SHIFTED K-THEORETIC POIRIER-REUTENAUER ALGEBRA

YINUO ZHANG

1. Introduction

Poirier and Reutenauer defined a Hopf algebra on the $\mathbb{Z}$-span of all standard Young tableaux in [10], which is later studied in [4, 11]. The Robinson-Schensted-Knuth insertion was used to relate the bialgebra to Schur functions. Schur function is a class of symmetric functions that can be determined by the summation of all semistandard Young tableaux of shape λ. With the help of the PR-bialgebra, the Littlewood-Richardson rule is established, which gives an explicit description on the multiplication of arbitrary Schur functions. The generalization of this approach has been used to develop the Littlewood-Richardson rule for other classes of symmetric functions. In [9], a K-theoretic analogue is developed using Hecke insertion, providing a rule for multiplication of the stable Grothendieck polynomials. Similarly, in [6], a shifted analogue is developed, providing a rule for multiplication of P-Schur functions. We use a shifted Hecke insertion, introduced in [8], to develop a shifted K-theoretic version of the Poirier-Reutenauer algebra and an accompanying Littlewood-Richardson rule.

Section 2 deals with the weak K-Knuth equivalence and its relationship with the shifted Hecke insertion. It is simultaneously a shifted analogue of Hecke insertion [1] and a K-theoretic analogue of Sagan-Worley insertion in [12]. In section 3, we introduced a shifted K theoretic analogue of the Poirier-Reutenauer algebra which was first introduced in [10]. In section 4, we define the weak shifted stable Grothendieck polynomials. This is class of symmetric functions we are working with. Finally, section 5 introduces a Littlewood-Richardson rule of the weak shifted stable Grothendieck polynomials. The Littlewood-Richardson rule gives us an explicit description of the product structure of the weak shifted stable Grothendieck polynomials.

2. Weak K-Knuth Equivalence and Shifted Hecke Insertion

The main result for this section is that the weak K-Knuth equivalence of words is preserved by operating the shifted Hecke insertion. Before stating this result, we will review some previous work on the weak K-Knuth equivalence, increasing shifted tableaux and shifted Hecke insertion.

2.1. **Word and weak K-Knuth equivalence.**

An *alphabet* is defined to be a nonempty set of symbols. Given an alphabet, one can define the *word* to be a finite list of symbols chosen from the alphabet. For example, the English alphabet is a finite alphabet and "elephant" is a word of this alphabet. The set of all positive integers is a infinite alphabet and "162743" is a word of this alphabet.

Definition 2.1. The *weak K-Knuth equivalence relation* on the alphabet $\{1,2,3, \dots\}$, denoted by $\hat{\equiv}$, is defined as the symmetric transitive closure of the following relations, where $\mathbf{u}$

and $\mathbf{v}$ are (possibly empty) words of positive integers, and $a < b < c$ are distinctive positive integers:

(1) $(\mathbf{u}, a, a, \mathbf{v}) \hat{\equiv} (\mathbf{u}, a, \mathbf{v})$,
(2) $(\mathbf{u}, a, b, a, \mathbf{v}) \hat{\equiv} (\mathbf{u}, b, a, b, \mathbf{v})$,
(3) $(\mathbf{u}, b, a, c, \mathbf{v}) \hat{\equiv} (\mathbf{u}, b, c, a, \mathbf{v})$,
(4) $(\mathbf{u}, a, c, b, \mathbf{v}) \hat{\equiv} (\mathbf{u}, c, a, b, \mathbf{v})$,
(5) $(a, b, \mathbf{u}) \hat{\equiv} (b, a, \mathbf{u})$.

Two words w and w' are said to be *K-Knuth equivalent*, denoted as $\hat{\equiv}$, if w can be obtained by applying the above relations finitely many times from w'.

Example 2.2. $124636 \hat{\equiv} 124363 \hat{\equiv} 124633 \hat{\equiv} 12463 \hat{\equiv} 21463 \hat{\equiv} 21436 \hat{\equiv} 24136$

This following lemma would be useful later on.

Lemma 2.3. *If $w \hat{\equiv} w'$, then $w|_I \hat{\equiv} w'|_I$ for any interval I.*

2.2. **Increasing Shifted Tableaux.** Each strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$ can be associated to the *shifted shape*, which is an array of boxes where the ith row has λ_i boxes and is indented $i-1$ units. A *shifted tableau* is a filling of the shifted shape with positive integers. A shifted tableau is *increasing* if the labels are strictly increasing from left to right along rows and top to bottom along columns. The *reading word* of a shifted tableau T, denoted as $\mathfrak{row}(T)$, is a word obtained from reading the labels of tableau T left to right, from bottom to top. Similarly, we say two tableaux T and T' are *weak K-Knuth equivalent* if $\mathfrak{row}(P_{SK}(T)) \hat{\equiv} \mathfrak{row}(P_{SK}(T'))$.

In this paper, unless otherwise stated, all the tableaux have shifted shape and we refer to them as tableaux.

