The CW-Complex of Translation Surfaces of Genus 2 with One Singularity

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May 5, 2015

Abstract

A translation surface X of genus 2 with one singularity of 6π can be described as a set of closed planar eight-bar linkages. A linkage L in this set can be regarded as a collection of eight vectors $\{\vec{v_1}, ..., \vec{v_8}\}$ in the plane whose sum equals zero. This paper will focus on the subset of unit-length linkages with an interior distance of one between distinct vertices. Each such linkage arises as the minimal vector decomposition of X. The configuration space of linkages satisfying these conditions is a CW complex whose geometrical and topological properties will be presented.

1 Introduction

A polygonal region K in $\mathbb{R}^2$ is the graph of an n-cycle embedded in $\mathbb{R}^2$ unioned with its interior. That is, it is a region in $\mathbb{R}^2$ whose boundary consists of straight edges joined at vertices. There is a labeling of the edges of K and an orientation on the edges such that there is an equivalence relation on the points of K which identifies edges of the same label such that if one edge is oriented from v_a to v_b and the other from v_c to v_d, then the points are identified by a linear map with $v_a \sim v_c$ and $v_b \sim v_d$. The quotient space $K/\sim = X$ is said to be obtained by pasting the edges of K together [7].

Another way to consider this is that the edges of a polygonal region K correspond to the arcs along which we cut the surface X. If X is a translation surface, then X is a topological surface which is flat everywhere, except at a finite number of cone points, or singular points, whose angles are each a multiple of 2π. For a translation surface X of genus 2 with one singularity of 6π we can cut X along arcs where some arcs must pass through the singular point such that when X is *unfolded* the resulting surface is a polygonal region $K \in \mathbb{R}^2$. In that case the interior angle sum of K must be at least 6π and so K must have at least eight edges.

The boundary of K can be described by a set of vectors in $\mathbb{R}^2$. The minimal vector decomposition of X is the smallest number of vectors needed to describe the boundary of a polygonal region K corresponding to the translation surface X. For a translation surface of genus 2 with one singularity of 6π, the minimal vector decomposition will describe the boundary of an octagon where all the vertices are identified to form X. We call this set of vectors a linkage. Thus a linkage L is the minimal vector decomposition of X if the vectors which form L define the edges of the polygonal region K that X is the quotient space of. For K a translation surface of genus 2, the vectors of L will form the set $\{\vec{v}_i\} \in \mathbb{R}^2$ for $i \in (1, 8)$ connected tip to tail counterclockwise. So $\sum \vec{v}_i = 0$ since K is connected and thus L is closed. The edge identification of K used to form X via pasting can be represented in L by letting $\vec{v}_i = -\vec{v}_{i+4(mod 8)}$ where the edges of K described by $\vec{v}_i$ and $\vec{v}_{i+4(mod 8)}$ are pasted together.

I will focus on the set of closed planar unit-length eight-bar linkages which arise as the minimal vector decomposition of a translation surface of genus 2 with a singularity of 6π. (A linkage is unit-length if it is formed from unit vectors, which is possible since K can be rescaled so that it has unit-length sides without affecting its quotient space X.)

To better understand these linkages and their configuration space, we next explore the implications of the above conditions on the structure of a linkage.

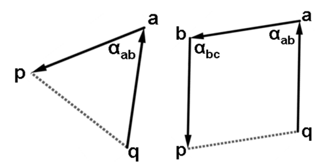
1.1 Interior Distance

The requirement that the interior distances between distinct vertices is no less than one determines the conditions on the angles between vectors. There are three cases of how to form linkages in the configuration space.

Consider the vectors describing a linkage. The vectors will be connected such that they form a chain. Let α_{ij} denote the angle between $\vec{v}_i$ and $\vec{v}_j$ where $\vec{v}_i$ is encountered first in the counter-clockwise orientation of the vectors. If $j = i + 4(mod 8)$ then $\alpha_{ij} = 0$.

Define q and p to be vertices at the start and end of the vector chain with the line segment connecting them denoted qp. Then the requirement that linkages have an interior distance no less than one means that for all sets of vector chains, $|qp| \geq 1$.

Proposition 1.1 *If* $|qp| \geq 1$, *then*
i) for a vector chain of two vectors $\{\vec{v}_a, \vec{v}_b\}$, $\alpha_{ab} \geq \frac{\pi}{3}$,
ii) for a vector chain of three vectors $\{\vec{v}_a, \vec{v}_b, \vec{v}_c\}$, $\alpha_{ab} + \alpha_{bc} \geq \pi$.



Proof

$i)$ Let $\vec{v}_a$ and $\vec{v}_b$ meet at the vertex a. Then the vertices q, a, p form the triangle qap.

If $|qp| = 1$, then the triangle qap is an equilateral triangle with unit length sides. Thus all interior angles are $\frac{\pi}{3}$, so $\alpha_{ab} = \frac{\pi}{3}$.

If $|qp| > 1$, then the triangle qap is an isosceles triangle. Since $|\vec{v}_a| = |\vec{v}_b| = 1$, qp is the longest side of triangle qap and so the angle opposite (α_{ab}) will be the largest interior angle of the triangle. Suppose $\alpha_{ab} < \frac{\pi}{3}$. Then the sum of the angles at vertex q and vertex p will be greater than $\frac{2\pi}{3}$ since the sum of all the angles must be π. This requires at least one of those two angles to be greater than $\frac{\pi}{3}$; contradicting that α_{ab} is the largest interior angle. Thus it must be that $\alpha_{ab} > \frac{\pi}{3}$.

Therefore $\alpha_{ab} \geq \frac{\pi}{3}$ if the vector chain has just two vectors.

$ii)$ Let $\vec{v}_a$ connect to $\vec{v}_b$ which connects to $\vec{v}_c$ at vertices a and b. Then the vertices q, a, b, p form a quadrilateral.

If $|qp| = 1$, then quadrilateral $qabp$ is a rhombus with unit-length sides. Opposite angles will be equivalent, yielding the relation: $2\alpha_{ab} + 2\alpha_{bc} = 2\pi$, so $\alpha_{ab} + \alpha_{bc} = \pi$.

If $|qp| > 1$, then for all possible values $\gamma = \alpha_{ab} + \alpha_{bc}$ there exists trapezoid $qabp$ (which will never be a rhombus) such that $\alpha_{ab} = \alpha_{bc} = \frac{\pi}{2} + \delta$ for some $\delta \in \mathbb{R}$. Since lines through vertices a and b, perpendicular to line qp will intersect qp at a' and b' respectively, the triangle qaa' and the triangle pbb' will be right triangles and so it must be that $\delta > 0$. Thus $\alpha_{ab} + \alpha_{bc} > \pi$ if $|qp| > 1$.

Therefore $\alpha_{ab} + \alpha_{bc} \geq \pi$ if the vector chain has just three vectors.

