymbol{I}$ and $\boldsymbol{M}_- = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{-1}\end{bmatrix} = \frac{1}{\sqrt{2}}\boldsymbol{\sigma_3}$. The explicit calculation is provided next in matrix notation:

$$\boldsymbol{M}_+|\psi\rangle_A = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}\left(\alpha\begin{bmatrix}0\\1\end{bmatrix} + \beta\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{\alpha}{\sqrt{2}}\begin{bmatrix}1\cdot 0 + 0\cdot 1\\ 0\cdot 0 + 1\cdot 1\end{bmatrix} + \frac{\beta}{\sqrt{2}}\begin{bmatrix}1\cdot 1 + 0\cdot 0\\ 0\cdot 1 + 1\cdot 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}$$

and similarly $\boldsymbol{M}_-|\psi\rangle_A = \frac{1}{\sqrt{2}}\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}$. Therefore:

$$\boldsymbol{M}_+|\psi\rangle_A \otimes |+\rangle_B + \boldsymbol{M}_-|\psi\rangle_{AB} \otimes |-\rangle_B = \frac{1}{2}\left(\begin{bmatrix}\beta\\ \alpha\end{bmatrix} \otimes \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}\beta\\ -\alpha\end{bmatrix} \otimes \begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}\beta+\beta\\ \beta-\beta\\ \alpha-\alpha\\ \alpha+\alpha\end{bmatrix} = \begin{bmatrix}\beta\\0\\0\\ \alpha\end{bmatrix} = \begin{bmatrix}0\\ \alpha\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}\beta\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B, \qquad [9]$$

as claimed above.

3. Quantifying entanglement with the Schmidt number

From Section 2.4 of Preskill's Notes, we know that any state of a bipartite system can be put in the Schmidt decomposition form, which is especially convenient to compute the degree of entanglement between the two parts. In this section, we apply this procedure to the initial and final states examined for the two-qubit system in the previous section.

This will serve to compute exactly the amount of entanglement introduced by the unitary transformation U.

First we remind the general procedure for representing a state through Schmidt decomposition. Remember that a general vector $|\psi\rangle_{AB}$ in $H_A \otimes H_B$ can be written in terms of a basis formed by all tensor-product combinations of basis $\{|i\rangle_A\}$ for A and $\{|\mu\rangle_B\}$ for B:

$$|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu} |i\rangle_A \otimes |\mu\rangle_B = \sum_{i,\mu} |i\rangle_A \otimes |i\rangle_B \qquad [10]$$

where we absorb the coefficients into the basis elements from B, by defining $|i\rangle_B \equiv \sum_\mu a_{i\mu} |\mu\rangle_B$. Besides, noting that we are interested in pure states of the combined system, we choose $|i\rangle_A$ to be the basis that diagonalizes $\boldsymbol{\rho}_A$. There is no reason *a priori* to believe that the new basis $|i\rangle_B$ is still orthonormal. However, we previously showed (Preskill Section 2.4), by comparing the eigenvalue expansion of $\boldsymbol{\rho}_A$ with its computation as a partial trace over B of $|\psi\rangle_{AB\,AB}\langle\psi|$, that they are in fact orthogonal; and can be made orthonormal through the rescaling: $|i'\rangle_B = p_i{}^{-1/2}$, where p_i are the eigenvectors of $\boldsymbol{\rho}_A$. This led us to the Schmidt decomposition of $|\psi\rangle_{AB}$ relative to the particular basis for A and B that we started with:

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i'\rangle_B. \qquad [11]$$

We also showed in a similar way that $\boldsymbol{\rho}_B$ has the same non-zero eigenvalues than $\boldsymbol{\rho}_A$. Then, we reasoned that, if $\boldsymbol{\rho}_A$ and $\boldsymbol{\rho}_B$ do not have non-zero degenerate eigenvalues, the Schmidt decomposition of $|\psi\rangle_{AB}$ above is uniquely determined by the subsystem density matrices. This is because the expression above tells us to pair up the basis vectors that correspond to the same eigenvalues. In other words, in this case the *arbitrary state* $|\psi\rangle_{AB}$ *itself* is determined exactly by the states of the two subsystems, given by $\boldsymbol{\rho}_A$ and $\boldsymbol{\rho}_B$.

