DISCUSSION ON GENERALIZED MULTI-ECHELON CROSS-DOCKING SYSTEM

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ABSTRACT. This paper explores the complexities and optimizations within generalized multi-echelon cross-docking systems, focusing on the balance between holding costs and backorder costs under variable demand scenarios. By incorporating probability theory and supply chain modeling, we present a detailed analysis of two-echelon, three-echelon cross-docking systems, and possible generic systems, emphasizing the optimization of inventory levels and the strategic distribution of goods from central warehouses to retailers. The investigation reveals the critical role of lead time and demand distribution in determining optimal order-up-to levels and discusses the implications of imbalance assumptions in multi-echelon structures. Through theoretical models and calculations, we propose strategies for achieving balanced stock-out probabilities across the supply chain, aiming to minimize the total expected cost under varying demand. The paper further discusses on potential strategies in scenarios where the imbalance assumption is violated, offering insights into the resilience and efficiency of supply chain designs. We conclude with the flexibility of the model highly depends on the lead time, holding/backorder costs and demand.

CONTENTS

1. INTRODUCTION

The domain of supply chain optimization, characterized by its inherent complexity and the stochastic nature of demand, presents a rich canvas for mathematical exploration. Within this

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domain, the strategy of cross-docking for supply chian as a critical mechanism aimed at enhancing logistical efficiency. Cross-docking, by facilitating the direct transfer of goods from inbound to outbound transportation with minimal storage in between, requires rigorous mathematical modeling to optimize its implementation, particularly in the context of multi-echelon supply chain systems.

1.1. Motivational Context

Contemporary supply chains are marked by a high degree of interconnectivity and dynamism, challenging traditional inventory management paradigms. The seminal work of Erkip, Hausman, and Nahmias [1] on multi-echelon inventory systems introduces a framework for considering demand correlations across different echelons, a critical aspect that significantly influences the optimization of inventory levels. This study extends the mathematical framework laid out by Erkip et al., with a specific focus on integrating cross-docking operations into generalized multi-echelon systems. The objective is to derive optimal inventory management strategies that not only acknowledge but leverage the complexity and stochastic nature of modern supply chains.

1.2. Theoretical Foundation

The mathematical underpinnings of multi-echelon inventory management were rigorously formalized by Clark and Scarf [2], setting the stage for subsequent explorations into complex supply chain structures. The incorporation of demand correlations, as further explored by Johnson and Thompson [3], adds another layer of complexity, necessitating advanced mathematical tools for system optimization.

Building on these foundational studies, the work of Erkip, Hausman, and Nahmias [1] provides a pivotal analysis of centralized ordering policies within multi-echelon inventory systems, taking into account the intricacies of correlated demands. Complementing this, Eppen and Schrage [4] delve into the optimization of centralized ordering policies in a multi-warehouse context, highlighting the benefits of centralized decision-making in managing lead times and random demand across warehouses.

1.3. Thesis Aims

This thesis aims to synthesize and extend the mathematical methodologies presented in the works of Erkip et al. [1] and Eppen and Schrage [4], with the goal of developing a comprehensive mathematical model for optimizing cross-docking operations in generalized multi-echelon supply chains.In the previous studies, main discussion adopted the two echelon system. Through rigorous mathematical analysis and optimization techniques, we seek to uncover strategies under increasing complexities of increasing echelon and flexibiliyt of system structure associated with cross-docking.

The thesis is organized as follows: Section 2 provides a basic review for probability theory, and detailed review of the mathematical literature on supply chain optimization, with a focus on multi-echelon systems and cross-docking operations. Section 3 introduces the mathematical model and formulation for the problem at hand specifically the multi-echelon cross-docking model with number of echelon greater than two. Section 4 introduces some topic of interest for future investigation.

2. Preliminary and Foundational Theorems

The whole discussion of supply chain is based on tradeoff between holding cost and backorder cost, and the cost function is given as charging amount of extra stock with holding cost $h > 0$ and charging for inadequate stock for a missing demand with backorder cost $b > 0$, under the random demand. Demand can be well discribed as a random variable.

2.1. Fundamental Probability

First, let's introduce basic definition and theorem in in probability theory [5].

Definition 2.1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the measurable sample space. A **random variable** X is a $\mathcal{A}-$measurable function from Ω into the $\mathbb{R}$. The **probability distribution** $\mathbb{P}_X$ of X is defined as

$$\mathbb{P}_X(B) := \mathbb{P}(X \in B) \ \ \forall B \in \mathcal{B}(\mathbb{R}).$$

The **cumulative distribution** of X is defined as

$$F_X(a) = \mathbb{P}_X((-\infty, a))$$

Definition 2.2. A **probability density function** f_X of random variable X is a nonnegative borel measurable function from $\mathbb{R}$ to $\mathbb{R}_+$ s.t.

$$\int_R f(x)dx = 1$$

Then a real random variable X has density f if

$$\mathbb{P}_X(B) = \int_B f(x)dx.$$

Now, the expectation is defined as

Definition 2.3. Let X be a random variable defined on $(\Omega, \mathcal{A}, \mathbb{P})$. The **expectation** of a random variable X is

$$\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(d\omega)$$

The **variance** of X is defined as

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Now, the notion of independence between different random variables.

Definition 2.4. The finite set of random variables $(X_n)_{n=1}^N$, that are defined on the same probability space, are said to be **independence** if for any subset $I \subset \{1, \cdots, N\}$,

$$\mathbb{P}(\forall j \in I, X_j \in B_j) = \prod_{j \in I} \mathbb{P}(X_j \in B_j), \ \ \forall B_j \in \mathcal{B}(\mathbb{R})$$

Theorem 2.5. *Let X, Y be two random variables defined on the same probability space, then*

(1) $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ *if all three expectations are defined.*
(2) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
(3) $\mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2COV[X, Y]$ *where* $COV[X, Y] := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
(4) Suppose X, Y are independent, $COV[X, Y] = 0$

The proof of this thoerem can be found in multiple probability texts, I would omit the proof here.

The idea of conditional probability are useful under this context, and we shall define it.

Definition 2.6. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the measurable sample space, and $B \in \mathcal{A}$ is an event with positive probabiltiy, then, the **conditional probability** $\mathbb{P}_{\cdot|B}$ is defined, for every $A \in \mathcal{A}$

$$\mathbb{P}_{\cdot|B}(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Let X random variable in $L^1(\Omega, \mathcal{A}, \mathbb{P})$. The **conditional expectation** of X given B, $\mathbb{E}[X|B]$ is defined as

$$\mathbb{E}[X|B] := \frac{\mathbb{E}[X\mathbb{1}_B]}{\mathbb{P}(B)}$$

$\mathbb{E}[X\mathbb{1}_B]$ is X's expectation with respect to $\mathbb{P}_{\cdot|B}$ probability measure.

2.2. Multivariate Extremum

After defining the probability and random variable to describe the indeterministic demand, we shall introduce some useful methods in finding optimization solution under certain constraints. [6]

Theorem 2.7 (Lagrange Multipliers)**.** *If f be a function from $\mathbb{R}^n$ to $\mathbb{R}$ has a relative local extremum when it subjected to m equality constraints,*

$$g_1(\mathbf{x}) = c_1, \dots, g_m(\mathbf{x}) = c_m,\ m < n$$

Then, exists scalar $\lambda_1, \dots, \lambda_m$ such that

$$\nabla f(\mathbf{x}^*) = \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*)$$

which $\mathbf{x}^*$ is the local extremum.

The extension for lagrange multipliers is extremely useful under the optimization scheme.

Theorem 2.8 (Karush–Kuhn–Tucker (KKT) conditions [7])**.** *Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}$, constraint to l equality constraints and m inequality constraints. Equivalently,*

$$\begin{aligned} &\max_{\mathbf{x}} z = f(\mathbf{x}) \\ &s.t.\ \mathbf{x} \in \mathbb{R}^n \\ &\qquad h_k(\mathbf{x}) = b_k,\ \forall k = 1, 2, \dots, l \\ &\qquad g_i(\mathbf{x}) \le c_i,\ \forall i = 1, 2, \dots, m \end{aligned}$$

Then, the following conditions must hold at an optimum $\mathbf{x}^*$

(1) $\nabla f(\mathbf{x}^*) = \sum_{k=1}^{l} \lambda_k \nabla h_k(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*)$ for some $\lambda_k, \mu_i \in \mathbb{R}$.
(2) $h_k(\mathbf{x}^*) = b_k$ for all $k = 1, 2, \dots, l$, and $g_i(\mathbf{x}^*) \le c_i$ for all $i = 1, \dots, m$.
(3) $\mu_i \ge 0$ and $\mu_i(c_i - g_i(\mathbf{x}^*)) = 0$ for each $i = 1, 2, \dots, m$.

