Dividing the Indivisible: A Comprehensive Analysis of the Banach-Tarski Paradox

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Abstract

The Banach-Tarski paradox discusses a series of questions about the decomposition of subsets of Euclidean spaces in different dimensions. The paradox states: "Given a solid ball in 3-dimensional Euclidean space, we can partition it into a finite number of pieces, so that we can rearrange them to get two solid balls congruent to the first ball." ["Banach-Tarski Paradox."]

There exists a generalization of this paradox, which states that if A and B are any two bounded subsets of $\mathbb{R}^{\nvDash}$, each having nonempty interior, then A and B are equidecomposable ["The Banach–Tarski..."]

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1 Introduction

1.1 History

1.1.1 History of Banach-Tarski paradox

The Banach-Tarski paradox was first stated in 1924. The idea of it is, as acknowledged by Banach and Tarski, based on decomposition of the unit interval example by Giuseppe Vitali that proved the existence of a set of real numbers that's not Lebesgue measurable. Moreover, the Banach-Tarski paradox is based on the work of Felix Hausdorff concerning the paradoxical decomposition of the sphere and the Hausdorff paradox of 1914.["The Banach–Tarski..."] The Hausdorff paradox states that there is a disjoint decomposition of the sphere $\mathbb{S}^{\nvDash}$ into four sets A,B,C,D such that A,B,C,$B \cup C$ are all congruent and D is countable ["Hausdorff Paradox."]. In addition, there is also a paper by Banach published in 1923 that contributes to the formation of the Banach-Tarski paradox ["The Banach–Tarski..."]. More importantly, the axiom of choice by Zermelo, a German mathematician, is crucial to the construction of Banach and Tarski's paper. The axiom of choice states: The product of a collection of nonempty sets is nonempty, and the Banach-Tarski paradox is said to be inherent, or even equivalent, to it ["What Does the..."].

1.1.2 Author's biography

Stefan Banach [Britannica, The Editors of Encyclopedia.] Born on 30 March 1892 in Poland, Stefan Banach was considered as one of the world's most influential mathematicians and was the one who lay the foundation of the modern functional analysis. His main contributions to mathematics include the 1932 book, Theory of Linear Operations (the first monograph on the general theory of functional analysis) etc.

Banach had demonstrated his outstanding talent of math since a young age. He attended a secondary school called IV Gymnasium in Krakow, where he started his math career. Immediately after graduation, along with his friend Steinhaus, he published their first joint work, and later on in 1919, he established a mathematical society with some other mathematicians and received an assistant ship at the Lwów Polytechnic in 1920. Subsequently, he soon became a professor at the Polytechnic and joined the Polish Academy of Learning, starting his work on Theory of Linear Operations.

Banach died on 31 August 1945. Having devoted to mathematics throughout his whole life, his contribution to math significantly prompted the development of math, and multiple well-known theorems are bearing his name, including Banach spaces, Banach algebras, Banach measures, the Banach–Tarski paradox, the Hahn–Banach theorem, the Banach–Steinhaus theorem, the Banach–Mazur game etc.

Alfred Tarski ["Alfred Tarski Facts"] Born on January 14, 1901, Tarski was a well-known Polish-American logician and mathematician. He attended the University of Warsaw in Poland and since 1942, he started to teach and conduct research at the University of California, Berkeley until his death in 1983.

At the age of 23, he and Banach proved that given the theorem of the axiom of choice, a ball can be cut into several pieces and be rearranged into two balls whose size are the same as the original number. This idea is later known as the Banach-Tarski paradox.

Beside the paradox, Tarski was also well-known for his contribution to model theory, meta-mathematics, and algebraic logic, and also contributed to areas such as abstract algebra, topology, geometry etc.

Other People's Opinions of the paradox After first stated in 1924, multiple other advancements were built on the original proof. The original proof states that six pieces must be needed in order to be put together to form two balls identical with the original ball. However, in 1947, Robinson proved that only five pieces are needed, and that five pieces is the minimum number (four pieces are actually sufficient as long as the center point is neglected). Then, in 2005, it's shown by Trevor M. Wilson that the pieces that the ball is decomposed into can be chosen in some ways that allow them to move continuously into place and not bumping into one another.

2 Basic Concept

2.1 Hotel Trick

Assume we have two hotels, hotel A and hotel B. There are finitely many rooms in hotel A and all rooms in hotel A are booked. In hotel B, there are infinitely many rooms and infinitely many people live in hotel B. Suppose there is a new guest who wants to book a room. The only choice for him is booking a room in hotel B. Although there are infinitely many people live in hotel B, the new guest still can book room in hotel B. We label every room in hotel B from 1 to ∞. Since there is a new guest move in, we need an empty room for him. The idea is making room 1 is empty for the new guest. Therefore, we move people who have already lived in hotel to the next room, which means we move people who originally lived in room 1 to room 2, who originally lived in room 2 to room 3 and so far. Then there is an empty room.