Example 2.4. The first two tableaux are increasing shifted tableaux, while the third one is not because the third column is weakly increasing. The reading word of the first two tableaux are 635124 and 471367, respectively.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 \\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline \multicolumn{1}{c|}{} & 4 & 7 & \multicolumn{1}{c}{} \\ \cline{2-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{3-3}\end{array}$$

An important observation can be made directly from the definition of increasing shifted tableaux.

Lemma 2.5. *There are only finitely many increasing shifted tableaux filled with a given finite alphabet.*

Proof. If the alphabet has n letters, each row and column of the tableau can be no longer than n. $\square$

2.3. **Shifted Hecke Insertion.** The rules for shifted Hecke insertion were introduced in [8]. It is simultaneously a shifted analogue of Hecke insertion [1] and a K-theoretic analogue of Sagan-Worley insertion [12]. From this point on, "insertion" will always refer to shifted Hecke insertion unless stated otherwise.

First, proceed with how to insert a positive integer x into a given shifted increasing tableau T. Start with inserting x into the first row of T. For each insertion, assign a box to record where the insertion terminates. This data will be used when the recording tableau is introduced in Subsection 2.4.

The rules for inserting x to T are as follows:

(1) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) results in an increasing tableau T', then T' is the resulting tableau. We say the insertion terminates at the new box.

Example 2.6. Inserting 4 into the first row of the left tableau gives us the right tableau below. The insertion terminates at box $(1, 4)$.

1	2	3
	3	5
		6

1	2	3	4
	3	5	
		6	

(2) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) does not result in an increasing tableau, then $T' = T$. If x is row inserted into a nonempty row, we say the insertion terminated at the box at the bottom of the column containing the rightmost box of this row. If x is row inserted into an empty row, we say that the insertion terminated at the rightmost box of the previous row. If x is column inserted, we say the insertion terminated at the rightmost box of the row containing the bottom box of the column x could not be added to.

Example 2.7. Adjoining 4 to the first row of the left tableau does not result in an increasing tableau. Thus the insertion of 4 into the first row of the tableau on the left terminates at (2,3) and gives us the tableau on the right.

1	2	4
	3	5

1	2	4
	3	5

Adjoining 2 to the (empty) second row of the tableau below does not result in an increasing tableau. The insertion ending in this failed row insertion terminates at $(1, 3)$.

1	2	3

Adjoining 3 to the end of the third column of the left tableau does not result in an increasing tableau. This insertion terminates at $(1, 3)$.

1	2	3
	3	

For the last two rules, suppose the row (resp. column) contains a box with label strictly larger than x, and let y be the smallest such box.

(3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row and y was not on the main diagonal, then insert y into the row below.

Example 2.8. Row inserting 3 into the first row of the left tableau results in the tableau below on the right. This insertion terminates at $(2, 3)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 \\ \cline{2-3}\end{array}$$

To insert 3 into the second row of the tableau below on the left, replace 4 with 3, and column insert 4 into the third column. The resulting tableau is on the right.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & \multicolumn{1}{|c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{|c}{} \\ \cline{3-3}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & 6 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{|c}{} \\ \cline{3-3}\end{array}$$

(4) If replacing y with x does not result in an increasing tableau, then do not change the row (resp. column). In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row, then insert y into the row below.

Example 2.9. If we insert 3 into the first row of the tableau below, notice that replacing 5 with 3 does not create an increasing tableau. Hence row insertion of 3 into the first row produces output integer 5, which is inserted into the second row. Replacing 6 with 5 in the second row does not create an increasing tableau. This produces output integer 6. Adjoining 6 to the third row does not result in an increasing tableau. Thus inserting 3 into the tableau below does not change the tableau. This insertion terminates at $(2, 3)$.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 6 \\ \cline{2-3}\end{array}$$

For any given word $w = w_1 w_2 \cdots w_n$, we define the *insertion tableau* of w, $P_{SK}(w)$, to be $(\cdots((\emptyset \xleftarrow{SK} w_1) \xleftarrow{SK} w_2) \cdots \xleftarrow{SK} w_n)$, where $\emptyset$ denotes the empty shape and $\xleftarrow{SK}$ denotes the insertion of a single letter.

Example 2.10. The sequence of tableaux obtained while computing $P_{SK}(2115432)$ is shown below. The tableau on the right is $P_{SK}(2115432)$.

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{|c}{} \\ \cline{2-3}\end{array}$$

For any interval I, we define $T|_I$ to be the tableau obtained from T by deleting all boxes with labels not in I and $w|_I$ to be the word obtained from w by deleting all letters not in I. We use $[k]$ to denote the interval $\{1, 2, ..., k\}$. The following simple lemma will be useful.

Lemma 2.11. *If $P_{SK}(w) = T$, then $P_{SK}(w)|_{[k]} = P_{SK}(w|_{[k]}) = T|_{[k]}$.*

Proof. Observe from from the insertion rules that letters greater than k never affect the placement or number of letters in $\{1, 2, \ldots, k\}$. $\square$

2.4. **Recording Tableaux.** In order to describe the recording tableau for shifted Hecke insertion of a word w, we need the following definition.

