SIZE OF PROJECTION OF VECTOR SPACE OVER $\mathbb{Z}_p^d$

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Abstract. The goal of the paper is to find the size of projection of vector space over $\mathbb{Z}_p^d$ by using the similar proof of Marstrand's projection theorem for one-dimensional projections.

1. Introduction

We discuss a special case of Marstrand's projection theorem in this paper. Let e be a unit vector in $\mathbb{R}^n$ and E $\subset \mathbb{R}^n$ a compact set. The projection $P_e(E)$ is the set $\{x \cdot e : x \in E\}$. We want to relate the Hausdorff dimensions of E and of its projections.

2. Marstrand's projection theorem

Definition 2.1. Let e be a unit vector in $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ a compact set. The projection $P_e(E)$ is the set $\{x \cdot e : x \in E\}$.

Definition 2.2. Fix $\alpha > 0$, and let E $\subset \mathbb{R}^n$. For $\epsilon > 0$, one defines
$H^\epsilon_\alpha(E)$=inf $(\sum_{j=1}^\infty r_j^\alpha)$, where the infimum is taken over all countable coverings of E by discs $D(x_j, r_j)$ with $r_j < \epsilon$.
It is clear that $H^\epsilon_\alpha(E)$ increases as ϵ decreases, and we define $H_\alpha(E) = \lim_{\epsilon\to 0} H^\epsilon_\alpha(E)$.
It is also clear that $H^\epsilon_\alpha(E) \leq H^\epsilon_\beta(E)$ if $\alpha > \beta$ and $\epsilon \leq 1$, thus $H_\alpha(E)$ is a nonincreasing function of α.

Remark 2.1. If $H^1_\alpha(E) = 0$, then $H_\alpha(E) = 0$. This follows readily from the definition, since a covering showing that $H^1_\alpha(E) < \delta$ will necessarily consist of discs of radius of radius $< \delta^{\frac{1}{\alpha}}$.

Remark 2.2. It is also clear that $H_\alpha(E) = 0$ for all E if $\alpha > n$, since one can then cover $\mathbb{R}^n$ by discs $D(x_j, r_j)$ with $\sum_j {r_j}^\alpha$ arbitrarily small.

Lemma 2.1. *There is a unique number α_0, called the Hausdorff dimension of E or dim E, such that $H_\alpha(E) = \infty$ if $\alpha < \alpha_0$ and $H_\alpha(E) = 0$ if $\alpha > \alpha_0$.*

Proof. Define α_0 to be the supremum of all α such that $H_\alpha(E) = \infty$. Since $H_\alpha(E)$ is a nonincreasing function of α, $H_\alpha(E) = \infty$ if $\alpha < \alpha_0$. Suppose $\alpha > \alpha_0$. Let $\beta \in (\alpha_0, \alpha)$. Define $M = 1 + H_\beta(E) < \infty$. If $\epsilon > 0$, then we have a covering by discs with
$\sum_j {r_j}^\alpha \leq \epsilon^{\alpha-\beta} \sum_j {r_j}^\beta \leq \epsilon^{\alpha-\beta} M$ which goes to 0 as $\epsilon \to 0$. Thus $H_\alpha(E) = 0$ □

Definition 2.3. L^1 Fourier transform
If $f \in L^1(\mathbb{R}^n)$, then its Fourier transform is $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int e^{-2\pi i x\xi} f(x)dx$$

More generally, let $M(\mathbb{R}^n)$ be the space of finite complex-valued measure on $\mathbb{R}^n$ with the norm

$$\| \mu \| = |\mu|\mathbb{R}^n,$$

where $|\mu|$ is the total variation. Thus $L^1(\mathbb{R}^n)$ is contained in $M(\mathbb{R}^n)$ via the identification $f \to \mu, d\mu = fdx$. We generalize the definition of Fourier transformation via

$$\hat{\mu}(\xi) = \int e^{-2\pi i x\xi} d\mu(x)$$

Definition 2.4. Define the $\alpha -$ *dimensional* energy of a (positive) measure μ with compact support by the formula

$$I_\alpha(\mu) = \iint |x-y|^{-\alpha}\, d\mu(x)\, d\mu(y)$$

We always assume that $0 < \alpha < n$.