Now we use mathematical term to explain Hotel Trick:
Assume A is an infinite set, $A \bigsqcup \{x\}, x \notin A$
choose $y_1 \in A$
$y_2 \in A \backslash \{y_1\}$
$y_3 \in A \backslash \{y_1, y_2\} \ \dots \ y_n \in A \backslash \{y_1, y_2 ... y_{n-1}\}$ as $n \longrightarrow \infty$
$\therefore \{y_1, y_2 ... y_n\} \subset A$

Suppose $f : A \cup \{x\} \longrightarrow A$, z is the element in $A \cup \{x\}$
1.When $f(z) = z$ if $z \neq x \quad then \quad z \in \{y_1, y_2 .... y_n\}$
2.When $f(z) = y_1$ if $z = x$
3.When $f(z) = y_n$ if $z = y_n$

2.2 HyperWebster

Assume there is a publishing company that wants to print a book containing all the words that can possibly created from the English alphabet A-Z (no matter that the words has the meaning or not). Since it is an infinitly collection of all the words, we wish to find a way to collect these words. Therefore, we divide all the words into 26 volumes by their first letter, then order them by alphabetical order. Then we have:

Volume 1: A,AA,AAA...,AB,ABA,ABAA...,AC,...,AZ,AZA,...
Volume 2: B,BA,BAA...,BB,BBA,BBAA...,BC,...,BZ,BZB,...
⋮
Volume 26: Z,ZA,ZAA...,ZZ,ZZA,ZZAA...,ZC,...,ZZ,ZZZ,...

Next, we realize the first letter in each word can be inferred from its volume and hence, the first letter of each word can be dropped. Then we form the new volumes, and new volumes look like the following:

Volume 1:A,AA,AAA,...,B,BA,BAA,...C,...,Z,ZA,...
Volume 2:A,AA,AAA,...,B,BA,BAA,...C,...,Z,ZA,...
⋮
Volume 26:A,AA,AAA,...,B,BA,BAA,...C,...,Z,ZA,...

Next, we find that each new volume looks the same, except the volume name. Therefore, we decide publishing the single volume instead of the whole book. Then we have the "HyperWebster" (each volume).

2.3 Axiom of Choice

2.3.1 Definition

In mathematics, the axiom of choice, or AC, is an axiom of set theory equivalent to the statement that the Cartesian product of a collection of non-empty sets in non-empty. It states that for every indexed family of nonempty sets there exists an indexed family of elements such that for every.

2.3.2 Formulation of axiom of choice

AC1: Any collection of nonempty sets has a choice function

AC2: Any indexed collection of sets has a choice function

AC3: For any relation R between sets A and B $\forall$ x $\in$ A $\exists$ y $\in$ B[R(x,y)] $\Rightarrow$ f[f:A$\rightarrow$B and $\forall$x $\in$A[R(x,fx)]]. In other words, every relation contains a function having the same domain.

AC4: Any surjective function has a right inverse.

We can call a choice set for a family of sets H, any subset T$\subseteq \bigcup$H for which each intersection T$\cap$X for X$\in$H has exactly one element. As a very simple example, let H=$\{0\}$,$\{1\}$,$\{2, 3\}$, then H has two choice sets $\{0, 1, 2\}$, $\{0, 1, 3\}$.

2.3.3 Independence and Consistency of Axiom of Choice

1) Independence: It was first proved by Fraenkel in 1922. He used a system of set theory with atoms and each atom is an individual. It contains no numbers but it is different from empty sets. The main method he used to prove is that he assumed there exists an infinite set S with atoms. And then he started the set with S first and added other subsets of S, we call this set Q(S), which is a model of set theory with atoms.

2) Consistency: It was proved by Kurt Godel by using the axiom of set theory—definability. He introduced a new hierarchy of sets, which also called constructible hierarchy. We can let P(X) is the powerful set of X, α is an ordinal, and λ is a limit ordinal. S_0=ϕ, $S_{\alpha+1} = P(V_\alpha)$, $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$.

His method is defined by a similar recursion on the ordinals, where Def(X) is set of all subsets of X. $L_0 = \phi$, $L_{\alpha+1} = Def(L_\alpha)$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha . L = \bigcup_\alpha L_\alpha$, α $\in$Ord.

2.4 Unmeasurable Set

2.4.1 Definition

In mathematics, a non-measurable set is a set which cannot be assigned a meaningful "size". The mathematical existence of such sets is construed to provide information about the notions of length, area and volume in formal set theory.

A subset V⊆[0,1] is called Vitali set, which means unmeasurable set. If V contains a single point from each coset of $\mathbb{Q}$ in $\mathbb{R}$

Coset: A set composed of all the products obtained by multiplying each element of a subgroup in turn by one particular element of the group containing the subgroup. In maths, if G is a group , H is a subgroup of G, and g is an element of G.Then $gH = \{gh : hanelementofH\}$ is the left coset of H in G with respect to g, and $Hg = \{hg : hanelementofH\}$ is the right coset of H in G with respect to g.