Definition 2.12. [8, Definition 5.16] A *set-valued shifted tableau* is defined to be a filling of the boxes of a shifted shape with finite, nonempty subsets of primed and unprimed positive integers such that:

1) The smallest number in each box is greater than or equal to the largest number in the box directly to the left of it, if that box exists.
2) The smallest number in each box is greater than or equal to the largest number in the box directly above it, if that box exists.
3) Any positive integer appears at most once, either primed or unprimed, but not both.
4) There are no primed entries on the main diagonal.

A set-valued shifted tableau is called *standard* if the set of labels is exactly $[n]$ for some n, each appearing either primed or unprimed exactly once.

Example 2.13. The tableaux below are set-valued shifted tableaux. The tableau on the right is standard.

$$\begin{array}{|c|c|c|}\hline 1 & 23' & 6 \\ \hline \multicolumn{1}{c|}{} & 4 & 8'9 \\ \cline{2-3}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3'4' & 6' \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{2-2}\end{array}$$

The *recording tableau* of a word $w = w_1w_2 \dots w_n$, denoted $Q_{SK}(w)$, is a standard set-valued shifted tableau that records where the insertion of each letter of w terminates. We define it inductively. Start with $Q_{SK}(\emptyset) = \emptyset$. If the insertion of w_k added a new box to $P_{SK}(w_1w_2 \dots w_{k-1})$, then add the same box with label k (k' if this box was added by column insertion) to $Q_{SK}(w_1w_2 \dots w_{k-1})$. If w_k did not change the shape of $P_{SK}(w_1w_2 \dots w_{k-1})$, we obtain $Q_{SK}(w_1w_2 \dots w_k)$ from $Q_{SK}(w_1 \dots w_{k-1})$ by adding the label k (k' if it ended with column insertion) to the box where the insertion terminated. If insertion terminated when a letter failed to insert into an empty row, label the box where the insertion terminated k'.

Example 2.14. Let $w = 352243$. We insert w letter by letter, writing the insertion tableau at each step in the top row and the recording tableau at each step in the bottom row.

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3}\end{array} = P_{SK}(w)$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3' \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline \multicolumn{1}{c|}{} & 5 & 6' \\ \cline{2-3}\end{array} = Q_{SK}(w)$$

In [8], a reverse insertion procedure is defined so that for each pair $(P_{SK}(w), Q_{SK}(w))$, the word w can be recovered. See [8] for details on reverse shifted Hecke insertion. This procedure gives the following result:

Theorem 2.15. *[8, Theorem 5.19] The map $w \mapsto (P_{SK}(w), Q_{SK}(w))$ is a bijection between words of positive integers and pairs of shifted tableaux (P, Q) of the same shape where P is an increasing shifted tableau and Q is a standard set-valued shifted tableau.*

In fact, the shifted Hecke insertion preserves the weak K-Knuth equivalence on words. The proof of the following statement can be found at [5], which was shown by using the *shifted K-theoretic jeu de taquin* introduced in [3]. For more information on the shifted K-theoretic jeu de taquin, see [2], [3], [13], and [5].
The theorem is a combined efforts of [5, Theorem 2.16] and [3, Theorem 7.8]. Since the

shifted K-theoretic jeu de taquin is only used to show that the shifted Hecke insertion is consistent with the weak K-Knuth equivalence on words, we omit the proof here.

Theorem 2.16. *[5, Corollary 2.18] If $P_{SK}(u) = P_{SK}(v)$, then $u \hat{\equiv} v$.*

From this point on, we refer to both the weak K-Knuth equivalence on words and the weak K-Knuth equivalence on insertion tableaux as "equivalence"

Remark 2.17. The converse of this statement does not hold. Consider the words 12453 and 124533, which are easily seen to be weakly K-Knuth equivalent. We compute that shifted Hecke insertion gives the following distinct tableaux.

$$P_{SK}(12453) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \qquad P_{SK}(124533) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

2.5. **Unique Rectification Targets.** As one can see from Remark 2.17, two K-Knuth equivalent words might have different insertion tableaux. The weak K-Knuth equivalence is "coarser" than the natural equivalence of insertion tableaux. This is a key difference between weak K-Knuth equivalence and Knuth equivalence. It is natural to ask if one can find some class of words that would have an unique insertion tableau given any shifted shape of tableau. The answer is yes, and it is crucial to our approach to prove the Littlewood-Richardson rule.

Definition 2.18. [2, Definition 3.5] T is a *unique rectification target*, or a URT, if it is the only tableau in its weak K-Knuth equivalence class. That is, T is a URT if and only if for every $w \hat{\equiv} \mathfrak{row}(T)$ we have $P_{SK}(w) = T$. If $P_{SK}(w)$ is a URT, we call the equivalence class of w a *unique rectification class*.

A more detailed discussion of URTs can be found at [2, 3] for shifted tableaux and straight shape tableaux.

In [2], Buch and Samuel describe a way to fill any shifted shape to create a URT. The *minimal increasing shifted tableau* M_λ of a shifted shape λ is the tableau obtained by filling the boxes of λ with the smallest values allowed in an increasing tableau. For example,

$$M_{(4,1)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array} \qquad M_{(5,3,1)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$$

are minimal increasing tableaux.