□

Since each linkage in the configuration space will have four pairs of parallel vectors, Proposition 1.1 provides some insight into the structure of a linkage L in the space. If L is in case (i), then $\vec{v}_a$ and $\vec{v}_b$ will not be one of the parallel vector pairs. If L is in case (ii) with $|qp| = 1$, then $\vec{v}_a$ and $\vec{v}_c$ will be parallel and could be one of the vector pairs describing L, but not necessarily, and $\vec{v}_b$, as in case (i) will not be paired with either $\vec{v}_a$ or $\vec{v}_c$.

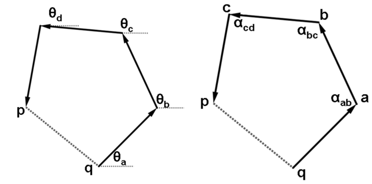
The third case is when there are four vectors $(\vec{v}_a, \vec{v}_b, \vec{v}_c, \vec{v}_d)$ in the vector chain with no subchain being in one of the previous two cases. To solve this case we will use a different notation for describing the relationship between the vectors in the chain. A unit vector $\vec{v}_\alpha$ is described by $e^{i\theta_\alpha}$ in the complex plane. The α_{ij} used in Proposition 1.1 can be recovered from the theta notion by the relation:

$$\alpha_{ij} = \pi - |\theta_j - \theta_i| .$$

Since vectors are taken in a counter-clockwise manner, this can be written as $\alpha_{ij} = \pi - (\theta_j - \theta_i)$.

Proposition 1.2 *If $|qp| \geq 1$, then*
iii) for a vector chain of four vectors $\{\vec{v}_a, \vec{v}_b, \vec{v}_c, \vec{v}_d\}$,

$$cos(\alpha_{ab}) + cos(\alpha_{bc}) + cos(\alpha_{dc}) - cos(\alpha_{ab} + \alpha_{bc}) - cos(\alpha_{bc} + \alpha_{cd}) + cos(\alpha_{ab} + \alpha_{bc} + \alpha_{cd}) \leq \tfrac{3}{2}.$$



Proof

Let $\vec{v}_{qp}$ describe the vector from point q to point p. Then $|qp| = |\vec{v}_{qp}| \geq 1$. The sum of the four vectors is $e^{i\theta_a} + e^{i\theta_b} + e^{i\theta_c} + e^{i\theta_d} = \vec{v}_{qp}$, so

$$\left|e^{i\theta_a} + e^{i\theta_b} + e^{i\theta_c} + e^{i\theta_d}\right|^2 = |\vec{v}_{qp}|^2 \geq 1.$$

$$\left|e^{i\theta_a} + e^{i\theta_b} + e^{i\theta_c} + e^{i\theta_d}\right|^2 = (e^{i\theta_a} + e^{i\theta_b} + e^{i\theta_c} + e^{i\theta_d})(e^{-i\theta_a} + e^{-i\theta_b} + e^{-i\theta_c} + e^{-i\theta_d}),$$

so expanding and plugging back into the inequality yields:

$$cos(\theta_b - \theta_a) + cos(\theta_c - \theta_a) + cos(\theta_c - \theta_b) + cos(\theta_d - \theta_a) + cos(\theta_d - \theta_b) + cos(\theta_d - \theta_c) \leq \tfrac{-3}{2}$$
$$cos(\pi - \alpha_{ab}) + cos(2\pi - \alpha_{bc} - \alpha_{ab}) + cos(\pi - \alpha_{bc}) + cos(3\pi - \alpha_{cd} - \alpha_{bc} - \alpha_{ab}) + cos(2\pi - \alpha_{cd} - \alpha_{bc}) + cos(\pi - \alpha_{dc}) \leq \tfrac{-3}{2}$$
$$-cos(\alpha_{ab}) + cos(\alpha_{bc} + \alpha_{ab}) - cos(\alpha_{bc}) - cos(\alpha_{cd} + \alpha_{bc} + \alpha_{ab}) + cos(\alpha_{cd} + \alpha_{bc}) - cos(\alpha_{dc}) \leq \tfrac{-3}{2}$$
$$cos(\alpha_{ab}) + cos(\alpha_{bc}) + cos(\alpha_{dc}) - cos(\alpha_{ab} + \alpha_{bc}) - cos(\alpha_{bc} + \alpha_{cd}) + cos(\alpha_{ab} + \alpha_{bc} + \alpha_{cd}) \leq \tfrac{3}{2}$$

□

These three cases are all of the cases since if there were a chain of five vectors, then since a linkage is closed, on one side of the line qp would be five vectors $(\vec{v}_a, \vec{v}_b, \vec{v}_c, \vec{v}_d, \vec{v}_e)$ and so three connected vectors would be on the other side and fall within case (ii). Similarly, case (i) implies that a chain of six vectors lies on the other side of the line qp.

1.2 Genus 2 Surface

If a linkage L is in the configuration space, L is closed and formed by eight vectors. Then L is an octagon with an interior angle sum of 6π. In order for L to correspond to a genus 2 surface with a singularity of 6π, each vertex of L must be identified with every other vertex of L.

As previously mentioned, L consists of four vectors and their additive inverses. Each vector and its inverse describes two edges in the polygonal region K which are pasted together to form the translation surface X. Since $\vec{v}_i = -\vec{v}_{i+4(mod 8)}$, each linkage in the configuration space can be described by the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ of four vectors and a matching.

Let L be represented by the word formed by the indices of the four vectors which describe it, $\{1, 2, 3, 4\}$, with every element appearing exactly twice. For example, if L is hypereliptic (180^o rotational symmetry), then the word 12341234 describes L, up to a relabeling of the vector pairs.

The way the vertices of a linkage L are identified is by looking at how the corresponding translation surface folds up. Each vector of L has an orientation and so the vertex that would be at the tip of $\vec{v_1}$ is the same as the vertex at the tail of $\vec{v_5}$ since by the matchings, the two vectors represent the same edge on the polygonal region described by L. This same vertex is at the tail of the vector following $\vec{v}_1$, say $\vec{v}_2$, and can be identified to another vertex on L which is at the tip of $\vec{v}_6$. This is done for all the vertices of L. All of the vertices must be identified together if L is in the configuration space. Figure 1 shows some examples of eight-bar linkages with their vertices identified.