2.4.2 Theorem

If V⊆[0,1] is a Vitali set, then V is not Lebesgue measurable.

Proof: Suppose Not. Let C=$\mathbb{Q}$ ∩ [-1,1] and let U=$\biguplus_{q\in C}$(q+V). Then U is a countable set of measurable sets. We know that [0,1] ⊆ U ⊆ [-1,-2]

$\therefore$ 1 ≤ m(U) ≤ 3 But m(U)=m($\biguplus_{q\in C}$ (q+V))=$\sum_{q\in C}$m (q+V)=$\sum_{q\in C}$m(V)

If m(V)=0, it will let m(U)=0, if m(V)>0, then m(U)=∞ $\therefore$ contradiction.

2.4.3 Lemma and Proof

Lemma 1 Let V ⊆[0,1] be a Vatali set, then the sets $\{q + V | q \in \mathbb{Q}\}$ are pairwise disjoint, and $\mathbb{R} = \biguplus_{q\in Q}(q + V)$

*proof*1 Suppose first that x

∈(q+V) ∩ (q'+V) for some q, q'∈ $\mathbb{Q}$. Then x=q+v and x'=q'+v' for some v, v'∈V. Then v=x+(-q) and v'=x+(-q'), so v and v' both lie in x+$\mathbb{Q}$. But V has only one point from each coset of $\mathbb{Q}$, so we conclude that v=v', and hence q=q'. This proves that the sets $\{q + V | q \in \mathbb{Q}\}$ are pairwise disjoint.

Next, observe that for any x∈ $\mathbb{R}$, there exists a point v∈V so that v∈x+$\mathbb{Q}$. Then v=x+q for some $q \in \mathbb{Q}$, so x=v+(-q), and hence $x \in v+(-q)$. It follows that $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(q + V)$

Lemma 2: Let V⊆[0,1] be a Vatali set, let $C = \mathbb{Q} \cap [-1, 1]$, and let $U = \biguplus_{q\in C}$ (q+V) Then [0,1] ⊆ U ⊆ [-1,2]

Proof 2: First, since V⊆[0,1], we know that q+V⊆ [-1,2] for all q∈[-1,1] and hence U⊆[-1,2]. To prove that [0,1]⊆U, let x∈[0,1]. Since V is a Vatali set, there exists a v∈V so that v∈x+Q. Then v=x+q for some $q \in \mathbb{Q}$. But v and x both lie in [0,1], so it follows that q=v-x lies in the interval [-1,1]. Thus $q \in C$ and $x \in q + V$, which proves that $x \in U$.

Proposition If V is a Vatali set then m_*(V)=0 and m*(V)>0

Proof: Let V be a Vatali set, let C=$\mathbb{Q} \cap$ [-1,1], let U=$\biguplus_{q \in C}(q+V)$.

Because [0,1] $\subseteq$ U $\subseteq$ [-1,2], so $1 \leq m_*(U) \leq m^*(U) \leq 3$.

But $m^*(U) \leq \sum_{q \in C} m^*(q+V) = \sum_{q \in C} m^*(V)$ and it follows that m*(V)>0.

As for the inner measure, recall that m_* is countable super-additive, i.e. $m_*(\biguplus_{n \in N} S_n) \geq \sum_{n \in N} m_*(S_n)$ for any sequence S_n of disjoint subjects of $\mathbb{R}$. It follows that $m_*(U) \geq \sum_{q \in C} m_*(q+V) = \sum_{q \in C} m_*(V)$, and hence m_*(V)=0

3 Main Proof

3.1 Paradoxical actions of group

Before we start to proof Banach-Tarski paradox, we need to know some basic concepts about groups. We need to know paradoxical actions of group, in order to better understand the proof of Banach-Tarski Paradox.

Definition 3.1.1. A **pairwise disjoint** is a collection of subsets of X is pairwise disjoint if no two sets share an element, i.e., their intersection is the empty set. For example $A = \{1, 2, 5\}$ and $B = \{3, 4, 6\}$ with $a, b \subset \mathbb{N}$
A, B is a pairwise disjoint subsets.

Definition 3.1.2. A group G is **free** if there exists a set F of generator such that every element of G can be expressed uniquely as a product of finitely many elements of F and their inverse (disregarding trivial variations)

Definition 3.1.3 Let a group G act on a set S. The action is **free**, if for all $g \in G \setminus \{e\}$ and for all $x \in S$, we have $gx \neq x$.