2.3. Basic Model Description

Back to the discussion about holding cost and backordering cost, the total cost is resulted from the indeterministic demand X be a nonnegative random variable with cumulative distribution F and the level of on-hand s. The actual cost function and expected cost function is described as

$$C(s) = h[s - x]^+ + b[x - s]^+ \tag{2.1}$$

$$G(s) := \mathbb{E}[C(s)] = h\mathbb{E}[(s - x)\mathbb{1}_{\{s \ge x\}}] + b\mathbb{E}[(x - s)\mathbb{1}_{\{s \le x\}}] \tag{2.2}$$

Assume that we have a f be the probability density function of X, then we have the following optimal solution to the minimize the expected cost. We can write the cost function (2.2) and write in terms of the density function.

Theorem 2.9 (Newsvendor Model in Continuous Demand)**.** *Let X be a real random variable with continuous density f, characterizing the demand, with continuous derivative. Let $h > 0$ be the unit holding cost, $b > 0$ be the unit backordering cost. Then, the optimal stock level $s^* \in \mathbb{R}_+$ of the cost function given by below*

$$\min_s\ G(s) = h \int_{-\infty}^{s} (s - x) f(x) dx + b \int_{s}^{\infty} (x - s) f(x) dx$$

satisfies $F(s^*) = \frac{b}{b+h}$.

[1] $[a]^+ := max\{a, 0\}$

To prove the newsvendor model, we need to use the following theorem about Leibniz's differentiation rules [8].

Theorem 2.10. *Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $F(x, t)$ and $\frac{\partial}{\partial x} F$ continuous over the $[u(t), v(t)] \times [a, b]$ for some $a, b \in \mathbb{R}$ and $u, v : \mathbb{R} \to \mathbb{R}$ are continuous and have continuous derivative on $t \in [a, b]$, then*

$$\frac{d}{dt} \int_{u(t)}^{v(t)} F(x,t)dx = F[v(t), t]v'(t) - F[u(t), t]u'(t) + \int_{u(t)}^{v(t)} \frac{\partial F(x,t)}{\partial t} dx$$

Proof. Let F, u, v be function as described, and let $G(u, v, t) = \int_{u(t)}^{v(t)} F(x, t)dx$. Then by chain rule,

$$\frac{\partial}{\partial t} G(u, v, t) = \frac{\partial G}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial G}{\partial v} \frac{\partial v}{\partial t} + \frac{\partial G}{\partial t}$$

By Fundamental Theorem of Calculus, the first two terms agree. Then, suffice to show $\frac{\partial G}{\partial t} = \int_{u(t)}^{v(t)} \frac{\partial F(x,t)}{\partial t} dx$.

$$\begin{aligned} \int_u^v \frac{\partial F(x,t)}{\partial t} dx &= \frac{d}{dt} \int_n^t ds \int_u^v \frac{\partial F(x,s)}{\partial s} dx \\ &= \frac{d}{dt} \int_n^t \int_u^v \frac{\partial F(x,s)}{\partial s} dx\ ds = (*) \end{aligned}$$

Since F, $\frac{\partial F(x,s)}{\partial s}$ are continuous over $[u, v] \times [a, b]$, the integral is boudned and apply Fubini's theorem.

$$\begin{aligned} (*) &= \frac{d}{dt} \int_u^v \int_n^t \frac{\partial F(x,s)}{\partial s} ds\ dx \\ &= \frac{d}{dt} \int_u^v [F(x,t) - F(x,n)]dx \\ &= \frac{d}{dt} \int_u^v F(x,t)dx \end{aligned}$$

□

We use the previous theorem2.10 to prove the Newsvendor calculation.

Proof. By Leibniz differentiation rules,

$$\frac{\partial}{\partial s} \int_{-\infty}^{s} (s-x)f(x)dx = [(s-s)f(s) - 0] + \int_{-\infty}^{s} f(x)dx = F(s)$$

$$\frac{\partial}{\partial s} \int_{s}^{\infty} (x-s)f(x)dx = [0 - (s-s)f(s)] - \int_{s}^{\infty} f(x)dx = F(s) - 1$$

Then,

$$\frac{\partial G}{\partial s} = hF(s) + b(F(s) - 1) = (h+b)F(s) - b = 0 \tag{2.3}$$

$$\Rightarrow F(s^*) = \frac{b}{h+b} \tag{2.4}$$

Further, since

$$\frac{\partial^2}{\partial s^2} G = \frac{d}{ds} F(s) = f(s) \geq 0$$

s^* provides the minimum. Further, when F takes value on nonnegative real, s^* is nonnegative. □

Under the discrete demand case with X a real random variable with probability mass function $p(x)$, Newsvendor solution has similar result.

Theorem 2.11. *(Newsvendor Model in Discrete Demand) Let X be a discrete random variable takes value in $\mathbb{Z}^*$* [1] *with probability mass function $p : \mathbb{Z}^* \to [0,1]$. Let $h > 0, b > 0$ be holding cost and backordering cost respectively. Then, the optimal stock level $s^* \in \mathbb{Z}^*$ to minimize the cost function 2.2 satisfies difference in expected cost function is positive.*

Proof. Consider the change of expected cost function 2.2, $\Delta G(s) := G(s+1) - G(s)$.

$$\begin{aligned}
\Delta G(s) =& G(s+1) - G(s) \\
=& \left[h \sum_{n=0}^{s+1} (s+1-n)p(n) + b \sum_{n=s+1}^{\infty} (n-s-1)p(n) \right] \\
&- \left[h \sum_{n=0}^{s} (s-n)p(n) + b \sum_{n=s}^{\infty} (n-s)p(n) \right] \\
=& h \left[(s-n)p(n)|_{n=s+1} + \sum_{n=0}^{s+1} p(n) \right] + b \left[-(n-s)p(n)|_{n=s} - \sum_{n=s+1}^{\infty} p(n) \right] \\
=& h \sum_{n=0}^{s} p(n) - b \sum_{n=s+1}^{\infty} p(n) \\
=& (h+b)F(s) - b
\end{aligned}$$

Here F is the cumulative distribution function of X, i.e. $F(s) = \mathbb{P}(X \leq s)$. Since the cost function is concave, and

$$G(s) = G(0) + \sum_{n=0}^{s-1} \Delta G(n)$$

Suppose exists $s^* \in \mathbb{Z}^*$ such that $F(s^*) = \frac{b}{h+b} \Leftrightarrow \Delta G(s^*) = 0$, then, $G(s^*) = G(s^*+1)$ are both minimized cost, and there are multiple optimal stock level. Suppose do not exist s^* an nonnegative integer that $\Delta G(s^*) = 0$, then s^* minimizes G implies $\forall s < s^*, \Delta G(s) < 0$, and $\forall s \geq s^*, \Delta G(s) > 0$. Then,

$$s^* := \min \{ n \in \mathbb{Z}^* | \Delta G(n) = (h+b)F(n) - b > 0 \} = \min \left\{ n | F(n) > \frac{b}{h+b} \right\}$$

□

2.4. Two-echelon Cross-docking Supply System

These content are mainly based on Professor Muckstadt's discussion [9]. The setting is as below, and the structure is shown on Figure 1. The system has a central warehouse (root node) that is responsible for allocating the on-hand demand into the lower streams. Suppose there are n affiliated retailers, denoted $r_1, \dots, r_n$.

Definition 2.12. The stock that is in the station are **on hand stocks**, the stock that has been assigned to the station, and sent to transit is called **pipeline stocks**.
The **inventory position** of a station is all the stocks this station i have on-hand, all the children

[1] $\mathbb{Z}^* := \{n \in \mathbb{Z} | n \geq 0\}$

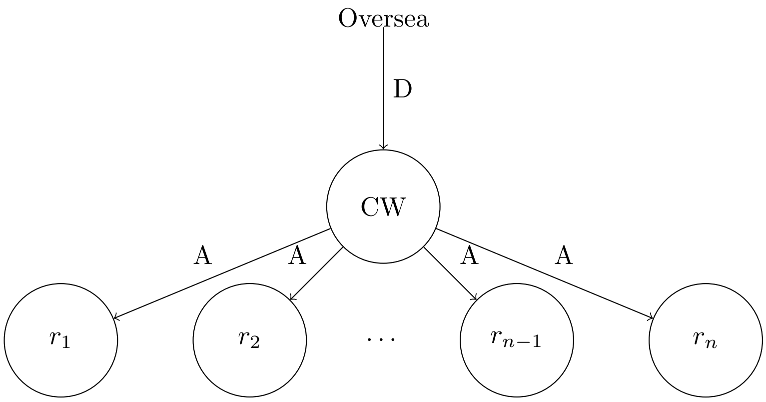

Figure 1. Two-Echelon System

stations' on-hand stock, and all the pipeline stock assigned to this station i and its children stations [2].