It turns out that if a linkage L is in the configuration space then, when two adjacent vectors of different labels appear in a certain order, if they happen to be next to each other at a different point in L, then they cannot appear in the reverse order. For example, L cannot be represented by the word 12342134 because this means that in the structure of L, $\vec{v}_1$ is followed by $\vec{v}_2$ but later on $\vec{v}_2$ is followed by $\vec{v}_1$. Checking the vertex identifications, the linkage with

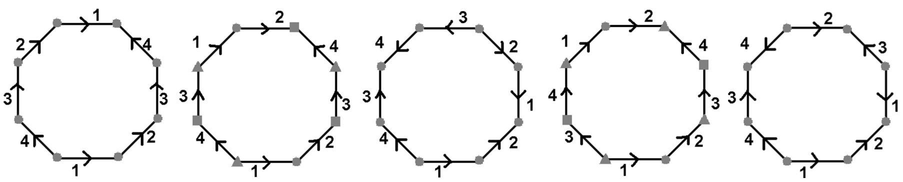


Figure 1: These octagons represent a linkage where the edge numbers correspond to vectors of that same label. The property that vectors of the same label are parallel is being ignored so as to focus on how to determine if a linkage is the decomposition of translation surface of genus 2 with a singularity of 6π. The arrow on an edge indicates the edge's orientation with respect to pasting, so some edges may be oriented in a different direction than the vector which describes it.

word 12342134 corresponds to a translation surface with three singularities of $\frac{9\pi}{4}$, $\frac{9\pi}{4}$ and $\frac{3\pi}{2}$ (as shown in Figure 1).

1.3 Allowable Linkages

Combining together all of the requirements for a linkage to be in the configuration space, there are only three types of linkages, up to a relabeling, which are allowed. These linkages have word representations 12341234, 12341423, and 12132434. Note, as seen in Figure 1, a linkage with word 12123434 satisfies the genus 2 requirement but not all interior distances between distinct vertices will be greater than or equal to one since 1212 corresponds to a closed four-bar linkage.

As a notational simplification, each of the allowed three linkage types can be written as being of type $(3,3,3,3), (3,2,2,1)$ and $(1,2,2,1)$. A linkage being of type (a,b,c,d) means that there are a edges in the linkage between the first edge and its second appearance and similarly for the other edges. When determining the separation between edges of the same label, the minimum separation is the one chosen. For example, a linkage of type 12341423 is of type $(3,2,2,1)$ but there are five edges between the third edge (3) and its second appearance, however starting at its second appearance yields a separation of two.

2 Constructing the Configuration Space

Let linkage L be in the configuration space. L is constructed from the vectors: $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Each linkage type can be described by a set of the α_{ij}, in a similar way as how only four of the eight vectors were needed to describe a linkage.

Proposition 2.1 *For each linkage type, only three angles are needed to know the entire structure of a linkage in the configuration space.*

Proof

A linkage in the configuration space is closed with eight unit-length sides, so it follows that only the angles between the vectors are needed to construct a linkage if its type is known, since that will provide information on the order of the vectors. Consider each linkage type separately:

$(3,3,3,3)$: If L is a linkage of type $(3,3,3,3)$, then L can be written as the word 12341234. The set of interior angles of L is $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{41}\}$ with each angle appearing twice in L. Since the sum of the interior angles in L is 6π, it follows that $\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} = 3\pi$. Then $\alpha_{41} = 3\pi - (\alpha_{12} + \alpha_{23} + \alpha_{34})$, so only $\alpha_{12}, \alpha_{23}, \alpha_{34}$ are needed to describe L.

$(3,2,2,1)$: If L is a linkage of type $(3,2,2,1)$, then L can be written as the word 12341423. The set of interior angles of L is $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{41}, \alpha_{14}, \alpha_{42}, \alpha_{31}\}$ with α_{23} appearing twice in L. The 414 part of L is in case (ii) of Proposition 1.1 and so $\alpha_{41} + \alpha_{14} = \pi$. It follows that only one of the two angles is needed to describe the 414 portion of L. Furthermore, if the vertex at α_{34} is labeled a and the vertex at α_{42} is labeled b, then let the vector from a to b be $\vec{v}_5$. Since the 414 part of L is in case (ii) of Proposition 1.1, $\vec{v}_5$ is parallel to $\vec{v}_1$. The word 4145 corresponds to a rhombus and the word 123523 corresponds to a hyperelliptic hexagon with equal-length sides. Therefore $\alpha_{12} = \alpha_{52}$ and $\alpha_{31} = \alpha_{35}$, so α_{12}, α_{23}, and α_{35} are the only angles needed to describe 123523. Additionally, $\alpha_{14} = \alpha_{54}$ and knowing α_{14} means knowing the entirety of 4145. Then $\alpha_{12}, \alpha_{23}, \alpha_{35}$ and α_{14} will describe L. Since $\alpha_{34} = \alpha_{35} + \alpha_{54} = \alpha_{35} + \alpha_{14}$, only $\alpha_{12}, \alpha_{23}, \alpha_{34}$ are needed to describe L.

$(1, 2, 2, 1)$: If L is a linkage of type $(1, 2, 2, 1)$, then L can be written as the word 12132434. The interior angles of L are all of unique label and form the set $\{\alpha_{12}, \alpha_{21}, \alpha_{13}, \alpha_{32}, \alpha_{24}, \alpha_{43}, \alpha_{34}, \alpha_{41}\}$. However, the structure of L is such that if $\vec{v}_5$ goes from the vertex at α_{13} to the one at α_{41} and $\vec{v}_6$ is defined similarly, only between α_{24} and α_{41}, then the words 1215, 3265, and 4346 all correspond to a rhombus and are thus in case (ii) of Proposition 1.1. As seen with linkages of type $(3, 2, 2, 1)$, only one interior angle is needed to describe a rhombus. Thus, only $\alpha_{12}, \alpha_{32}, \alpha_{34}$ are needed to describe L.

□

These sets of three angles can be used as coordinates in $\mathbb{R}^3$. Figure 2 shows which α_{ij} is used for each axis. Since each linkage type has a different construction, they cannot be plotted together using this method since the three coordinates provide no information about the structure of a linkage.

	$(3,3,3,3)$	$(3,2,2,1)$	$(1,2,2,1)$
x	α_{12}	α_{12}	α_{12}
y	α_{23}	α_{23}	α_{32}
z	α_{34}	α_{34}	α_{34}

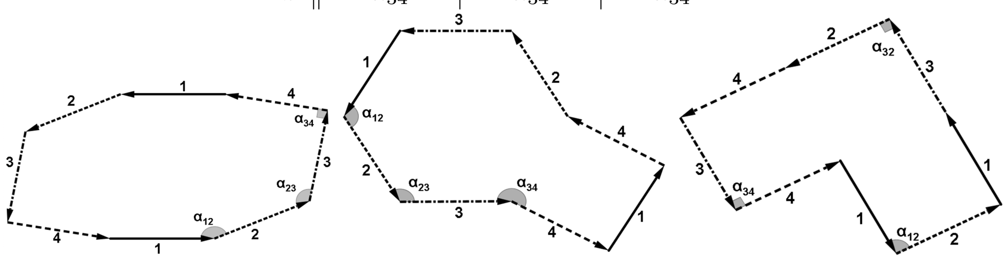


Figure 2: The table shows which α_{ij} was chosen to be used in the coordinate system for describing linkages of each type and which coordinate in $\mathbb{R}^3$ that α_{ij} corresponds to. An example of each linkage type is given with the three angles needed to describe it highlighted.