Another way to fill any shifted shape to create a URT is to consider the *superstandard shifted tableaux*. The superstandard shifted tableaux of a shifted shape $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ has the filling of $1, 2, \ldots \lambda_1$ for the first row, $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ for the second row, and etc. For example,

$$M_{(4,2)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array} \qquad M_{(5,2,1)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 6 & 7 & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 8 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$$

are superstandard increasing tableaux.

Theorem 2.19.

(1) [2, Corollary 7.2] *Minimal increasing shifted tableaux are URTs.*
(2) [3, Theorem 1.1] *Superstandard shifted tableaux are URTs.*

As a consequence, we see there are URTs for every shifted shape.

3. Shifted K-Poirier-Reutenauer Algebra

Poirier and Reutenauer first defined a Hopf algebra spanned by the set of standard Young tableaux in [10]. In [6], Jing and Li developed a shifted version of the classical Poirier-Reutenauer bialgebra, and Patrias and Pylyavskyy developed a K-theoretic analogue of the Poirier-Reutenauer bialgebra in [9]. In this section, a shifted K-theoretic analogue of the Poirier-Reutenauer algebra will be introduced.

A word h is said to be *initial* if the letters appearing in it are exactly the numbers in $[k]$ for some positive integer k. For example, 261534 and 31142 are initial, but 2632 is not.

Let $[[h]]$ denote the formal sum of all words in the weak K-Knuth equivalence class of an initial word h:

$$[[h]] = \sum_{h \hat{\equiv} w} w.$$

This is an infinite sum, however, the number of terms in $[[h]]$ of each length is finite. For example,

$$[[2413]] = 4213 + 2143 + 2413 + 24133 + 42133 + 42313 + 42131 + 421311 + \cdots .$$

Let $SKPR$ denote the vector space over $\mathbb{R}$ spanned by all sums of the form $[[h]]$ for some initial word h. To give an PR-algebra structure, a compatible product will be defined on $SKPR$.

Lemma 3.1. *We have*

$$[[h]] = \sum_{T} \left(\sum_{P_{SK}(w)=T} w \right),$$

where the sum is over all increasing shifted tableaux T whose reading word is in the weak K-Knuth equivalence class of h.

Proof. Fix a word h. For any given word w, it is weak K-Knuth equivalent to h if and only if $\mathfrak{row}(P_{SK}(w))$ is weak K-Knuth equivalent to h from Theorem 2.16. This concludes the proof. □

Note that the set of tableaux we sum over is finite by Lemma 2.5. Since given any h, the tableaux T equivalent to $P_{SK}(h)$ is defined on a finite alphabet. Hence, there are only finitely many increasing shifted tableaux T equivalent to $P_{SK}(h)$.

3.1. **Product structure.** Denote $w[n]$ as the word obtained from word w by increasing each letter by$[n]$. For example, if $w = 2413$, $w[2] = 4635$. Given two initial words h and h' on the alphabet $[n]$ and $[m]$ respectively, the *shuffle product* h and h', denoted as $\sqcup\!\sqcup$, is defined to be the concatenation of h and $h'[n]$. For example, $2413 \sqcup\!\sqcup 12 = 241356$.

Now we define the product of $[[h]]$ and $[[h']]$ in $SKPR$

$$[[h]] \cdot [[h']] = \sum_{w \hat{\equiv} h, w' \hat{\equiv} h'} w \sqcup w'[n].$$

where h is a word in the alphabet $[n]$ and h' is a word in the alphabet $[m]$.

The following theorem shows that this product is a binary operation on the vector space $SKPR$.

Theorem 3.2. *The product of any two initial words h and h' can be written as*

$$[[h]] \cdot [[h']] = \sum_{h''} [[h'']],$$

where the sum is over some set of initial words h''.

Proof. By Lemma 2.3, we know that once a word appears in the right-hand sum, its entire equivalence class must also appear. The result then follows. □

Need to come back and check the reference!

— Yn

Example 3.3. Let $h = 12$ and $h' = 12$. Then

$$[[12]] \cdot [[12]] = [[1234]] + [[4123]] + [[3124]] + [[31234]] + [[41234]] + [[34123]] + [[341234]].$$

By Lemma 3.1, $[[h]] \cdot [[h']]$ can be written as a sum over tableaux, which can be sorted into finitely many equivalence classes. This implies that we can write the product as an explicit sum over sets of tableaux.

Theorem 3.4. *Let h be a word in alphabet $[n]$, and let h' be a word in alphabet $[m]$. Suppose $\mathcal{T} = \{P_{SK}(h), {T_1}', {T_2}', \ldots, {T_s}'\}$ is the equivalence class containing $P_{SK}(h)$. Then*

$$[[h]] \cdot [[h']] = \sum_{T \in T(h \sqcup h')} \sum_{P_{SK}(w) = T} w,$$

where $T(h \sqcup h')$ is the set of shifted tableaux T such that $T|_{[n]} \in \mathcal{T}$ and $\mathfrak{row}(T)|_{[n+1,n+m]} \hat{\equiv} h'[n]$.