When both $\boldsymbol{\rho}_A$ and $\boldsymbol{\rho}_B$ represent pure states, the situation just described corresponds to no entanglement between the subsystems. More generally, we define the *Schmidt number* of $|\psi\rangle_{AB}$ to be the number of non-zero eigenvalues of $\boldsymbol{\rho}_A$ (or $\boldsymbol{\rho}_B$), and say that the bipartite pure state is *entangled* or *separable* if and only if this number is greater than one. The reason for this definition is that, then, a separable bipartite pure state will always be given by the direct product of pure states in the individual subsystems:

$$|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\chi\rangle_B$$

such that the reduced density operators will be pure: $\boldsymbol{\rho}_A = |\varphi\rangle_{A\,A}\langle\varphi|$ and $\boldsymbol{\rho}_B = |\chi\rangle_{B\,B}\langle\chi|$. In contrast, a state that cannot be expressed as such a direct product is entangled, and then the reduced density matrices represent mixed states.

Now, let us go back to our two-qubit example in order to confirm that our initial states is indeed unentangled:

$$|\psi\rangle_{AB} = (\alpha|0\rangle + \beta|0\rangle)_A \otimes |0\rangle_B \rightarrow$$

$$\boldsymbol{\rho}_A = [(\alpha|0\rangle + \beta|0\rangle)_A \otimes |0\rangle_B][{}_B\langle 0| \otimes {}_B(\alpha^*\langle 0| + \beta^*\langle 0|)]$$

$$= \left[\begin{pmatrix} 0 \\ \alpha+\beta \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] [(0 \quad 1) \otimes (0 \quad \alpha^* + \beta^*)]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha+\beta \end{bmatrix} [0 \quad 0 \quad 0 \quad \alpha^*+\beta^*] = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |\alpha|^2 + |\beta|^2 + \alpha\beta^* + \beta\alpha^* \end{bmatrix} \qquad [\,12\,]$$

Where the omitted entries equal zero, so that $\boldsymbol{\rho}_A$ is already diagonal and has only $|\alpha|^2 + |\beta|^2 + \alpha\beta^* + \beta\alpha^*$ as a non-zero (diagonal) entry. Therefore, the Schmidt number of the initial state is 1, and the state is unentangled.

In a similar way, we find the Schmidt number corresponding to the final state obtained after applying the unitary time evolution:

$$|\psi'\rangle_{AB} = \alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B \rightarrow$$

$$\boldsymbol{\rho'}_A = \left[\alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B\right]\left[\alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B\right]^\dagger$$

$$= \begin{bmatrix} \beta \\ 0 \\ 0 \\ \alpha \end{bmatrix} [\beta^* \quad 0 \quad 0 \quad \alpha^*] = \begin{bmatrix} |\beta|^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |\alpha|^2 \end{bmatrix} \qquad [\,13\,]$$

where again we omit the zero entries, and we have used the result $\alpha|0\rangle_A \otimes |0\rangle_B + \beta|1\rangle_A \otimes |1\rangle_B = \begin{bmatrix} \beta \\ 0 \\ 0 \\ \alpha \end{bmatrix}$ found in Eq. [9] in the previous section. Again, we can simply read out that $\boldsymbol{\rho'}_A$ has two non-zero eigenvalues ($|\beta|^2$ and $|\alpha|^2$). This confirms that the Schmidt number of the final state is two, so that the qubits are now entangled to each other as a result of the unitary evolution.

4. Conclusion

The simple example in this paper illustrates how quantum channels that transform a subsystem in a non-unitary way have the potential to introduce decoherence in the subsystem, due to interactions with the environment. Indeed, preventing this decoherence in a variety of systems in a major current research topic in quantum information, especially in the context of engineering hardware for quantum computation that can operate without "becoming classical" due to interactions with the environment. In this sense, small systems of a few degrees of freedom are easier to isolate and in fact represent every existing quantum computer existing to this day; while scaling these systems to a size large enough for powerful computations is a major challenge for the field in which technological progress is currently being made.

5. References

[1]. *Lecture Notes for Physics 229, Quantum Information and Computation*, by John Preskill (1998).