The assumptions of this model are

(1) For each station (CW and retailers), the holding cost is h per 1 unit, and the backorder cost occurs at retailer level with cost b per 1 unit accounts for the penalty of an order, but the stock from later day will satisfy the backorder from previous days[3], i.e all demand will be eventually fulfilled.
(2) Lead time: After CW order, there will be D day lead time to have the ordered stock arrive; the lead time of retailers are assume to be same, denoted A.
(3) Independent Normal Demand. Assume only one product is sold, and each retailer $j \in \{1, 2, \ldots, n\}$ has day demand $d_{j,t} \sim \mathcal{N}(\mu_j, \sigma_j^2)$ at time t, where $d_{j,t}$ are i.i.d over t, independent across different retailers.
(4) The CW always send order to the oversea suppliers by amount $\sum_{j=1}^{n} d_{j,t-1}$ to bring the system inventory position to a desired constant order-up-to level s at time t.
(5) Imbalance assumption holds, and the distribution strategy of intermediate state is such that the stock-out probability downstreams are equal. Equivalently, $\forall t \in \mathbb{Z}_+, \forall i, j \in \{1, 2, \ldots, n\}$

$$\mathbb{P}(s_{i,t} + x_{i,t} \leq \sum_{k=t}^{t+A} d_{i,k}) = \mathbb{P}(s_{j,t} + x_{j,t} \leq \sum_{k=t}^{t+A} d_{j,k})$$

$$\sum_{i=1}^{n} x_{i,t} = c, \quad x_{i,t} \geq 0 \; \forall i \in \{1, 2, \ldots, n\}$$

where $s_{i,t}$ is the inventory position of station i at time t. $x_{i,t}$ is the allocation made by CW at time t to r_i, c is the amount available to allocate. Note, since holding cost are the same across system, policy is leave no inventory upstream.
(6) The sequence of events happens at day t is described as: at the beginning of time t
    (a) CW make the order to supplier with $\sum_{j=1}^{n} d_{j,t-1}$. This order brings the inventory level of whole system back to s.

[2]For instance, inventory position of CW is all stock downstreams in the pipeline with lead time A and all stock on-hand for both CW and retailers and the intransite in the pipeline with lead time D.

[3]This can be replace by period. For simplicity, I will use day as a unit period of time.

(b) Order from CW to supplier a lead time D ago arrived CW by amount $\sum_{j=1}^{n} d_{j,t-D-1}$.
(c) CW will allocate on-hand stock to r_i based on the equal stockout probability policy(feasible by imbalance assumption). The amount allocated at time t is aim to balance the stock-out probability among r_i a lead time A later.
(d) Demand during time t occurs, and the total system stock level becomes $s-\sum_{j=1}^{n} d_{j,t}$ at the end of period t.
(e) Holding and backorder costs occur, the period ends.

Definition 2.13. $Y_0 := \sum_{t=1}^{D}\sum_j d_{j,t}$ is the total system demand over D time periods. $Y_j := \sum_{t=D+1}^{D+A+1} d_{j,t}$ is the retailer j demand over $A+1$ periods.

The summary of variable is shown on Table 1.

Variables	**Description**
h	Holding cost
b	Backordering cost
D	Lead time from oversea to CW
A	Lead time from CW to retailer r_i for $i \in 1, \cdots, n$
$d_{j,t}$	Demand of retailer j at time t
Y_0	Total System Demand over lead time period D
Y_j	Demand of r_j over lead time period $A+1$
s_t	CW's inventory position
$s_{j,t}$	r_j's inventory position
$x_{j,t}$	Stock allocated from CW to r_j at time t
s^*	Optimal order up-to level

TABLE 1. Summary of 2-echelon Model Variables

2.4.1. TWO-ECHELON SYSTEM AND IMBALANCE ASSUMPTION. The policy of distribution within the system is to balance the stock-out probability downstream. This leads to a lemma justifying the validity of such policy.

Lemma 2.14. *Let $h > 0$ be a holding cost and $b > 0$ be a backordering cost. Suppose there are n stations each with demand as nonnegative random variable with culmulative distribution F_i, and density f_i, with total of c elements able to distribute. Let each station has expected cost function 2.2. Let the total cost function*

$$G((s_1, s_2, \ldots, s_n)) = \sum_{i=1}^{n} G_i(s_i)$$

Then, $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ equalize the stock-out probability.

Proof. Denote $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ By the assumption, the following optimization problem is set up as below:

$$\min_{\mathbf{s}} G(\mathbf{s}) = \sum_{i=1}^{n} G_i(s_i) = \sum_{i=1}^{n} h\int_{-\infty}^{s_i}(s_i - x)f_i(x)dx + b\int_{s_i}^{\infty}(x - s_i)f_i(x)dx$$

$$\text{s.t.} \sum_{i=1}^{n} s_i \leq c$$

$$s_i \geq 0, \quad \forall i \in \{1, \ldots, n\}$$

Then, the KKT conditions in Theorem 2.8 gives

$$-\frac{\partial}{\partial s_i}\sum_{i=1}^{n} G_i = \lambda - \mu_i, \text{ for } \lambda \in \mathbb{R}, \quad \forall i \in \{1, \dots, n\} \tag{2.5}$$

$$\sum_{i=1}^{n} s_i \leq c, \; s_i \geq 0 \tag{2.6}$$

$$\mu_i \geq 0 \text{ and } \mu_i s_i = 0 \tag{2.7}$$

$$\lambda \geq 0 \text{ and } \lambda \left(c - \sum_{i=1}^{n} s_i \right) = 0, \quad \forall i \in \{1, \dots, n\} \tag{2.8}$$

Then, refering to the basic newsvendor cost function minimization, we have by (2.3)

$$\frac{\partial}{\partial s_i}\sum_{i=1}^{n} G_i(s_i) = \frac{d}{ds_i} G_i(s_i) = (h+b)F_i(s_i) - b \tag{2.9}$$

First, suppose that $\frac{\partial}{\partial s_i} G_i = 0$ for all i and $\sum_{i=1}^{n} F_i^{-1}\left(\frac{b}{b+h}\right) \leq c$. Thus,

$$\lambda = \mu_i = \mu_j, \; \forall i, j \in \{1, 2, \dots, n\}$$

Then, let $s_i^* = F_i^{-1}(\frac{b}{b+h})$, second condition (2.6) of feasibility has been satisfied. Set $\lambda = 0$, then condition 3 (2.7) and condition 4 (2.8) is trivially satisfied. By previous calculation, condition 1 (2.5) is satisfied. Then, $\mathbf{s} = (s_1^*, \dots, s_n^*)$ is the optimal soltion. Since stock-out probability under this optimal stock allocation has $\frac{b}{b+h}$, the allocation is balanced.

Now, suppose that $\sum_{i=1}^{n} \hat{s}_i > c$ where $\hat{s}_i = F^{-1}\left(\frac{b}{b+h}\right)$. Then, $s_i^* > 0 \Rightarrow \; \mu_i = 0$ by equation (by convexity), and by (2.9)

$$\frac{d}{ds_i} G_i(s_i) = (h+b)F_i(s_i) - b = \mu_i - \lambda = -\lambda \tag{2.10}$$

Thus,

$$\sum_{i=1}^{n} s_i^* = c \; \Rightarrow \lambda \text{ free.}$$

Thus, each station equalize their stock-out probability. □

The imbalance assumption states that if the stock-out probability for each is the same at time $t+D-1$, then there are always sufficent for CW to make allocation to r_i such that the total stock-out probability is same across r_i at time $t+D$. Particularly, the assumption is met given the following theorem.

Theorem 2.15. *Suppose the system is in balance at the beginning of period $t+D-1$. Then, it will be in balance following the CW allocation to the retailers in period $t+D$ if*

$$\sum_{i=1}^{n} d_{i,t-1} \geq \max_j \left\{ \sum_{i \neq j} d_{i,t+D-1} + d_{j,t+D-1}\left(1 - \frac{\sum_{k=1}^{n} \sigma_k}{\sigma_j}\right) \right\}$$

Proof. Suppose the system is in balance at the beginning of period $t+D-1$, and demand is normally distributed at each r_i with mean μ_i and standard deviation σ_i, then there exists a $k \in \mathbb{R}$ such that

$$s_{i,t+D-1} = (A+1)\mu_i + k\sqrt{A+1}\sigma_i, \forall i = 1, 2, \dots, n$$

where $s_{i,t+D-1}$ is defined to be the inventory position of r_i at the beginning of time $t+D-1$.