Proof. If w is a shuffle product of some $w_1 \hat{\equiv} h$ and $w_2 \hat{\equiv} h'[n]$, then by Lemma 2.11. $P_{SK}(w)|_{[n]} = P_{SK}(w|_{[n]}) = P_{SK}(w_1) \in \mathcal{T}$.

By definition, $\mathfrak{row}(P_{SK}(w)|_{[n+1,n+m]}) = \mathfrak{row}(P_{SK}(w))|_{[n+1,n+m]}$. By theorem 2.16, we have $\mathfrak{row}(P_{SK}(w)) \hat{\equiv} w$. Hence, $\mathfrak{row}(P_{SK}(w)|_{[n+1,n+m]}) \hat{\equiv} w|_{[n+1,n+m]} = w_2 \hat{\equiv} h'[n]$. Now using Lemma 3.1 and Theorem 3.2, we see that the product can be expanded in this way. □

Note that by Lemma 2.5 the set $T(h \sqcup h')$ is finite since all of the tableaux in it are on the finite alphabet $[n+m]$.

Example 3.5. Consider $h = 12$ and $h' = 12$. The set $T(12 \sqcup 12)$ consists of the seven tableaux below.

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \qquad \begin{array}{cccc} 1&2&3\\ &4& \end{array} \qquad \begin{array}{ccc} 1&2&4\\ &3& \end{array} \qquad \begin{array}{cccc} 1&2&3&4\\ &3&& \end{array} \qquad \begin{array}{cccc} 1&2&3&4\\ &4&& \end{array}$$

$$\begin{array}{ccc} 1&2&3\\ &3&4 \end{array} \qquad \begin{array}{cccc} 1&2&3&4\\ &3&4& \end{array}$$

For each of these tableaux, restricting to the alphabet $\{1,2\}$ gives the tableau $P_{SK}(12)$. Also, the reading word of each restricted to the alphabet $\{3,4\}$ is weak K-Knuth equivalent to $h'[2] = 34$.

Corollary 3.6. *If both T_1 and T_2 are URTs, then*

$$\left(\sum_{P_{SK}(u)=T_1} u\right) \cdot \left(\sum_{P_{SK}(v)=T_2} v\right) = \sum_{T \in T(T_1 \sqcup T_2)} \sum_{P_{SK}(w)=T} w,$$

where $T(T_1 \sqcup T_2)$ is the set of shifted tableaux T such that $T|_{[n]} = T_1$ and $\mathfrak{row}(T)|_{[n+1,n+m]} \hat{\equiv} \mathfrak{row}(T_2)[n]$.

4. Weak Shifted Stable Grothendieck Polynomials

4.1. **Weak shifted stable Grothendieck polynomials.** As an analogue of the weak set-valued tableau defined in [8], we define the weak shifted stable Grothendieck polynomial K_λ as a weighted generating function over weak set-valued shifted tableaux.

Definition 4.1. A *weak set-valued shifted tableau* is a filling of the boxes of a shifted shape with finite, nonempty multisets of primed and unprimed positive integers with ordering $1' < 1 < 2' < 2 < \cdots$ such that:

1. The smallest number in each box is greater than or equal to the largest number in the box directly to the left of it, if that box exists.
2. The smallest number in each box is greater than or equal to the largest number in the box directly above it, if that box exists.
3. There are no primed entries on the main diagonal.
4. Each unprimed integer appears in at most one box in each column.
5. Each primed integer appears in at most one box in each row.

Given any weak set-valued shifted tableau T, we define x^T to be the monomial $\prod_{i \geq 1} x_i^{a_i}$, where a_i is the number of occurrences of i and i' in T. For example, the weak set-valued tableaux T_1 and T_2 below have $x^{T_1} = x_1 x_3 x_4^3 x_5^2 x_6 x_7 x_8$ and $x^{T_2} = x_1 x_2 x_3 x_4^3$.

$$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4'4 & 45' \\ \hline \multicolumn{1}{c|}{} & 5 & 67' & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \qquad T_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4'4' \\ \hline \multicolumn{1}{c|}{} & 3 & 4' \\ \cline{2-3} \end{array}$$

Recall that we denote a shifted shape λ as $(\lambda_1, \lambda_2, \cdots \lambda_n)$, where n is the number of rows and λ_i is the number of boxes of ith row.

Definition 4.2. The *weak shifted stable Grothendieck polynomial* is

$$K_\lambda = \sum_T x^T,$$

where the sum is over the set of weak set-valued tableaux T of shape λ.

Remark 4.3. In tableaux formulation such as Definition 4.2, stable Grothendieck polynomials and their analogues typically have a sign $(-1)^{|T|-|\lambda|}$ for each monomial, where $|T|$ is the degree of x^T. We have chosen to suppress this sign for our definition K_λ as others have done e.g. [7]. It is easily reintroduced when necessary.