Definition 3.1.4. Let G be a group acting on a set Z. A subset $S \subset Z$ is **paradoxical** if there exist pairwise disjoint subsets $A_1, A_1, ...A_n$ and $B_1, B_2, ..., B_m \subset S$ and there exist $j_1, j_2, ..., j_n, k_1, k_2, ..., k_m$ in G such that

$$Z = \bigsqcup_{i=1}^{n} j_i A_i = \bigsqcup_{r=1}^{m} k_r B_r$$

If a set Z is paradoxical, then an action of a group G on the set Z is paradoxical. If the action of the group on itself by left multiplication is paradoxical.

Then we can say the group is paradoxical.

Definition 3.1.5. A **reduced word** is the word that contains no redundant pairs. If generators in a word is next to its inverse (xx^-1), it can be simplified by omitting the redundant pair:

$$xyxx^{-1} \longrightarrow xy$$

Paradoxical decomposition of group

Hausdorff Paradox is the famous example and the starting point of discovering paradoxical decomposition of group. Hausdorff demonstrated that a unit sphere can be divided into finitely many pieces, and then by some translations and rotations, we can get two unit spheres. We will talk more about Hausdorff in the following section.

Simple example: $\mathbb{F}_2$ is a free group on two free operators $a, b \in \mathbb{F}_2$. $W(a)$ is the set of all reduced words that start with a. Then the group can be decomposed into pairwise disjoint sets, such as:

$$\mathbb{F}_2 = \{e\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}).$$

$\{e\}$ is the identity element. $W(x)$ is the set of all reduced words that start with a; $W(a^{-1})$ is the set of all reduced words that start with a^{-1}; $W(b)$ is the set of all reduced words that start with b; $W(b^{-1})$ is the set of all reduced words that start with b^{-1}.
We claim $aW(a^{-1}) = W(a)$, then we have a paradoxical decomposition:

$$\mathbb{F}_2 = W(a) \sqcup aW(a^{-1}) = W(b) \sqcup bW(b^{-1})$$

Theorem 3.1.7. A group G is paradoxical if and only if there exists a free action on set X which is paradoxical.
Claim: G is paradoxical $\Longrightarrow X$ is paradoxical
Proof. We assume a G acts freely on X, there exists a pairwise disjoint subsets $A_1, A_2, ..., A_i$ and $B_1, B_2, ...B_j$ of G and there exist $g_1, g_2, ..., g_i \in G$ and $h_1, h_2, ..., h_j \in G$. Then, take $G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j$ be a paradoxical decomposition of G. By Axiom of Choice, we can select exactly one element from each orbit in $G \times X \to X$ to form a subset, denoted as $M \subset X$. Then,

$\bigcup_{g \in G} gM = X$, because every point in M can be transformed by $g \in G$ to all the points in X. Also, $\bigcup_{g \in G} gM$ is a disjoint partition of X. For some $g, h \in G$ and $x, y \in M$, we could have $gx = hy$. Because $h^{-1}gx = y$, and there exist $k \in G$ satisfies $kx = y$, then x and y are in the same orbit, and $x = y$. We know that the group action $G \times X \to X$ is a free action, therefore $g = h$. Finally, we can define:

$$\hat{A}_i = \bigsqcup_{g \in A_i} gM \ and \ \hat{B}_j = \bigsqcup_{g \in B_j} gM.$$

Therefore, X is paradoxical

□

Claim: X is paradoxical $\Longrightarrow G$ is paradoxical.

Proof. Consider an orbit O of a point $x \in X$. First, define an opera tion such that $gx * hx = (gh) * x$ be homomorphic. Since the action on O is a free action, it is bijective. Depending on the bijection, we can get

$$= X \bigcup_{i=1}^{n} g_i \hat{A}_i = \bigcup_{j=1}^{m} h_j \hat{B}_j$$

$$X = \bigcup_{i=1}^{n} g_i(A_i x) = \bigcup_{j=1}^{m} h_j(B_j x)$$

$$X = \bigcup_{i=1}^{n} (g_i A_i)x = \bigcup_{j=1}^{m} (h_j B_j)x$$

$$G = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j$$

Thus, paradoxical decomposition of O in the set Ximplies paradoxical decomposition of G.

□

3.2 Hausdorff paradox

Theorem 3.2.1

There are two rotations in $SO(3)$ which generate the free group on two generators.