The boundary of the configuration space will be when one of the inequalities in Propositions 1.1 and 1.2 is an equality. This occurs when there exists a distance of length one between at least two nonadjacent vertices within a linkage.

2.1 The Boundary

Each linkage L in the configuration space must satisfy certain requirements (previously mentioned). From these requirements a certain number of degrees of freedom can be assigned to L. If there exists an interior distance of length one between at least two nonadjacent vertices of L, then those vertices become locked at that position such that if any interior angle of L were to be changed to form linkage L', then L' will have a distance of one between the same vertices for which this is the case on L. The degrees of freedom of L are the number of angles which can be changed without breaking any locked distances that L may have. Since changing one angle will require at least one other angle to change, a linkage in the configuration space can have a maximum of three degrees of freedom.

The interior of the configuration space is the set of angles that satisfy the strict inequalities. It is the set of linkages which are described by the strict inequalities in Propositions 1.1 and 1.2. This set will be an open set bounded by the set of linkages in the configuration space with less than three degrees of freedom. An easy visualization is to picture where a point p is able to move. If p is on the interior of the configuration space then from p there is a point p' in the direction of a linear combination of the vectors $\vec{e}_i, \vec{e}_j, \vec{e}_k$ which is still on the interior, where $\vec{e}_i, \vec{e}_j, \vec{e}_k$ will always denote linearly independent unit vectors.

It then follows that a linkage with two degrees of freedom will correspond to point p on a face of the configuration space. From p, only traveling in the direction of some linear combination of $\vec{e}_i, \vec{e}_j$ will go from p to p' still on the face. The face does not have to be planar for this fact to be true. For instance: If a person is walking on a hill, they can only choose a linear combination of going forward or turning to the right to change their location and maintain being on the hill regardless of whether they are traveling upward or downward with respect to some reference position.

Using the same logic, it can be seen that a linkage on an edge of the configuration space will have one degree of freedom and if on a vertex it will have no degrees of freedom.

The next three subsections describe in detail the structure of the configuration space for each linkage type. This will be three separate configuration spaces for which their union is the configuration space which has been referred to up until now.

2.2 Linkage Type: $(3, 3, 3, 3)$

Let the linkages of type $(3, 3, 3, 3)$ be linkages with a word representation of 12341234, so the configuration space P_0 is plotted via a $(\alpha_{12}, \alpha_{23}, \alpha_{34})$-coordinate system. P_0 will have 12 faces, 30 edges, and 20 vertices. Specifically, P_0 will have sets of four faces which are either hexagonal, pentagonal, or quadrilateral. We will label the hexagonal faces F_{1_0} through F_{4_0}, F_{5_0} through F_{8_0} will be pentagonal, and the remaining faces F_{9_0} through F_{12_0} will be quadrilateral.

Since the vertices have no degrees of freedom, all of their interior angles are known with certainty and so their α_{ij} can be determined. Doing this for all of the vertices will yield a plot like that in Figure 3.

Vertex	Coordinate	Vertex	Coordinate
V_{1_0}	$(\frac{\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3})$	V_{11_0}	$(\frac{\pi}{3}, \pi, \frac{\pi}{3})$
V_{2_0}	$(\frac{4\pi}{3}, \frac{\pi}{3}, \pi)$	V_{12_0}	$(\pi, \pi, \frac{2\pi}{3})$
V_{3_0}	$(\frac{\pi}{3}, \pi, \frac{\pi}{3})$	V_{13_0}	$(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3})$
V_{4_0}	$(\pi, \frac{\pi}{3}, \frac{4\pi}{3})$	V_{14_0}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$
V_{5_0}	$(\pi, \pi, \frac{\pi}{3})$	V_{15_0}	$(\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3})$
V_{6_0}	$(\pi, \frac{\pi}{3}, \frac{2\pi}{3})$	V_{16_0}	$(\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3})$
V_{7_0}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \pi)$	V_{17_0}	$(\frac{4\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$
V_{8_0}	$(\frac{2\pi}{3}, \pi, \pi)$	V_{18_0}	$(\frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3})$
V_{9_0}	$(\pi, \frac{2\pi}{3}, \frac{\pi}{3})$	V_{19_0}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3})$
V_{10_0}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \pi)$	V_{20_0}	$(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3})$

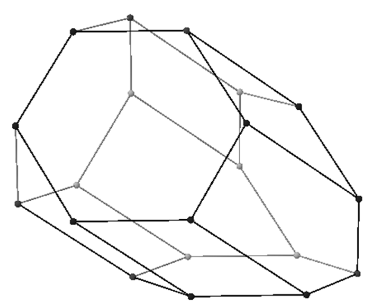


Figure 3: Coordinates for the vertices of P_0 and a plot of the vertices and edges.

To understand the properties of P_0, further investigation of the faces is needed. Knowing the curvature of each face will provide a better explanation as to the appearance of the configuration space in $\mathbb{R}^3$. The equations for each face are as follows:

$F_{1_0} : \alpha_{12} + \alpha_{23} + \alpha_{34} = \frac{8\pi}{3}$
$F_{2_0} : \alpha_{12} = \frac{\pi}{3}$
$F_{3_0} : \alpha_{23} = \frac{\pi}{3}$
$F_{4_0} : \alpha_{34} = \frac{\pi}{3}$

$F_{9_0} : \alpha_{12} + \alpha_{23} = 2\pi$
$F_{10_0} : \alpha_{23} + \alpha_{34} = 2\pi$
$F_{11_0} : \alpha_{12} + \alpha_{23} = \pi$
$F_{12_0} : \alpha_{23} + \alpha_{34} = \pi$

$F_{5_0} : -cos(\alpha_{12})+cos(\alpha_{23})-cos(\alpha_{34})-cos(\alpha_{12}+\alpha_{23})-cos(\alpha_{23}+\alpha_{34})-cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) = \frac{-3}{2}$
$F_{6_0} : cos(\alpha_{12})+cos(\alpha_{23})-cos(\alpha_{34})-cos(\alpha_{12}+\alpha_{23})+cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) = \frac{3}{2}$
$F_{7_0} : -cos(\alpha_{12})-cos(\alpha_{23})-cos(\alpha_{34})+cos(\alpha_{12}+\alpha_{23})+cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) = \frac{-3}{2}$
$F_{8_0} : -cos(\alpha_{12})+cos(\alpha_{23})+cos(\alpha_{34})+cos(\alpha_{12}+\alpha_{23})-cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) = \frac{3}{2}$

This indicates that all of the faces are planar, except for faces $F_{5_0}, F_{6_0}, F_{7_0}, F_{8_0}$. ($F_{5_0}$ is graphed in Figure 4). Due to the curvature of these faces with respect to the rest of P_0, P_0 is not convex.