Theorem 2.2. *If E is compact then the Hausdorff dimension of E coincides with the number*

$$sup\{\alpha : \exists \mu \in P(E) with I_\alpha(\mu) < \infty\}.$$

Proof. Denote the above supremum by s. If $\beta < s$, then E supports a measure with $\mu(D(x,r)) \leq Cr^\beta$. Then $H_\beta(E) > 0$, so $\beta \leq dimE$. So $s \leq dimE$. Conversely, if $\beta < dimE$, then E supports a measure with $\mu(D(x,r)) \leq Cr^{\beta+\epsilon}$ for $\epsilon > 0$ small enough. Then $I_\beta(\mu) < \infty$, so $\beta \leq s$, which shows that $dimE \leq s$. □

Theorem 2.3. *Let μ be a positive measure with compact support and $0 < \alpha < n$. Then*

$$\iint |x-y|^{-\alpha}\, d\mu(x)\, d\mu(y) = c_\alpha \int |\hat{\mu}(\xi)|^2 |\xi|^{-(n-\alpha)} d\xi,$$

where $c_\alpha = \frac{\gamma(\frac{n-a}{2})\pi^{a-\frac{n}{2}}}{\gamma(\frac{a}{2})}$.

Theorem 2.4. *Let μ be a positive measure with compact support and $0 < \alpha < n$. Then*

$$\iint |x-y|^{-\alpha}\, d\mu(x)\, d\mu(y) = c_\alpha \int |\hat{\mu}(\xi)|^2 |\xi|^{-(n-\alpha)} d\xi,$$

where $c_\alpha = \frac{\gamma(\frac{n-a}{2})\pi^{a-\frac{n}{2}}}{\gamma(\frac{a}{2})}$.

Proof. Suppose first that $f \in L^1$ is real and even, and that $d\mu(x) = \phi(x)dx$ with $\phi \in \mathbf{S}$ Then we have

$$\int f(x-y)\, d\mu(x)\, d\mu(y) = \int |\hat{\mu}(\xi)|^2 \hat{f}(\xi) d\xi$$

Now fix ϕ. Then both sides of the equation are seen to define continuous linear map from f $\in L^2$ to $\mathbb{R}$. Accordingly, the equation remains valid when $f \in L^1 + L^2$, $\phi \in \mathbf{S}$. We conclude if $d\mu(x) = \phi(x)dx$, $\phi \in \mathbf{S}$. □

Theorem 2.5. *Marstrand's projection theorem for one-dimensional projections Assume that $E \subset \mathbb{R}^n$ is compact and $dimE = \alpha$. Then*
(i)$\alpha \leq 1$ then for a.e. $e \in S^{n-1}$ we have $dimP_eE = \alpha$
(ii)$\alpha > 1$ then for a.e. $e \in S^{n-1}$ the projection P_eE has positive one-dimensional Lebesgue measure.

Proof. If μ is a measure supported on E, $e \in S^{n-1}$, then the projected measure μ_e is the measure on $\mathbb{R}$ defined by

$$\int f\, d\mu_e = \int f(x \cdot e)\, d\mu(x)$$

for continuous f. Notice that $\hat{\mu}_e$ may be readily be calculated from the is definition:

$$\hat{\mu}_e(k) = \int e^{-2\pi i k x \cdot e}\, d\mu(x)$$
$$= \hat{\mu}(ke).$$

Let $\alpha < dimE$ and let μ be a measure supported on R with $I_\alpha(\mu) < \infty$.We have then

$$\int |\hat{\mu}(ke)|^2 |k|^{-1+\alpha}\, dk d\sigma(e) < \infty$$

by Theorem 2.4 and and polar coordinates.
Thus, for a.e. e we have

$$\int |\hat{\mu}(ke)|^2 |k|^{-1+\alpha}\, dk < \infty(1)$$

It follows by Theorem 2.4 with $n = 1$ that for a.e. e the projected measure μ_e has finite α-dimensional energy. This and Theorem 2.3 give part (i), since μ_e is supported on the projected set $P_e E$. For part (ii), we note that if $dimE > 1$ we can take $\alpha = 1$ in (1). Thus $\hat{\mu}_e$ is in L^2 for almost all e. This condition implies that μ_e has an L^2 density, and in particular is absolutely continuous with respect to Lebesgue measure. Accordingly $P_e E$ must have positive Lebesgue measure. □

Remark 2.3. $dimP_e E \leq dimE$, this follows from the definition of dimension and the fact that the projection P_e is a Lipschitz function.

Remark 2.4. Theorem 2.3 has a natural generalization to k-dimensional instead of 1-dimensional projections, which is proved in the same way.