Example 4.4. We have

$$K_{(2,1)} = x_1^2x_2 + 2x_1x_2x_3 + 3x_1^2x_2^2 + 5x_1^2x_2x_3 + 5x_1x_2^2x_3 + \cdots ,$$

where the coefficient of $x_1^2x_2^2$ is 3 because of the tableaux shown below.

$$\begin{array}{|c|c|}\hline 11 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 12' \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 22 \\ \cline{2-2}\end{array}$$

4.2. K_λ **and fundamental quasisymmetric functions.** Before giving out the definition of quasisymmetric functions, one has to define the descent set for both word and tableau.

Given a word $w = w_1w_2\dots w_n$, a *descent set* of w, $\mathcal{D}(w)$, is defined to be $\{i : w_i > w_{i+1}\}$. For example, $\mathcal{D}(131442) = \{2, 5\}$. Similarly, one can define the descent set of a standard set-valued shifted tableau $\mathcal{D}(T)$ of T as the following.

$$\mathcal{D}(T) = \left\{ i \;:\; \begin{array}{l} \text{both } i \text{ and } (i+1)' \text{ appear} \\ \qquad\quad \text{OR} \\ i \text{ is strictly above } i+1 \\ \qquad\quad \text{OR} \\ i' \text{ is weakly below } (i+1)' \text{ but not in the same box} \end{array} \right\}.$$

Given a word w of length n, $\mathcal{D}(w) \subset [n-1]$. We can associate the *fundamental quasisymmetric function* of w as

$$f_{\mathcal{D}(w)} = \sum_{\substack{i_1 \le i_2 \le \dots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathcal{D}}} x_{i_1}x_{i_1}\dots x_{i_n}.$$

For example, the word $w = 352243$ has descent set $\mathcal{D}(w) = \{2, 5\}$. Its recording tableau is

$$Q_{SK}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3'4' \\ \hline \multicolumn{1}{c|}{} & 5 & 6' \\ \cline{2-3}\end{array}$$

Since 2 and 3′, 5, and 6′ appear in pairs, we can conclude that $\mathcal{D}(Q_{SK}(w)) = \{2, 5\}$. The associated fundamental quasisymmetric function is

$$\begin{aligned} f_{\{2,5\}} = {} & x_1^2x_2^3x_3 + x_1x_2x_3^3x_4 + \cdots + x_1^2x_2x_3^2x_5 + \cdots \\ & + x_2^2x_4^3x_5 + x_1x_3x_4^2x_5x_6 + \cdots + x_1x_2x_3x_4x_5x_6 + \cdots \end{aligned}$$

Note that $\mathcal{D}(Q_{SK}(w)) = \mathcal{D}(w)$. This is true in general. The proof is available in [5]

Theorem 4.5. *[5, Theorem 3.7] For any word $w = w_1w_2\dots w_n$, $\mathcal{D}(Q_{SK}(w)) = \mathcal{D}(w)$.*

Now we proceed to show that K_λ can be written as the sum of fundamental quasisymmetric functions. We will start by building the connection between the standard set-valued shifted tableau and the weak set-valued shifted tableau, since K_λ is directly defined on the weak set-valued shifted tableau of shape λ and a word w can be associated with a standard set-valued shifted tableau by considering its recording tableau.

Given a monomial $\sigma = x_{s_1}x_{s_2}\dots x_{s_r}$, where $s_1 \le s_2 \le \cdots \le s_r$, and a standard set-valued shifted tableau T, we *relabel* T *by* σ by replacing the ith smallest letter in T with s_i, primed if that letter had a prime to begin with.

Example 4.6. Given T below and $\sigma = x_1x_3x_5^2x_6x_7$, the relabeling $T(\sigma)$ of T with σ is the tableau on the right.

<table><tr><td>12</td><td>3′4</td><td>6</td></tr><tr><td></td><td>5</td><td></td></tr></table>

<table><tr><td>13</td><td>5′5</td><td>7</td></tr><tr><td></td><td>6</td><td></td></tr></table>

might have to change example
— Yn

A monomial $\sigma = x_{s_1}x_{s_2}\dots x_{s_r}$ is said to be *agree with* $\mathcal{D}$ if $s_i \le s_{i+1}$ with strict equality on $i \in \mathcal{D}$. For example, given $\mathcal{D} = \{1,3\} \subset [n-1]$ with $n = 4$, both $x_1x_2^2x_3, x_1x_3x_4x_6$ agree with $\mathcal{D}$.

Difference between $\subset$ and $\subseteq$
— Yn

The following lemma tells us how to translate a standard set-valued shifted tableau to a weakly set-valued shifted tableau with a given monomial.