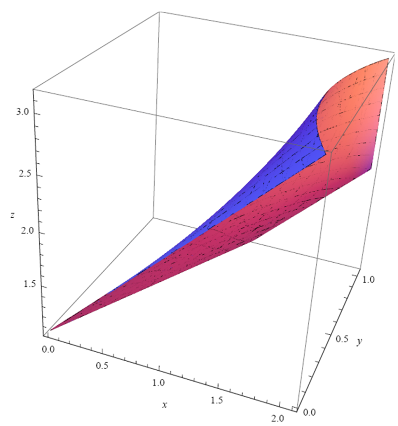


Figure 4: This is a plot of the equation describing the linkages on F_{5_0} with bounds determined by the vertices of the face. Faces F_{6_0}, F_{7_0}, F_{8_0} will have similar plots.

Proposition 2.2 *All the points $p \in \mathbb{R}^3$ satisfying the succeeding inequalities will correspond to a linkage L of type $(3, 3, 3, 3)$ which satisfies the requirements outlined in Section 1. These inequalities describe the subset of $\mathbb{R}^3$ which is bound by the faces of P_0.*

(1) $\alpha_{12} + \alpha_{23} + \alpha_{34} \leq \frac{8\pi}{3}$
(2) $\alpha_{ij} \geq \frac{\pi}{3}$ *for all* i, j
(3) $\pi \leq \alpha_{12} + \alpha_{23} \leq 2\pi$
(4) $\pi \leq \alpha_{23} + \alpha_{34} \leq 2\pi$

(5) $-cos(\alpha_{12})+cos(\alpha_{23})-cos(\alpha_{34})-cos(\alpha_{12}+\alpha_{23})-cos(\alpha_{23}+\alpha_{34})-cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) \geq \frac{-3}{2}$
(6) $cos(\alpha_{12})+cos(\alpha_{23})-cos(\alpha_{34})-cos(\alpha_{12}+\alpha_{23})+cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) \leq \frac{3}{2}$
(7) $-cos(\alpha_{12})-cos(\alpha_{23})-cos(\alpha_{34})+cos(\alpha_{12}+\alpha_{23})+cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) \geq \frac{-3}{2}$
(8) $-cos(\alpha_{12})+cos(\alpha_{23})+cos(\alpha_{34})+cos(\alpha_{12}+\alpha_{23})-cos(\alpha_{23}+\alpha_{34})+cos(\alpha_{12}+\alpha_{23}+\alpha_{34}) \leq \frac{3}{2}$

Proof

A linkage L of type $(3, 3, 3, 3)$ will be closed and have an interior angle sum of 6π with $\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} = 3\pi$. By Proposition 1.1 we know that $\alpha_{ij} \geq \frac{\pi}{3}$ for all i, j, so (2) is satisfied. This also means that

$$\alpha_{12} + \alpha_{23} + \alpha_{34} = 3\pi - \alpha_{41} \leq 3\pi - \frac{\pi}{3} = \frac{8\pi}{3},$$

so (1) is satisfied.

Additionally, Proposition 1.1 states that $\alpha_{ij} + \alpha_{jk} \geq \pi$ for all i, j, k. So

$$\alpha_{ij} + \alpha_{jk} = 3\pi - \alpha_{kl} - \alpha_{li} \leq 3\pi - \pi = 2\pi,$$

so (3) and (4) are satisfied.

Proposition 1.2 states that $cos(\alpha_{ij}) + cos(\alpha_{jk}) + cos(\alpha_{kl}) - cos(\alpha_{ij} + \alpha_{jk}) - cos(\alpha_{jk} + \alpha_{kl}) + cos(\alpha_{ij} + \alpha_{jk} + \alpha_{kl}) \leq \frac{3}{2}$. By following the process used to prove Proposition 1.2, it can be determined that (5), (6), (7), (8) are satisfied. The difference in signs arises from the vector orientations in L. That is (5), (6), (7), (8) will be of the form given in Proposition 1.2 if close attention was paid to whether α_{ij} was between $\vec{v}_i$ and $\vec{v}_j$ or if it was between $\vec{v}_{i+4(mod 8)}$ and $\vec{v}_{j+4(mod 8)}$ when determining the equation for the face each inequality comes from.

□

2.2.1 Fundamental Domain

Despite being the plot of all linkages of type $(3, 3, 3, 3)$ which satisfy all of the requirements outlined in Section 1, P_0 is not the proper configuration space for such linkages. Due to the symmetry of linkages of type $(3, 3, 3, 3)$, P_0 contains sets of points which correspond to the same linkage. This is because if a linkage $L \in P_0$ is represented by the word 12341234 and plots as the point$(\alpha_{12}, \alpha_{23}, \alpha_{34})$, then there exists linkage $L' \in P_0$ which plots as the point $(\alpha_{23}, \alpha_{34}, \alpha_{41})$ and thus is represented by the word 23412341, a cyclic permutation of the labeling of L. Every linkage $L \in P_0$ will plot as the set of points $\{(\alpha_{12}, \alpha_{23}, \alpha_{34}), (\alpha_{23}, \alpha_{34}, \alpha_{41}), (\alpha_{34}, \alpha_{41}, \alpha_{12}), (\alpha_{41}, \alpha_{12}, \alpha_{23})\}$, with not all the points necessarily unique. An affine transformation can be applied to an element of this set to get another element in the set. Define this transformation as:

$$T(\vec{\alpha}) = A\vec{\alpha} + b = \vec{\alpha'}.$$

Where

$$T\begin{pmatrix} \alpha_{ij} \\ \alpha_{jk} \\ \alpha_{kl} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{ij} \\ \alpha_{jk} \\ \alpha_{kl} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3\pi \end{pmatrix} = \begin{pmatrix} \alpha_{jk} \\ \alpha_{kl} \\ \alpha_{li} \end{pmatrix}.$$

Then $T^4(\vec{\alpha}) = id$. The set of fixed points of T is $\{(\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4})\}$ which is the set containing the point corresponding to a linkage which makes a regular octagon. This point is the midpoint of P_0. The transformation S can be applied to the coordinates of P_0 so as to make the point for the regular octagon be at the origin.

$$S\begin{pmatrix} \alpha_{ij} \\ \alpha_{jk} \\ \alpha_{kl} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{ij} \\ \alpha_{jk} \\ \alpha_{kl} \end{pmatrix} - \begin{pmatrix} \frac{3\pi}{4} \\ \frac{3\pi}{4} \\ \frac{3\pi}{4} \end{pmatrix} = \begin{pmatrix} \beta_{ij} \\ \beta_{jk} \\ \beta_{kl} \end{pmatrix}.$$

The transformation S can be used to change from the α-coordinate system of P_0 to a β-coordinate system for the linkages of type $(3, 3, 3, 3)$ and create the space P_{0_β}. The linear transformation $R = S \circ T \circ S^{-1}$ can be applied to the β-coordinates of P_0 to generate the set of points $\{(\beta_{12}, \beta_{23}, \beta_{34}), (\beta_{23}, \beta_{34}, \beta_{41}), (\beta_{34}, \beta_{41}, \beta_{12}), (\beta_{41}, \beta_{12}, \beta_{23})\}$ for each linkage. As with T, $R^4 = id$.