3. PRELIMINARIES

Definition 3.1. Given a function $f : \mathbb{Z}_p^2 \to \mathbb{C}$, its Fourier transformation is defined by

$$\hat{f}(m) = p^{-2} \sum_{x \in \mathbb{Z}_p^2} \chi(-x \cdot m) f(x)$$

Theorem 3.1. *Cauchy Schwarz Inequality* $|\sum_{i=1}^n u_i \bar{v}_i|^2 \leq \sum_{j=1}^n |u_j|^2 \sum_{k=1}^n |v_k|^2$

4. $\mathbb{Z}_p$ CASE

Consider $\mathbb{Z}_p^2$, where $p \equiv 3 \mod 4$

Definition 4.1. Consider E $\subseteq \mathbb{Z}_p^2$, define the projection $P_v(E)$=$\{x \cdot v : x \in E\}$, where $v \in \mathbb{Z}_p^2$.

Definition 4.2. Define λ_v by $\sum_{t \in \mathbb{Z}_p} \lambda_v(t)\ f(t)$=$\sum_{x \in \mathbb{Z}_p^2} f(x \cdot v)\ E(x)$, where $E(x)$ is the characteristic function on E and $f : \mathbb{Z}_p \to \mathbb{R}$, $x \mapsto x$ is a constant map.

Theorem 4.1. *Let* $\lambda_v(t) = |\{x \in E : x \cdot v = t\}|$, *then it is equivalent to the* λ_v *defined in Definition 4.2.*

Proof. Suppose $\sum_{t\in\mathbb{Z}_p} \lambda_v(t)\ f(t) = \sum_{x\in\mathbb{Z}_p^2} f(x\cdot v)\ E(x)$,
since $\sum_{x\in\mathbb{Z}_p^2} f(x\cdot v)\ E(x) = \sum_{t\in\mathbb{Z}_p} f(t) \sum_{x\cdot v=t} E(x) = \sum_{t\in\mathbb{Z}_p} f(t)\ |\{x\in E : x\cdot v = t\}|$,
$\sum_{t\in\mathbb{Z}_p} f(t)\ \lambda_v(t) = \sum_{x\in\mathbb{Z}_p^2} f(x\cdot v)\ E(x) = \sum_{t\in\mathbb{Z}_p} f(t)\ |\{x\in E : x\cdot v = t\}|$
so $\lambda_v(t) = |\{x\in E : x\cdot v = t\}|$.

Conversely, suppose $\lambda_v(t) = |\{x\in E : x\cdot v = t\}|$,
$\sum_{x\in\mathbb{Z}_p^2} f(x\cdot v)\ E(x) = \sum_t \sum_{x\cdot v=t} f(t)E(x) = \sum_t f(t) \sum_{x\cdot v=t} E(x) = \sum_t f(t)|\{x\in E : x\cdot v = t\}|$
$= \sum_{t\in\mathbb{Z}_p} f(t)\lambda_v(t)$ □

Theorem 4.2. $\sum_{t\in\mathbb{Z}_p} \lambda_v^2(t) = p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2$

Proof. By Plancherel Theorem $\sum_{t\in\mathbb{Z}_p} \lambda_v^2(t) = p \sum_{s\in\mathbb{Z}_p} |\hat{\lambda}_v(s)|^2$,
Directly from definition 4.2 and Fourier transformation,

$$\hat{\lambda}_v(s) = p^{-1} \sum_{t\in\mathbb{Z}_p} \chi(-t\cdot s)\lambda_v(t) = \sum_{x\in\mathbb{Z}_p^2} p^{-1}\chi(-x\cdot sv)E(x)$$

$$= p\cdot p^{-2} \sum_{x\in\mathbb{Z}_p^2} \chi(-x\cdot sv)E(x) = p\hat{E}(sv)$$

Therefore, $\sum_{t\in\mathbb{Z}_p} \lambda_v^2(t) = p \sum_{s\in\mathbb{Z}_p} |\hat{\lambda}_v(s)|^2 = p \sum_{s\in\mathbb{Z}_p} |p\hat{E}(sv)|^2 = p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2$. □

Lemma 4.3. $|V||E|^2 \leq \sum_{v\in V} |P_v(E)|\cdot p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2$

Proof. By definition 4.2, $\sum \lambda_v(t) = \sum E(x) = |E|$.
By Cauchy Schwarz Inequality, $|E|^2 = (\sum_{t\in\mathbb{Z}_p} 1\cdot\lambda_v(t))^2 \leq |P_v(E)|\cdot \sum_{t\in\mathbb{Z}_p} \lambda_v^2(t)$,
By theorem 4.2, $\lambda_v^2(t) = p \sum_{t\in\mathbb{Z}_p} |\hat{\lambda} v(s)|^2 = p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2$,
so $|V||E|^2 \leq \sum_{v\in V} |P_v(E)|\cdot p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2$ □