During period $t+D-1$,

(1) The order placed by CW to supplier in period t, by the total demand depleted at period $t-1$ arrived. The total amount arrived can be written as $\sum_i d_{i,t-1}$. Since the CW do not keep inventory, the total amount allocated to r_i is equal to the amount arrived in CW.
(2) Demand occurred by $d_{i,t+D-1}$ for each retailer r_i.

Suppose $x_{i,t+D}$ units are the allocation from CW to r_i in period $t+D$, then the inventory position for r_i is

$$s_{i,t+D} = \underbrace{s_{i,t+D-1}}_{\substack{\text{Inventory position}\\ \text{a day before}}} + \underbrace{x_{i,t+D}}_{\substack{\text{Allocated to}\\ r_i\text{to balance}}} - \underbrace{d_{i,t+D-1}}_{\substack{\text{demand lost}\\ \text{a day before}}} \tag{2.11}$$

$$= (A+1)\mu_i + k\sqrt{A+1}\sigma_i + x_{i,t+D} - d_{i,t+D-1} \tag{2.12}$$

Then, feasible allocation requires $x_{i,t+D} \geq 0$ for all i, satisfying

$$\sum_{i=1}^{n} x_{i,t+D} = \sum_{i=1}^{n} d_{i,t-1} \tag{2.13}$$

$$\exists k' \in \mathbb{R} \text{ s.t. } s_{i,t+D} = (A+1)\mu_i + k'\sqrt{A+1}\sigma_i, \forall i = 1,2,\ldots,n \tag{2.14}$$

Plugging (2.14) into (2.12), we have the following

$$x_{i,t+D} = (k'-k)\sqrt{A+1}\sigma_i + d_{i,t+D-1}, \forall i = 1,2,\ldots,n. \tag{2.15}$$

Based on (2.13),

$$\sum_{i=1}^{n} x_{i,t+D} = \sum_{i=1}^{n} d_{i,t-1} = (k'-k)\sqrt{A+1}\sum_{i=1}^{n}\sigma_i + \sum_{i=1}^{n} d_{i,t+D-1}$$

Solve for $(k'-k)$, we have

$$k'-k = \frac{\sum_{i=1}^{n} d_{i,t-1} - \sum_{i=1}^{n} d_{i,t+D-1}}{\sqrt{A+1}\sum_{i=1}^{n}\sigma_i} \tag{2.16}$$

Substitue (2.16) to (2.15), we have

$$x_{i,t+D} = \left(\sum_{i=1}^{n} d_{i,t-1} - \sum_{i=1}^{n} d_{i,t+D-1}\right)\frac{\sigma_i}{\sum_{j=1}^{n}\sigma_j} + d_{i,t+D-1} \tag{2.17}$$

Set (2.17) be nonnegative, we have

$$\sum_{i=1}^{n} d_{i,t-1} \geq \sum_{i=1}^{n} d_{i,t+D-1} - d_{i,t+D-1}\frac{\sum_{j=1}^{n}\sigma_j}{\sigma_i}, \forall i.$$

Thus, all $x_{i,t+D}$ are nonnegative if

$$\sum_{i=1}^{n} d_{i,t-1} \geq \max_j \left\{\sum_{i\neq j} d_{i,t+D-1} + d_{j,t+D-1}\left(1 - \frac{\sum_{i=1}^{n}\sigma_i}{\sigma_j}\right)\right\}.$$

□

Proposition 2.16. *Suppose retailers following the previous assumption, and the coefficient of variation is sufficiently small, the equal-probability allocation can be made almost surely, given the previous period is equalized.*

Proof. Let d_i denote the demand of retailer i. x_i be the allocation made from CW to the retailer i. Then,

$$\mathbb{P}(x_i \geq 0, \forall i) \geq 1 - \mathbb{P}(\exists j \in \{1, \cdots, n\} | x_j < 0) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(x_i < 0)$$

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{x_i \geq 0\}\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(x_i < 0)$$

Then with $d_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ and $\frac{\sigma_i}{\mu_i} < \epsilon$ for some small $\epsilon_i > 0$, [4]

$$\mathbb{P}(x_i < 0) = \mathbb{P}\left(\frac{x_i - \mu_i}{\sigma_i} < -\frac{\mu_i}{\sigma_i}\right) = \Phi\left(-\frac{\mu_i}{\sigma_i}\right) = \Phi\left(-\frac{1}{\epsilon_i}\right)$$

Then,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{x_i \geq 0\}\right) \geq 1 - \sum_{i=1}^{n} \Phi\left(-\frac{1}{\epsilon_i}\right) \to 1$$

as $\epsilon_i \to 0$ by the characteristic of cumulative distribution. ☐

2.4.2. Describing the optimal inventory position. Now, we have sufficient information to claim a optimal inventory level that CW should bring up each time.

Theorem 2.17. *Suppose the model satisfies previous assumptions and coefficient of variation of demand is small, then the optimal stock level for the whole system is*

$$s^* = (D + A + 1) \sum_{i=1}^{n} \mu_i + z \left(D \sum_{i=1}^{n} \sigma_i^2 + (A+1) \left(\sum_{i=1}^{n} \sigma_i \right)^2 \right)^{\frac{1}{2}}$$

To prove this theorem, we shall start from auxiliary definition and proposition.

Proposition 2.18. *Let $F_{v_j}(\cdot)$ be the cumulative distribution function of random variable defined by $v_j = Y_j + Y_0 \frac{\sigma_j}{\sum_i \sigma_i}$. Then, v_j is normal with*

$$\text{mean of } (A+1)\mu_j + D \sum_{i=1}^{n} \mu_i \frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i}$$

$$\text{and variance } (A+1)\sigma_j^2 + D \sum_{i=1}^{n} \sigma_i^2 \left(\frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i} \right)^2$$

Proof. First calculate the expectation of v_j.

$$\begin{aligned}
\mathbb{E}(v_j) &= \mathbb{E}\left(Y_j + Y_0 \frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i} \right) \\
&= \mathbb{E}(Y_j) + \frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i} \mathbb{E}(Y_0) \\
&= (A+1)\mu_j + D \sum_{i=1}^{n} \mu_i \frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i}
\end{aligned}$$

[4]Note the normal assumption is not required here, and can be extend to broderer cases.

Then, the variance of v_j. To be noted, each demand variable are independent across time.

$$\begin{aligned}\mathrm{Var}(v_j) &= \mathrm{Var}\left(Y_j + Y_0\frac{\sigma_j}{\sum_{i=1}^n \sigma_i}\right)\\ &= \mathrm{Var}(Y_j) + \mathrm{Var}\left(\sum_{t=1}^{D}\sum_{i=1}^{n} d_{i,t}\right)\left(\frac{\sigma_j}{\sum_{i=1}^n \sigma_i}\right)^2 + \mathrm{Cov}\left(\sum_{t=D+1}^{D+A+1} d_{j,t}, \frac{\sigma_j}{\sum_{i=1}^n \sigma_i}\sum_{t=1}^{D}\sum_{i=1}^{n} d_{i,t}\right)\\ &= (A+1)\sigma_j^2 + D\sum_{i=1}^{n}\sigma_i^2\left(\frac{\sigma_j}{\sum_{i=1}^n \sigma_i}\right)^2\end{aligned}$$

All covariance is zero since they are independent. □

Now, we can proof for the theorem.

Proof of Theorem 2.17. The net system inventory in period $D+1$ prior to allocating CW stock is $s - Y_0$.

Note: This only includes system stocks at CW and downstream(i.e., supplier to CW pipeline stocks are excluded). Since here we fiexed at a time t, the following will write $s_{i,t} = s_i$ for simplicity. After allocating CW stock, all stock downstream from CW is allocated to retailers, thus $\sum_{j=1}^n s_j = s - Y_0$. By the imbalance assumption, there is a single $k \in \mathbb{R}$ such that

$$s_j = (A+1)\mu_j + k(A+1)^{1/2}\sigma_j,\ \forall j \in \{1, 2, \ldots, n\}$$

Using $\sum_{j=1}^n s_j = s - Y_0$, solve for k and

$$s_j = (A+1)\mu_j + \left(s - Y_0 - (A+1)\sum_{i=1}^{n}\mu_i\right)\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} \geq 0.$$

Let

$$c_j = (A+1)\mu_j - (A+1)\sum_{i=1}^{n}\mu_i\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} \tag{2.18}$$

$$v_j = Y_j + Y_0\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} \text{ and} \tag{2.19}$$

$$z_j = s_j - Y_j = r_j\text{'s net inventory at the period's end.} \tag{2.20}$$

With substitution,

$$z_j = s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} + c_j - v_j$$

Notice that among c_j, v_j, only v_j is a composition of random variables. And therefore, randomness in z_j only come from demands in v_j. Then, $z_j < 0$ is a stockout, with backorder cost $-bz_j$, if $z_j > 0$ is a stock-in with holding cost hz_j. Refering back to the original newsvendor model2.9, we have

$$G_j\left(s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} + c_j\right)|\{F = F_{v_j}\}$$