Proposition 2.3 *P_0 can be split into four quadrants where each quadrant will contain at least one of the coordinate representatives from the set of points representing a linkage $L \in P_0$. One of these quadrants will form the fundamental domain, P_1, of P_0.*

Proof Begin by dividing the boundary of P_0 into four pieces. We have that:

$$T(F_{1_0}) = F_{2_0}, T^2(F_{1_0}) = F_{3_0}, T^3(F_{1_0}) = F_{4_0};$$

$$T(F_{5_0}) = F_{6_0}, T^2(F_{5_0}) = F_{7_0}, T^3(F_{5_0}) = F_{8_0};$$

$$T(F_{9_0}) = F_{10_0}, T^2(F_{9_0}) = F_{11_0}, T^3(F_{9_0}) = F_{12_0}.$$

The boundary of P_0 can then be divided into the four sets given by $\{F_{i_0}, F_{j_0}, F_{k_0}\}$, for $i \in (1,4), j \in (5,8), k \in (9,12)$, based on which faces are connected to one another.

Consider the set $\{F_{1_0}, F_{5_0}, F_{9_0}\}$. Connect the vertices $V_{1_0}, V_{2_0}, V_{4_0}, V_{5_0}, V_{8_0}, V_{13_0}, V_{16_0}, V_{17_0} V_{18_0}$ of this set to the midpoint of P_0. This will define the boundary of the space $P_1 \in P_0$.

For coordinate simplicity, apply R to the linkages in P_1 to get the space P_{1_β}. Then the edges connecting the vertices $(V_{1_\beta}, V_{2_\beta}, V_{4_\beta}, V_{5_\beta}, V_{8_\beta}, V_{13_\beta}, V_{16_\beta}, V_{17_\beta}, V_{18_\beta})$ to the origin can be described by the vertex. That is, every linkage on the edge between the origin and a vertex $V \in P_{1_\beta}$ is of the form tV where $t \in (0,1)$. Since R is linear, $R(tV) = tR(V)$. Thus tV is represented by a set of four points in P_{0_β} and those four points will appear on the edges connecting the origin to the image of V under R^n.

Every point in P_{1_β} is equivalent to $t\rho$ where ρ is a point on the boundary of P_{1_β} and so ρ is on one of the faces $F_{1_\beta}, F_{5_\beta}, F_{9_\beta}$. Since the faces are sent to other faces not in P_{1_β} under the transformations R, R^2, R^3; for all $t\rho \in P_{1_\beta}$ there exists a point in $R(P_{1_\beta}), R^2(P_{1_\beta}), R^3(P_{1_\beta})$.

Applying R^{-1} to P_{1_β} shows that for all $\rho \in P_1$ there exists a point in $T(P_1), T^2(P_1), T^3(P_1)$ and so P_1 is the fundamental domain of P_0 and the desired configuration space for linkages of type $(3,3,3,3)$.

□

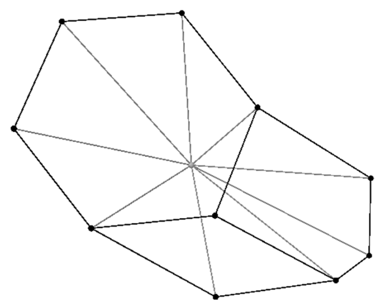


Figure 5: P_1.

2.3 Linkage Type: $(3,2,2,1)$

Let linkages of type $(3,2,2,1)$ with a word representation of 12341423, with the configuration space P_2, be plotted using a $(\alpha_{12}, \alpha_{23}, \alpha_{34})$-coordinate system. P_2 will have 8 faces, 18 edges, and 12 vertices. Specifically, P_2 will have two hexagonal faces F_{1_2} and F_{2_2}, and the remaining faces F_{3_2} through F_{8_2} will be quadrilateral.

Linkages of type $(3,2,2,1)$ do not have the same amount of symmetry as those of type $(3,3,3,3)$ do, so every point in P_2 corresponds to a unique linkage and there is no fundamental domain for the space which needs to be found. Figure 6 gives a plot of the vertices and edges of P_2.

The equations for the faces of P_2 are:

$F_{1_2} : \alpha_{12} + \alpha_{23} + \alpha_{34} = \frac{7\pi}{3}$
$F_{2_2} : \alpha_{12} + \alpha_{23} + \alpha_{34} = \frac{8\pi}{3}$
$F_{3_2} : \alpha_{23} = \pi$
$F_{4_2} : \alpha_{12} + \alpha_{23} = \pi$

$F_{5_2} : \alpha_{12} = \pi$
$F_{6_2} : \alpha_{23} = \frac{\pi}{3}$
$F_{7_2} : \alpha_{12} = \frac{\pi}{3}$
$F_{8_2} : \alpha_{12} + \alpha_{23} = \frac{5\pi}{3}$

This indicates that all of the faces are planar and P_2 will be convex if the interior is well-behaved.

Vertex	Coordinate	Vertex	Coordinate
V_{1_2}	$(\frac{\pi}{3}, \pi, \pi)$	V_{7_2}	$(\pi, \frac{\pi}{3}, \frac{4\pi}{3})$
V_{2_2}	$(\frac{\pi}{3}, \pi, \frac{4\pi}{3})$	V_{8_2}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3})$
V_{3_2}	$(\frac{2\pi}{3}, \pi, \frac{2\pi}{3})$	V_{9_2}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3})$
V_{4_2}	$(\frac{2\pi}{3}, \pi, \pi)$	V_{10_2}	$(\pi, \frac{2\pi}{3}, \frac{2\pi}{3})$
V_{5_2}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3})$	V_{11_2}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3})$
V_{6_2}	$(\pi, \frac{\pi}{3}, \pi)$	V_{12_2}	$(\pi, \frac{2\pi}{3}, \pi)$

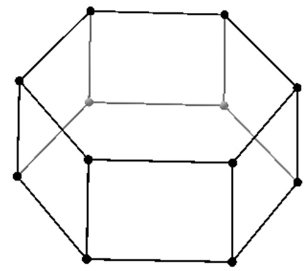


Figure 6: Coordinates for the vertices of P_2 and a plot of the vertices and edges.