Proof. There are many ways of getting a pair of independent rotations of $\mathbb{S}^2$.In this paper,we use matrices to define a pair of independent rotation explicitly

$$A_\sigma = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

$$A_\sigma^- = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A_\gamma^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

These are rotations by angle across $(\frac{1}{3})$ around z-axis and x-axis. The idea is in three-dimensional Euclidean space, we can rotate objects in four different directions. Each of these four matrices represents one direction respectively; therefore,we form $A_\sigma^\pm A_\gamma^\pm$. To form these four matrices we also need the determinant of each matrix equals to 1 or -1. To show that:

$$\det(A_\sigma) = \det(A_\sigma^-) = \det(A_\gamma) = \det(A_\gamma^-) = 1 \cdot (\frac{1}{3} \cdot \frac{1}{3}) - (-\frac{2\sqrt{2}}{3} \cdot \frac{2\sqrt{2}}{3}) = \frac{1}{9} + \frac{8}{9} = 1$$

In order to show four matrices are representing four different directions and each direction can not be overlapped others. We need to show they are orthogonal.
These matrices are orthogonal,by the definition of the orthogonal matrix,we need to show $A_\sigma = A_\sigma^- = A_\sigma{}^{-1}$, $A_\gamma = A_\gamma^- = A_\gamma{}^{-1}$
Obviously,$A_\sigma = A_\sigma^- = A_\sigma{}^T$, $A_\gamma = A_\gamma^- = A_\gamma{}^T$
We just need to prove $A_\sigma \cdot A_\sigma^- = I$,$A_\gamma \cdot A_\gamma^- = I$ (Here I is the identity matrix)

$$\begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We have showed that these matrices are orthogonal,which means $A_\sigma^- = A_\sigma{}^{-1}$, $A_\gamma^- = A_\gamma{}^{-1}$. This implies A_σ and A_σ^- are pairwise opposite direction, A_γ and

A_γ^- are pairwise opposite directions.

Next,we wish to show that there does not exist non-trivial reduced word in $\sigma^\pm, \gamma^\pm$ equals the identity. Assume τ be a non-trivial word in the free group on two generators. We want to show $q(\tau)$is a non-trivial rotation, where q is a homomorphism of $\mathbb{F}_2$ (dense of $SO(3)\times SO(3)$) that sends generators to A_σ and A_γ. In order to simplify notations of non-trivial rotation, we denote $q(\tau)$ again by τ. Conjugating τ by A_γ we may assume that τ ends by $A_\gamma^{\pm 1}$ on the right. To prove the theorem, it is suffice to prove that

$$\tau \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}$$

$\Rightarrow a, b, c$ are integers,b is not divisible by 3 and k is the length of τ

In this proof, we will start using induction on k. If $|\tau| = 1$, then $\tau = A_\sigma^{\pm 1}$ and we start induction. When $k = 1$

$$L.H.S = \tau \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \pm\frac{2\sqrt{2}}{3} \\ 0 \end{pmatrix}$$

$$R.H.S = \frac{1}{3^k} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix}, when \quad k = 1, a = 1, b = 2, c = 0$$

$\because$ L.H.S=R.H.S
$\therefore$ Proved when k = 1

Now let τ be equal to $A_\sigma^{\pm 1}\tau'$ or $A_\gamma^{\pm 1}\tau'$,assume k - 1 satisfies:

$$\tau' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

For a', b', c' are integers and b' is not divisible by 3.
Next we want to prove $\frac{1}{3^k}$ also satisfied the induction.
Assume τ=$A_\sigma^{\pm 1}\tau'$:

$$L.H.S = \tau \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}$$

$$R.H.S = \frac{1}{3}\begin{pmatrix} 1 & \mp 2\sqrt{2} & 0 \\ \pm 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \frac{1}{3^{k-1}}\begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix} = \frac{1}{3^k}\begin{pmatrix} a' \mp 4b' \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$

$\because$ L.H.S=R.H.S
$\therefore$

$$\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix} = \begin{pmatrix} a' \mp 4b' \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix}$$

Assume $\tau{=}A_\gamma^{\pm 1}\tau'$

$$L.H.S = \tau\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}$$

$$R.H.S = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & \mp 2\sqrt{2} \\ 0 & \pm 2\sqrt{2} & 1 \end{pmatrix} \cdot \frac{1}{3^{k-1}}\begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix} = \frac{1}{3^k}\begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}$$

$\because$ L.H.S=R.H.S
$\therefore$

$$\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix} = \begin{pmatrix} 3a' \\ (b' \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}$$

By induction,we can get answers of a, b, c

$$a = a' \mp 4b', b = b' \pm 2a', c = 3c' \quad or \quad a = 3a', b = b' \mp 2c', c = c' \pm 4b'$$

and for all a,b,c are integers. From these equations we can not show b is divisible by 3 or not. (We want to show that b is not divisible by 3.) Therefore,we have four cases to consider. Assume τ can be written as $A_\sigma^{\pm 1}A_\gamma^{\pm 1}\tau''$, $A_\sigma^{\pm 1}A_\sigma^{\pm 1}\tau''$, $A_\gamma^{\pm 1}A_\sigma^{\pm 1}\tau''$, $A_\gamma^{\pm 1}A_\gamma^{\pm 1}\tau''$.For τ'' is some possibly empty word. Assume τ'' satisfies the following:

$$\tau''\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-2}}\begin{pmatrix} a'' \\ \sqrt{2}b'' \\ c'' \end{pmatrix}$$

where $a'', b'', c'' \in \mathbb{Z}$ and b'' is not divisible by 3, and $k-2$ is the length of the τ''.