Since F_{v_j} is a normal distribution by proposition 2.18, the expected cost per period for r_j is given by

$$h\int_{-\infty}^{s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i}+c_j}\left(s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} + c_j - v\right)dF_{v_j}(v) + b\int_{s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i}+c_j}^{\infty}\left(v - s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} - c_j\right)dF_{v_j}.$$

By the newsvendor minimization in theorem2.9, the minimum cost occuring when

$$F_{v_j}\left(s\frac{\sigma_j}{\sum_{i=1}^n \sigma_i} + c_j\right) = \frac{b}{b+h}$$

Then, denote $\Phi^{-1}\left(\frac{b}{b+h}\right) = z$, where Φ is the inverse standard normal distribution. Then, since the random variable v_j is normal,

$$\begin{aligned} z &= \frac{s\frac{\sigma}{\sum_{i=1}^{n}\sigma} + c_j - \mathbb{E}(v_j)}{\sigma v_j} \\ &= \frac{s - (D+A+1)\sum_{i=1}^{n}\mu_i}{[(A+1)(\sum_{i=1}^{n}\sigma_i)^2 + D\sum_{i=1}^{n}\sigma_i^2]^{\frac{1}{2}}} \end{aligned}$$

Note that z is independent of r_j, so s^* can be solved as

$$s^* = (D+A+1)\sum_{i=1}^{n}\mu_i + z\left(D\sum_{i=1}^{n}\sigma_i^2 + (A+1)\left(\sum_{i=1}^{n}\sigma_i\right)^2\right)^{\frac{1}{2}}$$

□

3. Main Discussion: Three-echelon System with equalizing stock-out probability policy

With the analysis of multi-echelon model in previous section, it is natural to consider the more complex system with similar assumption and strategies. Therefore, the main discussion of this thsis is

(1) What does the generalized structure's optimal inventory level (s^*) under the equalized-allocation for each intermediate echelon?
(2) What is the condition for generalized structure to have imbalance assumption?
(3) What is the probability of breaking the imbalance assumption? What is the strategy under breaking the assumption?

First, we shall start with the three-echelon system, which builds heavily on the two-echelon model. As shown below in Figure 2, the three-echelon system with oversea supplier send product to the central warehouse with D. Suppose that there are m intermediate stations r_i, $i \in \{1, \ldots, m\}$ that recieve assigned stock from CW. Each intermediate station r_i has n_i lower stream retailers r_{ij}, $j \in \{1, \ldots, n_i\}$. These retailers $r_{i\cdot}$ only receive stock sent by r_i.

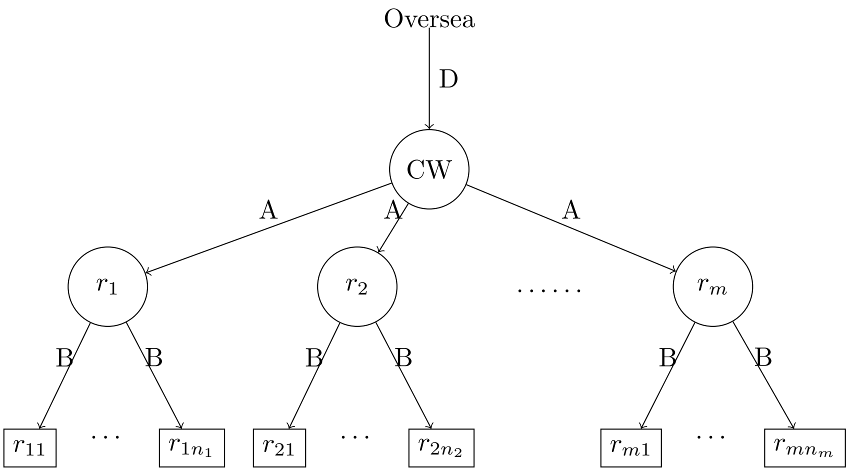

Figure 2. Three-Echelon Cross-Docking System

3.1. Introduction and assumptions of three-echelon model

The summary of variables is shown on Table 2.

Variables	Description
h	Holding cost
b	Backordering cost
D	Lead time from oversea to CW
A	Lead time from CW to retailer r_i for $i \in 1, \cdots, m$
B	Lead time from r_i to retailer r_{ij} for $j \in 1, \cdots, n_i$ for all i
$d_{ij,t}$	Demand of retailer r_{ij} at time t
Y_0	Total System (CW and downstream) Demand over lead time period D
Y_i	Demand of r_i (r_i and its retailers) over lead time period A
Y_{ij}	Demand of retailer r_{ij} over lead time period $B+1$
s	System order up-to level
s_t	CW's inventory position
$s_{i,t}$	r_i's inventory position
$s_{ij,t}$	r_{ij}'s inventory position
$x_{i,t}$	Stock allocated from CW to r_i at time t
$x_{ij,t}$	Stock allocated from r_i to r_{ij} at time t

Table 2. Summary of 3-echelon Model Variables

The assumptions of this model are

(1) For each station (CW, intermediate stations, and retailers), the holding cost is h per 1 unit, and the backorder cost occurs at retailer level with cost b per 1 unit.
(2) The lead time between supplier and CW is D; lead time between CW to r_i is A for all $i \in \{1, \ldots, m\}$; and lead time between r_i to r_{ij} retailers denoted B for all i, j.
(3) Assume only one product is sold throughout system, retailer r_{ij} has day demand $d_{ij,t} \sim \mathcal{N}(\mu_{ij}, \sigma_{ij}^2)$ at time t, where $d_{ij,t}$ are i.i.d over all t, i, j.
(4) Let $s \in \mathbb{R}^*$ be the system order-up-to level. At the end of everyday, CW send order to the oversea suppliers by amount $\sum_{i=1}^{m}\sum_{j=1}^{n_i} d_{ij,t-1}$ to bring the system inventory position to a desired constant order-up-to level s at time t.
(5) Suppose imbalance assumption holds for each level, the distribution strategy of intermediate stations and CW is such that the stock-out probability downstreams are equal.
(6) The sequence of events happens at day t is described as:
At the beginning of time t
   (a) CW make the order to supplier with $\sum_{j=1}^{n} d_{j,t-1}$. Supplier directly send corresponding amount to the pipeline. This order brings the inventory level s_t of whole system back to s.
   (b) Order from CW to supplier a lead time D ago arrived CW by amount
$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} d_{ij,t-1}$$
For each station downstream, the order from a lead time ago arrived. For r_i, the allocation $x_{i,t-A}$ arrived. For r_{ij}, the allocation $x_{ij,t-B}$ arrived.
   (c) CW will allocate the on-hand to r_i based policy of equalizing stockout probability downstream. The amount allocated at time t is denoted as $x_{i,t}$

(d) Demand during time t occurs, and the total system stock level becomes

$$s - \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij,t}$$

at the end of period t.

(e) Holding and backorder costs occur, the period ends.

Definition 3.1. Define the demand over lead time. $Y_0 := \sum_{t=1}^{D} \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij,t}$ is the total system demand over D time periods. $Y_i := \sum_{t=D+1}^{D+A} \sum_{j=1}^{n_i} d_{ij,t}$ is the intemediate station i and affiliated demand over A time periods. $Y_{ij} := \sum_{t=D+A+1}^{D+A+B+1} d_{ij,t}$ is the downstream retailer r_{ij}'s total demand over lead time $B + 1$.

Suppose s is total system level after order in the morning made to overseas suppliers but before the day's demands are incurred downstream. On day t, an order $\sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij,t-D-1}$ is received at CW, which must be allocated to branch $i = 1 \cdots m$.