Proposition 2.4 *All the points $p \in \mathbb{R}^3$ satisfying the succeeding inequalities will correspond to a linkage L of type $(3,2,2,1)$ which satisfies the requirements outlined in Section 1. These inequalities describe the subset of $\mathbb{R}^3$ which is bound by the faces of P_2.*

(1) $\frac{7\pi}{3} \leq \alpha_{12} + \alpha_{23} + \alpha_{34} \leq \frac{8\pi}{3}$
(2) $\pi \leq \alpha_{12} + \alpha_{23} \leq \frac{5\pi}{3}$
(3) $\frac{\pi}{3} \leq \alpha_{12} \leq \pi$
(4) $\frac{\pi}{3} \leq \alpha_{23} \leq \pi$

Proof

A linkage L of type $(3,2,2,1)$ will be closed and have an interior sum of 6π with $\alpha_{12}+\alpha_{23}+\alpha_{34}+\alpha 41 = 3\pi$. Due to its structure, an additional vector, $\vec{v}_5$, can be added to the linkage which will be parallel to $\vec{v}_1$ and create the six-bar linkage represented 123523 and the four-bar linkage represented by 4145. Then by symmetry, $\alpha_{12}+\alpha_{23}+\alpha_{35} = 2\pi$ and $\alpha_{54}+\alpha 41 = \pi$, where $\alpha_{34} = \alpha_{35}+\alpha_{54}$.

By Proposition 1.1, $\alpha_{ij} \geq \frac{\pi}{3}$ for all i,j. Then $\alpha_{41} = \pi - \alpha_{54} \leq \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Subtracting α_{41} from $\alpha_{12}+\alpha_{23}+\alpha_{34}+\alpha 41 = 3\pi$ yields $\frac{7\pi}{3} \leq \alpha_{12}+\alpha_{23}+\alpha_{34} \leq \frac{8\pi}{3}$, so (1) is satisfied. Similarly, $\alpha_{12}+\alpha_{23} = 2\pi - \alpha_{35} \leq 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Proposition 1.1 also gives that $\alpha_{ij}+\alpha_{jk} \geq \pi$ for all i,j,k. Combining this with $\alpha_{12}+\alpha_{23} \leq \frac{5\pi}{3}$ yields (2). We know that $\alpha_{12}+\alpha_{35} \geq \pi$, so $\alpha_{23} = 2\pi - (\alpha_{12}+\alpha_{35}) \leq \pi$. This satisfies (4) since $\alpha_{ij} \geq \frac{\pi}{3}$ for all i,j. Switching α_{12} and α_{23} gives (3). □

2.4 Linkage Type: $(1,2,2,1)$

Let linkages of type $(1,2,2,1)$ with a word representation of 12132434, with the configuration space P_3, be plotted using a $(\alpha_{12},\alpha_{32},\alpha_{34})$-coordinate system. P_3 will have 6 faces, 12 edges, and 8 vertices. Specifically, all of the faces of P_3 will be quadrilateral.

The relations between the faces, edges, and vertices mentioned in Table **??** can be used to determine that all of the faces will be quadrilateral.

Linkages of type $(1,2,2,1)$ do not have the same amount of symmetry as those of type $(3,3,3,3)$ do, so every point in P_3 corresponds to a unique linkage and there is no fundamental domain for the space which needs to be found. Figure 7 gives a plot of vertices and edges of P_3.

The equations for the faces of P_3 are:

Vertex	Coordinate	Vertex	Coordinate
V_{1_3}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3})$	V_{5_3}	$(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3})$
V_{2_3}	$(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$	V_{6_3}	$(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$
V_{3_3}	$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$	V_{7_3}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3})$
V_{4_3}	$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3})$	V_{8_3}	$(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$

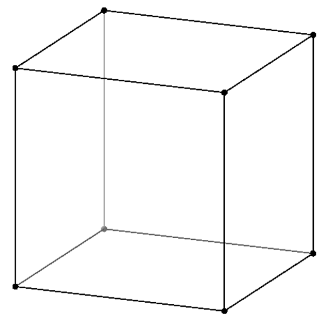


Figure 7: Coordinates for the vertices of P_3 and a plot of the vertices and edges.

$F_{1_3} : \alpha_{32} = \frac{\pi}{3}$
$F_{2_3} : \alpha_{12} = \frac{2\pi}{3}$
$F_{3_3} : \alpha_{12} = \frac{\pi}{3}$
$F_{4_3} : \alpha_{34} = \frac{2\pi}{3}$
$F_{5_3} : \alpha_{34} = \frac{\pi}{3}$
$F_{6_3} : \alpha_{32} = \frac{2\pi}{3}$

This indicates that all of the faces are planar and P_2 will be convex if the interior is well-behaved.

Proposition 2.5 *All the points $p \in \mathbb{R}^3$ satisfying the succeeding inequalities will correspond to a linkage L of type $(1, 2, 2, 1)$ which satisfies the requirements outlined in Section 1. These inequalities describe the subset of $\mathbb{R}^3$ which is bound by the faces of P_3.*

(1) $\frac{\pi}{3} \leq \alpha_{12} \leq \frac{2\pi}{3}$ (2) $\frac{\pi}{3} \leq \alpha_{32} \leq \frac{2\pi}{3}$ (3) $\frac{\pi}{3} \leq \alpha_{34} \leq \frac{2\pi}{3}$

Proof
A linkage L of type $(1, 2, 21)$ will be closed with a symmetric structure that causes $\alpha_{12} + \alpha_{21} = \pi$ and $\alpha_{43} + \alpha_{34} = \pi$. By Proposition 1.1, $\alpha_{ij} \geq \frac{\pi}{3}$ for all i, j. Combining this fact with the above properties yields $\frac{\pi}{3} \leq \alpha_{12} \leq \frac{2\pi}{3}$ (1) and $\frac{\pi}{3} \leq \alpha_{34} \leq \frac{2\pi}{3}$ (3).

The structure of L allows for an additional vector $\vec{v}_5$ to be added which will be parallel to $\vec{v}_2$. A vector $\vec{v}_6$ parallel to $\vec{v}_3$ can also be added to L. This will form three quadrilaterals represented by 1215, 3265, and 4346. This means that $\alpha_{32} + \alpha_{26} = \pi$, so (2) is satisfied by subtracting $\alpha_{26} \geq \frac{\pi}{3}$.