Lemma 4.7. *[5, Lemma 3.8] If T is a standard set-valued shifted tableau and $\sigma = x_{s_1}x_{s_2}\dots x_{s_r}$ agree with $\mathcal{D}(T)$, then $T(\sigma)$ is a weakly set-valued shifted tableau.*

To see how to translate a weak set-valued tableau T to a standard set-valued tableau, we define the *standardization* of a weak set-valued tableau T. The standardization $\mathfrak{st}(T)$ is defined to be refinements of the order of entries of T given by reading each occurrence of k in T from left to right and each occurrence of k' in T from top to bottom, using the total order $(1' < 1 < 2' < 2 < \cdots)$. Notice that $\mathfrak{st}(T)$ will be a standard set-valued shifted tableau. For example, we have

$T =$
<table><tr><td>1 2</td><td>3 3 3</td><td>3</td><td>7</td><td>9</td></tr><tr><td></td><td>4 5</td><td>6′</td><td>8′</td><td></td></tr><tr><td></td><td></td><td>7</td><td>8′</td><td></td></tr></table>

$\mathfrak{st}(T) =$
<table><tr><td>1 2</td><td>3 4 5</td><td>6</td><td>11</td><td>14</td></tr><tr><td></td><td>7 8</td><td>9′</td><td>12′</td><td></td></tr><tr><td></td><td></td><td>10</td><td>13′</td><td></td></tr></table>

Now we can write K_λ as a sum of fundamental quasisymmetric functions.

Theorem 4.8. *For any fixed increasing shifted tableau T of shape λ,*

$$K_\lambda = \sum_{P_{SK}(w)=T} f_{\mathcal{D}(w)}.$$

Proof. Let W be a weak set-valued shifted tableau of shape λ with entries $s_1 \le s_2 \le \ldots \le s_n$, some of which may be primed. By Theorem 2.15, there is a unique word w such that $P_{SK}(w) = T$ and $Q_{SK}(w) = \mathfrak{st}(W)$. We will show that x^W agrees with $\mathcal{D}(w)$ by showing that x^W agrees with $\mathcal{D}(\mathfrak{st}(W))$. By Lemma 4.7, every σ that agrees with $\mathcal{D}(w)$ corresponds to some W, so this would complete the proof.

Let j be strictly above $j+1$ in $\mathfrak{st}(W)$. If $s_j = s_{j+1}$, by the weakly increasing property, we see they must be in the same column. This violates condition (4) in the definition of weak set-valued tableaux, so $s_j < s_{j+1}$.

Next assume j and $(j+1)'$ both appear in $\mathfrak{st}(W)$. By the definition of standardization, s_j and s_{j+1} cannot be the same number, as primed entries of the same value are standardized first. Therefore, $\sigma_j < \sigma_{j+1}$.

Finally, let $(j+1)'$ be weakly above j' and not in the same box. If $s_j = s_{j+1}$, they cannot appear on the same row of W, as this would violate condition (5) in the definition of weak set-valued shifted tableaux. Moreover, if ${s_{j+1}}'$ were strictly above ${s_j}'$, it would have to be strictly smaller after standardization, which cannot occur. Therefore, $s_j < s_{j+1}$, which completes our proof. $\square$

Since quasisymmetric function is the "building block" of symmetric functions, showing that K_λ can be written as the sum of quasisymmetric functions is a positive sign of K_λ being symmetric. In section 3.3 of [5], the symmetry of K_λ was shown.

5. A Littlewood-Richardson Rule

Give some idea on what the Littlewood-Richardson Rule is?
— Yn

Recall that $SKPR$ denotes the vector space over $\mathbb{R}$ spanned by all sums of the form $[[h]]$ for some initial word h. We previously define a compatible product on $SKPR$. Now we construct an algebra homomorphism that takes a weak K-Knuth equivalence class of initial word $[[h]]$ to a sum of fundamental quasisymmetric functions.

Theorem 5.1. *The linear map ϕ :SKPR$\to$ QSym defined by*

$$\phi([[h]]) = \sum_{w \hat{=} h} f_{\mathcal{D}(w)}$$

is an algebra homomorphism.

Proof. The well-definedness is obvious. To see why ϕ preserves the product, consider h_1 on $[n]$ and h_2 on [m]. We use $Sh(w', w''[n])$ to denote the set of all the elements of the shuffle product of w and w'. First, [7, Proposition 5.9] implies

$$f_{\mathcal{D}(w')} \cdot f_{\mathcal{D}(w'')} = \sum_{w \in Sh(w', w''[n])} f_{\mathcal{D}(w)},$$

where the sum is over all shuffles of $w', w''[n]$. Recall that from Theorem 3.4, we have

$$\begin{aligned}
\phi([[h_1]] \cdot [[h_2]]) &= \phi\left(\sum_{w' \hat{=} h_1, w'' \hat{=} h_2} w' \sqcup w''\right) \\
&= \sum_{w' \hat{=} h_1, w'' \hat{=} h_2} \sum_{w \in Sh(w', w''[n])} f_{\mathcal{D}(w)} \\
&= \sum_{w' \hat{=} h_1, w'' \hat{=} h_2} f_{\mathcal{D}(w')} \cdot f_{\mathcal{D}(w'')} \\
&= \left(\sum_{w' \hat{=} h_1} f_{\mathcal{D}(w')}\right)\left(\sum_{w'' \hat{=} h_2} f_{\mathcal{D}(w'')}\right) = \phi([[h_1]])\phi([[h_2]]).
\end{aligned}$$