Suppose no inventory carried on CW, then after the allocation to the m stations is made and before the demand occurs, we have

$$\sum_{i=1}^{m} s_i = s - Y_0$$

Similarly, the lowerstream follows similarly. that

$$\sum_{j=1}^{n_i} s_{ij,t} = s_{i,t-A} - Y_i$$

3.2. Imbalance Assumption for 3-echelon System

Theorem 3.2. *Suppose the system is in balance at beginning of $t+D+A-1$ and previous days, then there will be in balance following for each allocation to lower stream in period $t + D + A$ if*

$$\begin{aligned}&\left(\sum_{j=1}^{n_i} d_{ij,t+D-1} - \sum_{j=1}^{n_i} d_{ij,t+D+A-1}\right) \frac{\sigma_{ij}}{\sum_{j=1}^{n_i} \sigma_{ij}} + \\ &\left[\sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij,t-1} - \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij,t+D-1}\right] \frac{\sigma_{ij}}{\sum_{j=1}^{n_i} \sigma_{ij}} \frac{\Sigma_i}{\sum_{i=1}^{m} \Sigma_i} + d_{ij,t+D+A-1}\end{aligned} \geq 0 \tag{3.1}$$

where

$$\Sigma_i^2 = A \sum_{j=1}^{n_i} \sigma_{ij}^2 + (B+1) \left(\sum_{j=1}^{n_i} \sigma_{ij}\right)^2$$

Proof. We use the very similar analysis as imbalance analysis in 2-echelon case. Since, by assumption, system is balance at time $t + D + A - 1$, then for any $i \in \{1, \dots, m\}$ exists $k_i \in \mathbb{R}$ such that

$$\begin{aligned} s_{ij,t+D+A-1} &= (B+1)\mu_{ij} + k_i \sqrt{B+1}\sigma_{ij} \\ s_{ij,t+D+A} &= s_{ij,t+D+A-1} + x_{ij,t+D+A} - d_{ij,t+D+A-1} \end{aligned}$$

for all $j \in \{1, \dots, n_i\}$. Fixed an i, and suppose system is balance at $t + D + A$, then it satsifies that

(1) $x_{ij,t+D+A} \geq 0$ for all $j \in \{1, \dots, n_i\}$

(2) Exists $k_i' \in \mathbb{R}$ such that for all j

$$s_{ij,t+D+A} = (B+1)\mu_{ij} + k_i'\sqrt{B+1}\sigma_{ij}$$

(3) $\sum_{j=1}^{n_i} x_{ij,t+D+A} = x_{i,t+D}$ stock arrived at r_{ij} in $t+D+A$ is the total allocation from r_i made at $t+D$, and these sum up to the total allocation received.

This is equivalent to

$$s_{ij,t+D+A} \Rightarrow x_{ij,t+D+A} = (k_i' - k_i)\sqrt{B+1}\sigma_{ij} + d_{ij,t+D+A-1}$$

Then, we can sum all up and get

$$x_{i,t+D} = \sum_{j=1}^{n_i} x_{ij,t+D+A} = (k_i' - k_i)\sqrt{B+1}\sum_{j=1}^{n_i}\sigma_{ij} + \sum_{j=1}^{n_i} d_{ij,t+D+A-1}$$

Solve for $(k_i' - k_i)$, we have expression of $x_{ij,t+D+A}$

$$\left(x_{i,t+D} - \sum_{j=1}^{n_i} d_{ij,t+D+A-1}\right)\frac{\sigma_{ij}}{\sum_{j=1}^{n_i}\sigma_{ij}} + d_{ij,t+D+A-1} \geq 0 \tag{3.2}$$

being the condition for r_i to balance downstreams.

Then, use the very similar approach, consider the CW's allocation to balance downstreams. Let $(F_i)_{i=1}^m$ denote the synthesized demand over lead time and policy for $(r_i)_{i=1}^m$ consider its downstreams. Let M_i, Σ_i^2 denote mean and variance of F_i. Since the allocation made at time $t+D+A-1$ at CW enable r_i to balance at time $t+D+2A-1$, (), we can write F_i following normal distribution with

$$M_i = (A+B+1)\sum_{j=1}^{n_i}\mu_{ij}$$

$$\Sigma_i^2 = A\sum_{j=1}^{n_i}\sigma_{ij}^2 + (B+1)\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2$$

Then, suppose the system is balanced yesterday, there exists a $q \in \mathbb{R}$ such that for all $i \in \{1, \dots, m\}$

$$s_{i,t+D+A-1} = M_i + q\Sigma_i$$

$$s_{i,t+D+A} = s_{i,t+D+A-1} + x_{i,t+D+A} - \sum_{j=1}^{n_i} d_{ij,t+D+A-1}.$$

Suppose system is balance at $t+D+A$, then it satsifies that

(1) $x_{i,t+D+A} \geq 0$ for all $i \in \{1, \dots, m\}$
(2) Exists $q' \in \mathbb{R}$ such that for all i

$$s_{i,t+D+A} = M_i + q'\Sigma_i$$

(3) $\sum_{i=1}^m x_{i,t+D+A} = \sum_i^m \sum_{j=1}^{n_i} d_{ij,t+A-1}$

This is equivalent to

$$
\begin{aligned}
x_{i,t+D+A} &= (q'-q)\Sigma_i + \sum_{j=1}^{n_i} d_{ij,t+D+A-1} \\
\xRightarrow{\text{Sum over } i} \sum_{i=1}^{m} x_{i,t+D+A} &= (q'-q)\sum_{i=1}^{m}\Sigma_i + \sum_{i=1}^{m}\sum_{j=1}^{n_i} d_{ij,t+D+A-1} \\
&= \sum_{i}^{m}\sum_{j=1}^{n_i} d_{ij,t+A-1}
\end{aligned}
$$

Solve for $(q'-q)$, and force the allocation be feasible, we have

$$
x_{i,t+D+A} = \sum_{j=1}^{n_i} d_{ij,t+D+A-1} + \left(\sum_{i}^{m}\sum_{j=1}^{n_i} d_{ij,t+A-1} - \sum_{i}^{m}\sum_{j=1}^{n_i} d_{ij,t+D+A-1}\right)\frac{\Sigma_i}{\sum_{i=1}^{m}\Sigma_i} \geq 0 \tag{3.3}
$$

being the condition for CW to make feasible allocation that equalize downstream stockout probability.

Since $x_{i,t+D}$ in (3.2) is the allocated made by CW before $t+D+A-1$, (3.3) holds. Then, (3.2) with substitution gives

$$
\begin{aligned}
x_{i,t+D} = &\left(\sum_{j=1}^{n_i} d_{ij,t+D-1} - \sum_{j=1}^{n_i} d_{ij,t+D+A-1}\right)\frac{\sigma_{ij}}{\sum_{j=1}^{n_i}\sigma_{ij}} + \\
&\left[\sum_{i=1}^{m}\sum_{j=1}^{n_i} d_{ij,t-1} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} d_{ij,t+D-1}\right]\frac{\sigma_{ij}}{\sum_{j=1}^{n_i}\sigma_{ij}}\frac{\Sigma_i}{\sum_{i=1}^{m}\Sigma_i} + d_{ij,t+D+A-1}
\end{aligned} \geq 0 \tag{3.4}
$$

□

3.3. Analyzing 3-echelon System

Given that System has determined a policy internally which is to allocation such that downstreams are always have same stock-out probability. The optimal order-up-to level s^* can be found depending on the costs, lead time, and demand distribution.

Theorem 3.3. *Let system be a 3-echelon cross-docking system satisfying assumption in section* 3.1*, and assume the imbalance assumption hold, then the optimal order-up-to level is*

$$
\begin{aligned}
s^* = (A+B+D+1)\sum_{i=1}^{m}\mu_i + z\sqrt{(A+B+1)(\sum_{i=1}^{n}\Lambda_i)^2 + D\sum_{i=1}^{m}\Lambda_i^2} \\
= (A+B+D+1)\left(\sum_{i=1}^{m}\sum_{j=1}^{m_i}\mu_{ij}\right) \\
+ z\left\{\left(\sum_{i=1}^{m}\left[(B+1)\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2 + A\sum_{j=1}^{n_i}\sigma_{ij}^2\right]^{1/2}\right)^2\right. \\
\left. +D\left((B+1)\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2\right] + A\left[\sum_{i=1}^{n}\sum_{j=1}^{n_i}\sigma_{ij}^2\right]\right)\right\}^{1/2}
\end{aligned}
$$

where for any $i \in \{1, \ldots, m\}$,

$$\Lambda_i = \sqrt{(B+1)\left(\sum_{j=1}^{n_i} \sigma_{ij}\right)^2 + A\sum_{j=1}^{n_i} \sigma_{ij}^2}$$

Proof. Assume the assumption about 3-echelon model holds, then fixed time start at t. Suppose the allocation has finished and system proceeds

Step 1. At time $\mathbf{t + A}$

$(r_i)_{i=1}^m$ receive stocks $(x_{i,t})_{i=1}^m$. By no inventory leave upstream, and equalizing stockout probability strategy for retailers $(r_{ij})_{j=1}^{n_i}$, this allocation made by r_i at time t aims to equalize the stockout probability under time $t+A+B$. With imbalance assumption, the common stockout probability $p_i \in [0, 1]$ at retailer r_{ij} among r_i in terms of s_i is:

$$p_i{}^5 = \mathbb{P}\left(\frac{\nu_{ij} - \mathbb{E}(\nu_{ij})}{\mathbb{V}(\nu_{ij})} > \frac{s_i - (A+B+1)\sum_{j=1}^{n_i} \mu_{ij}}{\sqrt{(B+1)\left(\sum_{j=1}^{n_i} \sigma_{ij}^2\right) + A\sum_{j=1}^{n_i} \sigma_{ij}^2}}\right) \tag{3.5}$$

where ν_{ij} is the random component contribute to the retail level applied in the Newsvendor model in 2-echelon model. Specifically, refering to Proposition 2.18 with $v_{ij} = Y_{ij} + Y_i \frac{\sigma_{ij}}{\sum_{j=1}^{n_i} \sigma_{ij}}$