1.When τ=$A_{\sigma}^{\pm 1}A_{\gamma}^{\pm 1}\tau''$.
Using what we have calculated before, we can get $a' = 3a'' \Rightarrow b = b' \pm 6a'' \Rightarrow$ a' is divisible by 3. By assumption, we assume b' is not divisible by 3, b has an element(b') that is not divisible by 3. Therefore, b is not divisible by 3.

2.When τ=$A_{\sigma}^{\pm 1}A_{\sigma}^{\pm 1}\tau''$.
In this case, last two steps are $A_{\sigma}^{\pm 1}$,which implies $b' = b'' \pm 2a'', a' = a'' \mp 4b''$; therefore, $b = b' \pm 2a' = (b'' \pm 2a'') \pm 2(a'' \mp 4b'') = b'' \pm 4a'' - 8b'' = b'' + b'' \pm 4a'' - 8b'' - b'' = (2b'' \pm 4a'') - 9b'' = 2b' - 9b''$. Obviously, $9b''$ is divisible by 3 , by assumption we assume b' is not divisible by 3.Therefore $b = 2b' - 9b''$ has an element($2b'$) that is not divisible by 3, b is not divisible by 3.

3. When τ=$A_{\gamma}^{\pm 1}A_{\sigma}^{\pm 1}\tau''$.
In this case, last two steps are $A_{\gamma}^{\pm 1}A_{\sigma}^{\pm 1}$, which implies $b = b' \mp 2c', c' = 3c''$. Obviously, $b = b' \mp 6c''$,$6c''$ is divisible by 3. But by assumption, b' is not divisible by 3, b has an element(b') that is not divisible by 3. Therefore, b is not divisible by 3.

4. When τ=$A_{\gamma}^{\pm 1}A_{\gamma}^{\pm 1}\tau''$
In this case, last two steps are $A_{\gamma}^{\pm 1}A_{\gamma}^{\pm 1}$, which implies $b' = b'' \mp 2c'', b = b' \mp 2c', c' = c'' \pm 4b''$; therefore, $b = b' \mp 2c' = (b'' \mp 2c'') \mp 2(c'' \pm 4b'') = b'' \mp 4c'' - 8b'' = b'' + b'' \mp 4c''(8b'' + b'') = (2b'' \mp 4c'') - 9b'' = 2b' - 9b''$. Obviously, $-9b''$ is divisible by 3.By assumption, b' is not divisible by 3. Therefore there is an element in $b(2b')$ that is not divisible by 3, b is not divisible by 3.
□

Theorem 3.2.2 (Hausdorff paradox). There exists a countable subset in a sphere S^2 such that its complement in S^2 is $SO(3)$-paradoxical.

Proof. Suppose we create a free group of $SO(3)$. According to Theorem 2.1, each element in that free group is a rotation except the identity element (e), let N be the set of all points in S^2, which are fixed by some elements in this group. N is a countable set. By definition, all elements in $S^2 \backslash N$ are not fixed under the action, which means it is invariant under the action of free group;therefore,there exists a free group action on $S^2 \backslash N$. By Theorem 1.2 since all free group with two generators are paradoxical, the set $S^2 \backslash N$ is paradoxical. □

3.3 Banach-Tarski Paradox

The original Banach-Tarski Paradox amounts to a decomposition of a unit ball into finitely many pieces, rearranging these pieces into two unit balls. Since, the group of rotations preserves the origin, this would not be sufficient to achieve traditional Banach-Tarski Paradox.

Definition 3.3.1 Let G be a group acting on a set X. Two subsets $A, B \subset X$ are **equidecomposable**, if there exists a pairwise disjoint subsets $A_1, A_2, ...A_n \subset A$ and $B_1, B_2, ..., B_n \subset B$ and $g_1, g_2, ..., g_n \in G$ such that:

$$A = \bigsqcup_{i=1}^{n} A_i \ and \ B = \bigsqcup_{j=1}^{n} B_j$$

and $g_i(A_i) = B_i$ for all $1 \leq i \leq n$.
In order to check that equidecomposability is an equivalence relation, we need to check three parts: for any group G acting on set A,B,C are arbitrary subsets of X such that:
1.A is equidecomposable to A.
2. if A is equidecomposable to B, then B is equidecomposable to A.
3. if A is equidecomposable to B, and B is equidecomposable to C, then, A is equidecomposable to C.
$Proof1$: Because every set is a subset of itself and every group has an identity. A is a subset of A, there exists $e \in G$ which is the identity element of G, then we have $A = eA$. Therefore, $A \sim A$.
$Proof2$: Let $A = A_1 \sqcup A_2 \sqcup ... \sqcup A_n$ and $B = B_1 \sqcup B_2 \sqcup ... \sqcup B_n$ and $g_1, g_2, ..., g_n \in G$ such that $g_k(A_k) = B_k$ for all $k \in \mathbb{N}$ and $k \leq n$. There exists a unique inverse element $g_k^{-1} \in G$. Then, we have $g_k^{-1}(B_k) = A_k$. Hence, $A \sim B \Longrightarrow B \sim A$.
$Proof3$: Let $A = A_1 \sqcup A_2 \sqcup ... \sqcup A_n$ and $B = B_1 \sqcup B_2 \sqcup ... \sqcup B_n$ and $g_1, g_2, ...g_n \in G$ such that $g_k(A_k) = B_k$, for all $k \in \mathbb{N}$ and $k \leq n$. Let $B = P_1 \sqcup P_2 \sqcup ... \sqcup P_m$ and $C = Q_1 \sqcup Q_2 \sqcup ... \sqcup Q_m$ and $h_1, h_2, ..., h_m \in G$ such that $h_i(P_i) = Q_i$ for all $i \in \mathbb{N}$ and $i \leq m$. Let $H_{ki} = B_k \cap Q_i$. Since $\bigsqcup_{k=1}^{n} B_k$ and $\bigsqcup_{i=1}^{m} Q_i$ are both partition of B, $\bigsqcup_{k=1}^{n}\bigsqcup_{i=1}^{m} H_{ki}$ is a partition of B. Employing the same method, we can partition A into $\bigsqcup_{k=1}^{n}\bigsqcup_{i=1}^{m} A_k$ and C into $\bigsqcup_{k=1}^{n}\bigsqcup_{i=1}^{m} C_{ki}$. Then, for some $g, h \in G$, we can have $A_{ki} = g(B_{ki})$ and $B_{ki} = h(C_{ki})$. By employing the closure property of group, for some $h \in G$, we can get

$\bigsqcup_{k=1}^{n}\bigsqcup_{i=1}^{m} A_{ki} = \bigsqcup_{k=1}^{n}\bigsqcup_{i=1}^{m} h(C_{ki})$. Hence, we have $A \sim C$.

Proposition 3.3.2 Let G be a group that acts on S. If X is paradoxical, and X is equidecomposable to Y, then Y is paradoxical.
Proof : Let $X = \bigcup_{k=1}^{n} K_i(A_i) = \bigcup_{j=1}^{m} j_i(B_i)$. According to the definition each A_i and A_j are disjoint, but $k_i(A_i)$ and $h_j(A_j)$ are not necessarily disjoint. Therefore, define $\bigcup_{i=1}^{n} k_i(M_i) = \bigcup_{i=1}^{n} k_i(A_i)$, except when $k_i(A_i) \cap k_j(A_j) \neq \emptyset$ for each $i < j$, define $A_i = C_i \sqcup D_i$ and $A_j = C_i \sqcup D_j$ such that $k_i(C_i) = k_j(C_i)$ and $k_i(D_i) \cap k_j(D_j) = \emptyset$, we remove $k_i(C_i)$ from $\bigcup_{i=1}^{n} k_i(M_i)$. Inductively, each repeated elements and only the repeated elements are discarded from $\bigcup_{i=1}^{n} k_i(M_i)$. Thus, we constructed a partition of X, $X = \bigsqcup_{i=1}^{n} k_i(M_i)$. Similarly, based on $X = \bigcup_{i=1}^{m} h_i(B_i)$, we can construct another partition, $X = \bigsqcup_{i=1}^{m} h_i(N_i)$. Since for each k_i and h_i, there exist k_i^{-1} and h_i^{-1} in G. Therefore, $\bigsqcup_{i=1}^{n} M_i$ and $\bigsqcup_{i=1}^{m} N_i$ are both equidecomposable to X. According to the transitive property of equidecomposability, Y is equidecomposable to both $\bigsqcup_{i=1}^{n} M_i$ and $\bigsqcup_{i=1}^{m} N_i$. Thus, X is paradoxical.

□

Proposition 3.3.3 Let D be a countable subset of S^2. Then S^2 and $S^2 \backslash D$ are $SO(3)$ equidecomposable.
Proof: Since D is a countable subset, S^2 is uncountable, there exists a axis L in S^2 that does not intersect D. Define A as the collection of α. There exists a point p in D, such that, if we rotate p by angle of $n\alpha$ where $n \in \mathbb{N}$, p is still in D. A is countable because D is countable. Thus, there exists an angle θ in $[0, 2\pi)$ such that $\theta \notin A$. Define $r_\theta(x)$ to be the image of rotating the point x around L by θ degree. Thus, we have $r_\theta^0(D) = D$ and for all $h, k \in \mathbb{N}$ we have $r_\theta^h(D)$ intersected with $r_\theta^k(D) =$ is the empty set. Define $D' = \bigsqcup_{i=0}^{\infty} r_\theta^i(D)$.