$\square$

Theorem 5.2. *Letting $\lambda(T)$ denote the shape of T, we have*

$$\phi([[h]]) = \sum_{\mathfrak{row}(T) \hat{=} h} K_{\lambda(T)}.$$

Proof. By Theorem 4.8, we see

$$\phi([[h]]) = \sum_{w \hat{=} h} f_{\mathcal{D}(w)} = \sum_{T \hat{=} P_{SK}(h)} \sum_{P_{SK}(w)=T} f_{\mathcal{D}(w)} = \sum_{T \hat{=} P_{SK}(h)} K_{\lambda(T)} = \sum_{\mathfrak{row}(T) \hat{=} h} K_{\lambda(T)}.$$

□

With the algebra homomorphism ϕ defined above, we can show the Littlewood-Richardson rule for K_λ by using Theorem 5.2.

Theorem 5.3. *Let T be a URT of shape μ. Then we have*

$$K_\lambda K_\mu = \sum_\nu c^\nu_{\lambda,\mu} K_\nu,$$

where $c^\nu_{\lambda,\mu}$ is given by the number of increasing shifted skew tableaux R of shape ν/λ such that $P_{SK}(\mathfrak{row}(R)) = T$.

Proof. In addition to T, fix a URT T' of shape λ. Then by Theorems 3.6 and 5.2, we have

$$\begin{aligned} K_\lambda K_\mu &= \phi([[\mathfrak{row}(T')]])\phi([[\mathfrak{row}(T)]]) \\ &= \phi([[\mathfrak{row}(T')]] \cdot [[\mathfrak{row}(T)]]) \\ &= \sum_{R \in T(T' \sqcup T)} \sum_{P_{SK}(w)=R} w \\ &= \sum_{R \in T(T' \sqcup T)} K_{\lambda(R)}, \end{aligned}$$

where $T(T' \sqcup T)$ is the set of shifted tableaux R such that $R|_{[|\lambda|]} = T'$ and $P_{SK}(\mathfrak{row}(R|_{[|\lambda|+1,|\lambda|+|\mu|]})) = T$, giving our result. □

This rule, up to sign, coincides with the rules of Clifford, Thomas and Yong [3, Theorem 1.2] and Buch and Samuel [2, Corollary 4.8].

6. Acknowledgments

This research was done with Zachary Hamaker, Adam Keilthy, Rebecca Patrias, Lillian Webster, and Shuqi Zhou as part of the 2015 REU program at the University of Minnesota, Twin Cities, and was supported by RTG grant NSF/DMS-1148634. I would like to thank Pasha Pylyavskyy, Vic Reiner for their mentorship and support.

References

[1] Anders Skovsted Buch, Andrew Kresch, Mark Shimozono, Harry Tamvakis, and Alexander Yong. Stable Grothendieck polynomials and K-theoretic factor sequences. *Mathematische Annalen*, 340(2):359–382, 2008.

[2] Anders Skovsted Buch and Matthew J Samuel. K-theory of minuscule varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014.

[3] Edward Clifford, Hugh Thomas, and Alexander Yong. K-theoretic Schubert calculus for $OG(n, 2n+1)$ and jeu de taquin for shifted increasing tableaux. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(690):51–63, 2014.

[4] Gérard HE Duchamp, Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Noncommutative symmetric functions vii: free quasi-symmetric functions revisited. *Annals of combinatorics*, 15(4):655–673, 2011.

[5] Zachary Hamaker, Adam Keilthy, Rebecca Patrias, Lillian Webster, Yinuo Zhang, and Shuqi Zhou. Shifted Hecke insertion and the K-theory of $OG(n, 2n+1)$. *arXiv preprint arXiv:1510.08972*, 2015.

[6] Naihuan Jing and Yunnan Li. The shifted Poirier-Reutenauer algebra. November 2014. pre-print arXiv:1411.2131.

[7] Thomas Lam and Pavlo Pylyavskyy. Combinatorial Hopf algebras and K-homology of Grassmanians. *International Mathematics Research Notices*, page rnm125, 2007.

[8] Rebecca Patrias and Pavlo Pylyavskyy. Dual filtered graphs. *arXiv preprint arXiv:1410.7683*, 2014.

[9] Rebecca Patrias and Pavlo Pylyavskyy. K-theoretic Poirier-Reutenauer bialgebra. *arXiv preprint arXiv:1404.4340*, 2014.

[10] Stéphane Poirier and Christophe Reutenauer. Algebres de Hopf de tableaux. *Ann. Sci. Math. Québec*, 19(1):79–90, 1995.

[11] Victor Reiner and Muge Taskin. The weak and Kazhdan-Lusztig orders on standard Young tableaux. In *FPSAC PROCEEDINGS 2004 ACTES SFCA 2004 Vancouver CANADA*, page 281. Citeseer, 2004.

[12] Bruce E. Sagan. Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley. *J. Comb. Theory, Ser. A*, 45(1):62–103, 1987.

[13] Hugh Thomas and Alexander Yong. Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm. *Advances in Applied Mathematics*, 46(14):610 – 642, 2011.