Then, for $\hat{z} \sim \mathcal{N}(0,1)$

$$\begin{aligned} p_i = \mathbb{P}(Stockout) &= \mathbb{P}\left(\hat{z} > \frac{s_i - (A+B+1)\sum_{j=1}^{n_i} \mu_{ij}}{\sqrt{(B+1)(\sum_{j=1}^{n_i} \sigma_{ij}^2) + A\sum_{j=1}^{n_i} \sigma_{ij}^2}}\right) \\ &= 1 - \Phi\left(\frac{s_i - (A+B+1)\sum_{j=1}^{n_i} \mu_{ij}}{\sqrt{((B+1)(\sum_{j=1}^{n_i} \sigma_{ij}^2) + A\sum_{j=1}^{n_i} \sigma_{ij}^2)}}\right) \end{aligned}$$

Step 2. At time $\mathbf{t + A + B}$ Define $G_{ij} :=$ expected cost at a retailer r_{ij} at time $t+A+B$. $G_{lk} :=$ expected cost at a retailer r_{lk} at time $t + A + B$. Then, according to the same decomposition of demand for r_i to r_{ij} with the 2-echelon model in equation (2.18 and 2.19), define $\omega_{ij} := s_i \frac{\sigma_{ij}}{\sum_j \sigma_{ij}} + c_{ij}$, and $c_{ij} = \mathbb{E}(\nu_{ij}) - (A+B+1)\sum_j \mu_{ij} \frac{\sigma_{ij}}{\sum_j \sigma_{ij}}$. The cost function is described as

$$G_{ij} = h\int_{-\infty}^{\omega_{ij}} (\omega_{ij} - \nu_{ij}) dF\nu_{ij} + b\int_{\omega_{ij}}^{\infty} (\nu_{ij} - \omega_{ij}) dF\nu_{ij}$$

If take the derivative of G_{ij} respect to s_i, we have

$$\frac{\partial G_{ij}}{\partial s_i} = \frac{\partial G_{ij}}{\partial \omega_{ij}} \frac{\partial \omega_{ij}}{\partial s_i} \tag{3.6}$$

$$= \frac{\sigma_{ij}}{\sum_{j=1}^{n_i} \sigma_{ij}} \left[(h+b)F_{\nu_{ij}}(\omega_{ij}) - b\right] \tag{3.7}$$

Claim: optimal allocation of CW equalize downstream stock-out probability.

---

[4] Since we have fixed time, the time scripts are omitted when it is clear.

[5] $p_i = \mathbb{P}(\text{stockout at retailer } r_{ij} \text{ at time } t + A + B)$

Define $G_i := \sum_{j=1}^{n_i} G_{ij}$. Then, for any $i \in \{1, \ldots, m\}$,

$$\begin{aligned}\frac{\partial G_i}{\partial s_i} &= \sum_{j=1}^{n_i} \frac{\partial G_{ij}}{\partial s_i} \\ &= (h+b)\left[\Phi\left(\frac{s_i - (A+B+1)\sum_{j=1}^{n_i}\mu_{ij}}{\sqrt{(B+1)(\sum_j \sigma_{ij}^2) + A\sum_j \sigma_{ij}^2)}}\right)\right] - b\end{aligned}$$

Notice by equation(3.7),

$$\frac{\partial G_i}{\partial s_i} = \cdots = \frac{\partial G_j}{\partial s_j} = 0$$

The optimization of whole system is

$$\begin{aligned}\min_{\mathbf{s}} \ &\sum_{i=1}^{m} G_i(s_i) \\ \text{s.t.} \ &\sum_{i=1}^{m} s_i = s - Y_0\end{aligned}$$

where $\mathbf{s} = (s_1, \ldots, s_m)$. Then by lagrange multiplier in Theorem 2.7,

$$\frac{\partial}{\partial s_i}\left(\sum_{i=1}^{m} G_i\right) = \lambda$$

for all $i = 1, \cdots, m$, where λ is a lagrange multiplier.

This implies optimal allocation satisfies equal stock-out probability.

Step 3. Solving for s. Under imbalance assumption, there exists a $J \in \mathbb{R}$ such that

$$s_i = (A+B+1)\sum_{j=1}^{n_i}\mu_{ij} + J\sqrt{(B+1)\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2 + A\sum_{j=1}^{n_i}\sigma_{ij}^2} \quad \forall i \in \{1, \cdots, m\}$$

Define

$$\begin{aligned}\mu_i &= \sum_{j=1}^{n_i}\mu_{ij} \\ \Lambda_i &= \sqrt{(B+1)\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2 + A\sum_{j}\sigma_{ij}^2}.\end{aligned}$$

Then, $s - Y_0 = \sum_{i=1}^{m} s_i$, we solve for

$$J = \frac{s - Y_0 - (A+B+1)(\sum_{i=1}^{m}\mu_i)}{\sum_{i=1}^{m}\Lambda_i}$$

Now s_i can be express as the following,

$$s_i = (A+B+1)\mu_i + \frac{s - Y_0 - (A+B+1)(\sum_{i=1}^{m}\mu_i)}{\sum_{i=1}^{m}\Lambda_i}\Lambda_i$$

Apply Newsvendor optimization for s in Theorem 2.9,

$$F_{\nu_i}\left(s\frac{\Lambda_i}{\sum_{i=1}^m \Lambda_i}+c_i\right)=F_{\nu_k}\left(s\frac{\Lambda_k}{\sum_{i=1}^m \Lambda_i}+c_k\right),$$

where

$$c_i=(A+B+1)\left[\mu_i-\left(\sum_{i=1}^m \mu_i\right)\frac{\Lambda_i}{\sum_{i=1}^m \Lambda_i}\right]$$

and ν_i is normally distributed with

$$\mathbb{E}[\nu_i]=(A+B+1)\mu_i+D\left(\sum_{i=1}^m \mu_i\right)\frac{\Lambda_i}{\sum_{i=1}^m \Lambda_i}$$

and

$$\mathbb{V}(\nu_i)=\left[(A+B+1)(\sum_{i=1}^m \Lambda_i)^2+D\sum_{i=1}^m \Lambda_i^2\right]\left(\frac{\Lambda_i}{\sum_{i=1}^m \Lambda_i}\right)^2.$$

Note that we used the same decomposition of dynamic flow as in equation 2.18 and equation 2.19. Then,

$$\begin{aligned}\mathbb{P}(\text{stock-out at } r_i)&=\mathbb{P}\left(\nu_i>s\frac{\Lambda_i}{\sum_{i=1}^m \Lambda_i}+c_i\right)\\&=\mathbb{P}\left(\hat{z}>\frac{s-(A+B+D+1)(\sum_{i=1}^m \mu_i)}{\sqrt{(A+B+1)(\sum_{i=1}^m \Lambda_i)^2+D\sum_{i=1}^m \Lambda_i^2}}\right)\\&=\frac{b}{h+b}\end{aligned}$$

Then, we can extend this using a similar formula:

$$\begin{aligned}s^*=&(A+B+D+1)\sum_{i=1}^m \mu_i+z\sqrt{(A+B+1)\left(\sum_{i=1}^n \Lambda_i\right)^2+D\sum_{i=1}^m \Lambda_i^2}\\=&(A+B+D+1)\left(\sum_{i=1}^m\sum_{j=1}^{n_i}\mu_{ij}\right)\\&+z\left\{\left(\sum_{i=1}^m\left[(B+1)\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2+A\sum_{j=1}^{n_i}\sigma_{ij}^2\right]^{1/2}\right)^2\right.\\&\left.+D\left((B+1)\left[\sum_{i=1}^m\left(\sum_{j=1}^{n_i}\sigma_{ij}\right)^2\right]+A\left[\sum_{i=1}^m\sum_{j=1}^{n_i}\sigma_{ij}^2\right]\right)\right\}^{1/2}\end{aligned}$$

□

By the three echelon model, we can inductively obtain optimal stock level s^* arbitrary multi-echelon cross-docking model under similar assumption and strategy.