According to the above, we can get $S^2 = (S^2 \backslash D') \cup D'$. Rotating D' one time by an angle of θ, then, we have $r_\theta(D') = D' \backslash r_\theta^0(D) = D' \backslash D$.

Thus, $S^2\backslash D = (S^2\backslash D')\cup(D'\backslash D) = (S^2\backslash D')\cup(r_\theta(D'))$. By the definition of a group, there is an identity element e in $SO(3)$, so we have $e(S^2\backslash D') = S^2\backslash D'$ and$r_\theta(D') = r_\theta(D')$ such that $r_\theta \in SO(3)$. Therefore, $S^2 \sim S^2\backslash D$ under action of $SO(3)$.

□

Proposition 3.3.4 S^2 is $SO(3)$ paradoxical, e.g S^2 is paradoxical under the action of $SO(3)$ because there exists $S^2\backslash D$ paradoxical and equidecomposable to S^2.

Next, we are ready to proof the actual Banach-Tarski Paradox in a ball.

Theorem 3.3 c t.5 Every ball in $\mathbb{R}^3$ can be paradoxical decomposed by rotations and translations.
Proof : Let Q_r donotes the ball with radius of r which is the length in $\mathbb{R}^3$. Since S^2 is paradoxical under the action of $SO(3)$, there exist pairwise disjoint subsets, $A_1, A_2, ..., A_n$ in S^2, and pairwise disjoint subsets, $B_1, B_2, ..., B_m$ in S^2, and there exist $h_1, h_2, ..., h_n, k_1, k_2, ..., k_m$ in $SO(3)$ such that $S^2 = \bigcup_{i=1}^{n} h_i(A_i) = \bigcup_{i=1}^{m} k_i(B_i)$. Define $\hat{A}_i = \{dp : d \in (0, r], p \in A_i\}$ and $\hat{B}_i = \{dp : d \in (0, r], p \in B_i\}$. Thus, $\hat{A}_1, \hat{A}_2, ..., \hat{A}_n \subset Q_r$ are pairwise disjoint and $\hat{B}_1, \hat{B}_2, ..., \hat{B}_m \subset Q_r$ are pairwise disjoint. Correspondingly, we have $Q_r\backslash O = \bigcup_{i=1}^{n} h_i(\hat{A}_i) = \bigcup_{i=1}^{m} k_i(\hat{B}_i)$, where $O = (0, 0, 0)$ is the origin in $\mathbb{R}^3$. Therefore, $Q_r\backslash O$ is paradoxical. In order to prove the theorem, we need to show $Q_r\backslash O \sim Q_r$.

Assume the Q_r is a unit ball with radius of 1. Let L be a line through the point $(0, 0, \frac{1}{2})$ which does not intersect the origin O. Since the distance between the origin and the line L is $\leq \frac{1}{2}$, the image of all rotation of O is within Q_r. Let $R^n(O)$ denotes the image of rotating the origin by an angle with a rational radian measure n times around L. We have $R^0(O) = O$ and $R^n(O) \neq O$ for all $n \in \mathbb{N}$. Define $D = \bigcup_{i=0}^{\infty} R^i(O)$, we have $R(D) = D\backslash R^0(O) = D\backslash O$.

The same as Proposition 2.3, we have $Q_r = (B_r\backslash D)\cup D$ and $Q_r\backslash O = (Q_r\backslash D) \cup (D\backslash O) = (B_r\backslash D)\backslash R(D)$. There exist $e, R \in SO(3)$, so $Q_r \sim Q_r\backslash O$. Since $Q_r\backslash O$ is paradoxical under the action of $SO(3)$, according to the proposition 2.2, we have Q_r is paradoxical. Therefore, all balls in $\mathbb{R}^3$ are paradoxical.

□

4 Conclusion

We have mentioned that Banach-Tarski Paradox is worked in case of a 3-D ball, but what about in 2-Dimension? 1-Dimension? The answer is no. According to Stan Wagon, if we separate a 2-D plane into pieces and try to apply Banach-Tarski paradox on it, we will see that Banach-Tarski paradox does not work because the pieces does not contain a free non-commutative group. Due to the same reason, Banach-Tarski paradox also does not work in a line, which is 1-D.

The axiom of choice seems right when we see the definition, if we have some boxes with balls inside, we can definitely pick a ball from each box. Banach-Tarski paradox seems wrong when we see the definition, how can we use the same materials to make one object becomes two? Based on the axiom of choice, group theory, Stan Wagon's "The Banach-Tarski Paradox", Kate Juschenko's material we successfully provided a proof for Banach-Tarski paradox in 3-dimensions. Learning from history, there are many "weird" mathematical findings that at first seems ridiculous but finally useful for the society. This lead us to a thinking that is there any properties of matter that humans do not find yet but allow the paradox becomes a theory? We left this question to our readers.