Theorem 4.4. *Let V be set of all direction of $\mathbb{Z}_p^2$ and $S_1 = \{x\in\mathbb{Z}_p^2 : \parallel x \parallel = 1\}$ where $\parallel x \parallel = x_1^2 + x_2^2$. Let r be a non-square in $\mathbb{Z}_p*$, then $V = S_1 \cup S_r$*

Proof. Let ζ be the set of squares in $\mathbb{Z}_p*$. Claim: ζ is a group.
$a\in\zeta, b\in\zeta$ implies $a = t^2, b = v^2$, then $ab = t^2v^2 = (tv)^2 \in \zeta$
$a\in\zeta$, implies $a = t^2$. Since $a\cdot a^{-1} = 1, a^{-1} = \frac{1}{t^2} \in \zeta$
$1\in\zeta$ since 1 is a square of itself.
Let $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p\ x \mapsto ax$ where $a \notin \zeta$, Φ is an isomorphism.
Let $\Psi : \mathbb{Z}_p* \to \zeta\ x \mapsto x^2$, since $ker(\Psi) = \{-1, 1\}$, $\frac{\mathbb{Z}_p}{ker(\Psi)} \cong \zeta$ implies $|\zeta| = \frac{p-1}{2}$
So if we prove Φ(sq)=not sq, Φ(not sq)=sq, then we have done for the theorem.
Take $x\in S_1$, then $\parallel tx \parallel = t^2 \parallel x \parallel = t^2$.
Suppose $y\in\mathbb{Z}_p^2$ and $\parallel y \parallel$ is a square, then let $\parallel y \parallel = s^2$ for some s $\in\mathbb{Z}_p$, $s\neq 0$.
Let $x = \frac{y}{s}$, then $\parallel x \parallel = \frac{\|y\|}{s^2} = 1$ implies $y = sx$
Suppose $y\in\mathbb{Z}_p^2$ and $\parallel y \parallel$ is not a square. We want to write $y = tx$ for some $x\in S_r$.

Consider $\parallel \frac{y}{t} \parallel = \frac{\|y\|}{t^2}$. To make this equal to r, we must find t such that $\frac{\|y\|}{r} = t^2$. Since $\parallel y \parallel$ and r is not in ζ, $\frac{\|y\|}{r} \in \zeta$.
Therefore $V = S_1 \cup S_r$. □

Remark 4.1. We have the fact that $\sum_{y\in\mathbb{Z}_p^2} |\hat{E}(y)|^2 = p^{-2}|E|$

Theorem 4.5. $|P_v(E)| = p \cdot \dfrac{1}{1 + \frac{p^2}{(p+1)|E|} - \dfrac{1}{p+1}}$, *if* $|E| > p$

Proof. By Lemma 4.3,

$$\begin{aligned}
|V||E|^2 &\le \sum_{v\in V} |P_v(E)| \cdot p^3 \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2 \\
&\le \max_{v\in V} |P_v(E)| \cdot p^3 \sum_{v\in V} \sum_{s\in\mathbb{Z}_p} |\hat{E}(sv)|^2 \\
&\le \max_{v\in V} |P_v(E)| \cdot (p^3 \sum_{v\in V} \sum_{s\neq 0} |\hat{E}(sv)|^2 + p^3 \sum_{v\in V} \sum_{s=0} |\hat{E}(sv)|^2) \\
&\le \max_{v\in V} |P_v(E)| \cdot (2p^3 \sum_{x\in\mathbb{Z}_p^2} |\hat{E}(x)|^2 + p^3 \sum_{v\in V} |\hat{E}(\vec{0})|^2) \cdot p^{-4} \\
&\le \max_{v\in V} |P_v(E)| \cdot (2p^3 \cdot p^{-2}|E|^2 - 2p^3 \cdot p^{-4}|E|^2 + p^{-1}|V||E|^2)
\end{aligned}$$

$$2(p+1)|E|^2 \le \max_{v\in V} |P_v(E)| \cdot (2p|E| + 2(p+1)p^{-1}|E|^2 - 2p^{-1}|E|^2)$$

$$\begin{aligned}
\max_{v\in V} |P_v(E)| &\ge \frac{2(p+1)|E|^2}{2p|E| + 2(p+1)p^{-1}|E|^2 - 2p^{-1}|E|^2} \\
&\ge \frac{p \cdot 2(p+1)|E|^2}{2(p+1)|E| + 2p^2|E| - 2|E|^2} \\
&= p \cdot \frac{1}{1 + \frac{p^2}{(p+1)|E|} - \dfrac{1}{p+1}}
\end{aligned}$$

□

5. Reference

Thomas H. Wolf, "LECTURES IN HARMONIC ANALYSIS",pp.58-61

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