Corollary 3.4. *Let a multi-echelon model be k-echelon system. From each echelon in level m to level n, there is l_n lead time. There is a oversea supplier send amount ordered by level 1 echelon over lead time l_1, and once a intermediate station receive the stock, they will allocate all of them to all its affiliated branch to balance downstream stock-out probability. Only one product is sold,*

and demand d. only occur at retailer level following normal distribution independent across day and station. Let $h > 0, b > 0$ be holding and backordering cost. Let s be the order-up-to level, which CW order each end of day by amount of demand consumed in previous period. Then the optimal order up-to-level is given by the recursive calculation as following:
Set parameter i be the level, and set $z = \Phi^{-1}\left(\frac{b}{h+b}\right)$ Create array of synthesized demand mean and variance, then compute the synethesized deman by its corresponing branch under that level. Start from $i = k$, then record $\vec{\sigma_k} = (\sigma_{km})_{m\in I_k}$ where I_k is the index set of level k. Let I_{ij} be the child index set of station j in level i. Then,

(1) *If $i = k-1$, then record $\sigma_{ij} = \sqrt{(l_k+1)\left(\sum_{p\in I_{k-1,j}}[\vec{\sigma_k}]_p\right)^2 + l_{k-1}\sum_{p\in I_{k-1,j}}[\vec{\sigma_k}]_p^2}$.*[6]
(2) *Then record $\vec{\sigma}_{k-1} = (\sigma_{k-1,j})_j$ for j being a station in level $k-1$.*
(3) *Update $i = i-1$.*

Now, for any level $i < k-1$,

(1) *Record $\sigma_{ij} = \sqrt{(1+\sum_{n=i+1}^{k} l_n)\left(\sum_{p\in I_{ij}}[\vec{\sigma}_{i+1}]_p\right)^2 + l_i\sum_{p\in I_{ij}}[\vec{\sigma}_{i+1}]_p^2}$*
(2) *Record $\vec{\sigma}_i = (\sigma_{ij})_j$ for j being a station in level i.*
(3) *Update $i = i-1$.*
(4) *Stop when $i = 0$*

s^* is given by

$$\left(1+\sum_{i=1}^{k} l_i\right)\left(\sum_{j\in I_k}\mu_j\right) + z\sigma_1 \tag{3.8}$$

Note that here it only account for one Central Warehouse, and $\vec{\sigma}_1$ is a scalar.

3.4. Discussion of Breaking Imbalance Assumption

In an arbitrary echelon supply system with cross-docking policies with normal demand as discussed before, we have imbalance assumption that implies we can always be able to implement the equal-stockout probability allocation. However, in some scenario, there is possible imbalance cases happens. For completeness, we should also discuss the optimal strategy under the imbalance case.

Suppose the demand at retailer level follows independent normal distribution with culmulative distribution F_i for retailer r_i. Holding cost and backordering cost $h, b > 0$. Zoom into the small branch of the system, and we only focus on the downstream retailers. Suppose the upper echelon has n retailers and denote $N = \{1, \ldots, n\}$. Suppose c is the amount arrived for upper echelon, and by strategy, they should assign x_i to each retailer affiliated to this station, for each $i \in N$, s.t

$$\sum_{i=1}^{n} x_i = c$$
$$\exists k \in [0,1] \text{ s.t. } \mathbb{P}(\text{stockout in retailer i}) = k \text{ for } \forall i \in N$$

[6] $[\vec{x}]_j$ is defined to be the j th element in vector x.

Suppose the equal stock-out strategy is impossible, equivalently the previous allocation is infeasible. Then, exists a index set I such that for any subset $A \subset I$ such that

$$\nexists \, \mathbf{x} = \text{sequence of } (x_i)_{i \in (N \setminus I) \cup A} \text{ satisfies:} \tag{3.9}$$
$$x_i \geq 0 \;\; \forall i \in (N \setminus I) \cup A \tag{3.10}$$
$$\sum_{i \in (N \setminus I) \cup A} x_i = c \tag{3.11}$$
$$F_i(s_i + x_i) = F_j(s_j + x_j) \; \forall \, i, j \in (N \setminus I) \cup A \tag{3.12}$$

Define this I be the smallest balance set. Consider the original optimization problem without constraint given by imbalance in equation(3.9), equation(3.10), equation (3.11), and equation (3.12).

$$\begin{aligned} \min_{\mathbf{x}} G(\mathbf{x}) &= \sum_{i=1}^{n} G_i(s_i + x_i) \\ \text{s.t. } \sum_{i=1}^{n} & x_i = c \\ & x_i \geq 0 \; \forall i \in N \end{aligned}$$

Since by equation (2.10)

$$\begin{aligned} \frac{\partial}{\partial x_i} G = \frac{\partial}{\partial x_i} G_i &= \frac{\partial G_i}{\partial (x_i + s_i)} \frac{\partial (x_i + s_i)}{x_i} \\ &= (h + b) F_i(s_i + x_i) - b \end{aligned}$$

Thus, KKT condition gives

$$-\left((h + b) F_i(s_i + x_i^*) - b\right) = \lambda - \mu_i \; \forall i \in N \tag{3.13}$$
$$\sum_{i=1}^{n} x_i^* = c, \;\; x_i \geq 0 \tag{3.14}$$
$$\mu_i \geq 0, \;\; \mu_i x_i^* = 0 \;\; \forall i \in N \tag{3.15}$$

Rewrite condition 3.13, for any $i \in N$,

$$F_i(s_i + x_i^*) = \frac{b - \lambda}{h + b} + \frac{\mu_i}{h + b} \tag{3.16}$$

Introduce the assumption that balance allocation is not feasible for N. Then, let subset $I \subset N$ be the smallest balance set. Consider the solution $\hat{\mathbf{x}}$ given by

$$\begin{aligned} &\text{if } i \in I, \; \hat{x}_i = 0 \\ &\text{If } i \notin I, \; F_i(\hat{x}_i + s_i) = F_j(\hat{x}_j + s_j) \; \forall i, j \in N \setminus I \end{aligned}$$

Claim: **$\hat{\mathbf{x}}$ satisfies the KKT condition.** By condition (3.15),

$$\mu_i = 0, \; \forall i \in N \setminus I \tag{3.17}$$
$$\Rightarrow F_i(s_i + \hat{x}_i) = \frac{b - \lambda}{h + b} = F_j(s_j + \hat{x}_j), \; \forall i, j \in N \setminus I \tag{3.18}$$
$$\mu_i \text{ free for } i \in I \tag{3.19}$$
$$\Rightarrow \mu_i := (h + b) F_i(s_i) - b + \lambda, \; \forall i \in I \tag{3.20}$$

Denote the equal stockout probability of $i \in N \setminus I$ by p given by $\hat{x}$. Assessing the KKT condition with this set of $\lambda, (\mu_i)_{i=1}^n$.

$$(3.13) \Leftrightarrow \begin{cases} -((h+b)p - b) = \lambda & \text{for} \quad i \in N \setminus I \\ -(h+b)F_i(s_i) + b = \lambda - \mu_i = \lambda - (h+b)F_i(s_i) + b - \lambda & \text{for} \quad i \in I \end{cases}$$

$$(3.14) \Leftrightarrow x_i \geq 0, \ \sum_{i=1}^{n} \hat{x}_i = c$$

$$(3.15) \Leftrightarrow \begin{cases} \mu_i = 0 \Rightarrow x_i \mu_i = 0 & \text{for} \quad i \in N \setminus I \\ \mu_i = (h+b)F_i(s_i) - b + \lambda = (h+b)(F_i(s_i) - p) > 0 & \text{for} \quad i \in I \end{cases}$$

Therefore, $\hat{\mathbf{x}}$ satisfies the KKT condition, and it minimizes G. Therefore, when allocation can not be made to balance, remove those with low initial stock-out probability, and distribute to equalize the rest of station's stock-out probability whenever it is feasible.

4. Question remains.

Under the model described, we are able to identify a specific optimal order up-to level. However, some assumptions can be further assessed. For instance, under our modeling of suboptimality by making the decision each time fram, might not necessarily lead to the overall optimality in the long run. Some question that I would like to investigate is

(1) Is a constant order up to level the optimal strategy if the lower stream demands remain following the same distribution over time.
(2) When the system is have holding cost over the system, then is it optimal to allocate all stock arrived for the intermediate stations to the lower streams? Is holding upstream having any benefit in long run?

There are some implication to the model investigated. From traditional supply chain that transit inventory directly from supplier to each retailer, comparing to the cross-docking system, sufferes from more variation in demand to the achieve the same level of stock-out under at retailer level, and therefore more cost with the lead time required. Further, the optimal stock-level is also determined by both holding/backordering cost, lead time, and demand and their interaction downstream over time and location. Therefore, the conclusion for a optimal structure under a generic case may be very difficult. By adjusting the system, there might be different realistic scenario that prefers specific structures.

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