□

3 Equivalence Relation on the Linkages

Each linkage which is in either P_1, P_2, or P_3 corresponds to the minimal vector decomposition of a translation surface of genus 2. Let K be the polygonal region that has edges defined by the linkage $L \in \bigcup_{n=1}^{3} P_n$. Then the translation surface corresponding to L is obtained by pasting the edges of K together. The word $(12341234, 12341423, 12132434)$ used to describe L can be replaced by a labeling scheme $(klmnk^{-1}l^{-1}m^{-1}n^{-1}$, $klmnk^{-1}n^{-1}l^{-1}m^{-1}$, $klk^{-1}ml^{-1}nm^{-1}n^{-1})$ for K. The exponent of each edge indicates the orientation of the edge. When pasting, edges k and k^{-1} may be pasted together, but not edges k and k. Figure 8 shows the pasting process for a polygonal region with labeling scheme $klk^{-1}ml^{-1}nm^{-1}n^{-1}$ corresponding to a linkage of type $(1, 2, 2, 1)$.

Before all of the edges of K are pasted together, any interior line connecting two vertices can be cut along to create two polygonal regions K_1 and K_2. These two regions can then

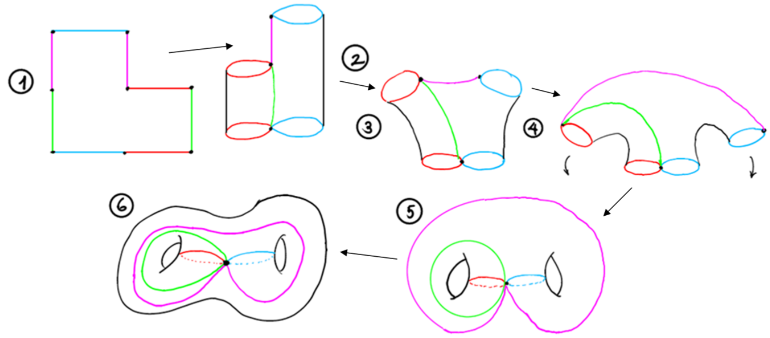


Figure 8: This diagram shows how pasting a polygonal region corresponding to a linkage of type $(1, 2, 2, 1)$ will result in a genus 2 surface. (Image by Christopher Judge.)

be pasted together along any pair of edges with the same label, as in Figure 9, to make the polygonal region K'. Such a cutting and pasting will change the geometry of K but will not change the translation surface obtained by pasting together all of the edges of K [7]. That is, the topological space obtained from pasting together all of the edges of K will be the same as that obtained from K' and so the two regions are said to be equivalent.

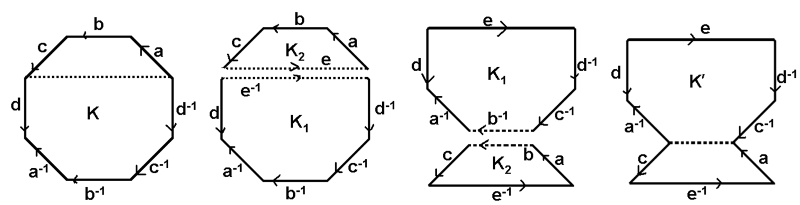


Figure 9: This diagram shows how a polygonal region K with a labeling scheme of $abcda^{-1}b^{-1}c^{-1}d^{-1}$ can be cut and then pasted back together at a different pair of edges to form the region K' with the labeling scheme $ac^{-1}d^{-1}eda^{-1}ce^{-1}$.

Proposition 3.1 *Given a linkage $L \in \bigcup_{n=1}^{3} P_n$, there exists a polygonal region K with edges defined by L. K can be cut along any interior line segment joining two vertices ℓ and pasted together to form the polygonal region K' with a different labeling scheme than that of K. The edges of K' can be described by a linkage L'. Then $L' \in \bigcup_{n=1}^{3} P_n$ if and only if $|\ell| = 1$.*

Proof

Let $L' \in \bigcup_{n=1}^{3} P_n$. The edges of L' will all be of length one since L' is created by eight unit vectors. L' is the linkage describing the edges of the polygonal region K' which was obtained by cutting-and-pasting the region K, described by L. Since K was cut along the line ℓ and has a different labeling scheme than K', ℓ must appear in the labeling scheme of K'. So K' has an edge labeled ℓ which corresponds to a vector in L' and so ℓ must be of length one.

Suppose that ℓ is the line segment in K which was cut along such that K' is obtained. Thus when K was cut apart along ℓ, the resulting two polygonal regions had unit length sides since the edges of K can be described by $L \in \bigcup_{n=1}^{3} P_n$. When these two regions are pasted together to form K', the edges which comprise K' will be of unit length since cutting-and-pasting has no effect on the length of existing edges and so L' will consist of unit vectors. Furthermore, the two regions obtained from cutting K can be described by linkages with an interior distance between distinct vertices greater than or equal to one since $L \in \bigcup_{n=1}^{3} P_n$. Pasting these two regions together will preserve this property, so L' will have interior distances between distinct vertices which are no less than one. The action of cutting-and-pasting is geometrical and will not affect the topological space, so K' will also paste together to be a genus 2 surface. This means that L' is the minimal vector decomposition of a translation surface of genus 2, so $L' \in \bigcup_{n=1}^{3} P_n$.

□

By Proposition 3.1, there exists a mapping via cutting-and-pasting which sends any linkage $L \in \bigcup_{n=1}^{3} P_n$ to a linkage $L' \neq L$ also in $\bigcup_{n=1}^{3} P_n$, shown in Figure 10. The same notion of equivalence between two polygonal regions K and K' which paste together to be the same topological space, can be extended to the linkages which describe K and K'.

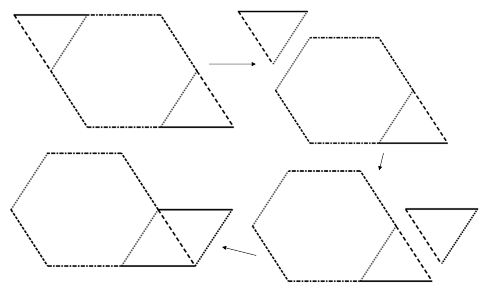


Figure 10: This is an example of how to get a linkage on $F_{1_2} \in P_2$ from a linkage on $F_{1_0} \in P_1$.

Proposition 3.2 *Given the linkages $L, L' \in \bigcup_{n=1}^{3} P_n$ describing the polygonal regions K, K'. The cutting-and-pasting techniques described above can be regarded as an equivalence relation, where $L \sim L'$ if K and K' are equivalent.*

Proof Let the polygonal regions K, K', K'' described by the linkages L, L', L'' be equivalent.

Reflexive: $L \sim L \; \forall \; L \in \bigcup_{n=1}^{3} P_n$.

K is equivalent to K since K can be cut along some line and then pasted together upon that same line. It follows that L will describe K, so $L \sim L$.

Symmetric: $L \sim L'$ implies $L' \sim L$ for all $L, L' \in \bigcup_{n=1}^{3} P_n$.

$L \sim L'$ implies K is equivalent to K'. Then K' is equivalent to K, so $L' \sim L$

Transitive: $L \sim L'$ and $L' \sim L''$ implies $L \sim L''$ for all $L, L', L'' \in \bigcup_{n=1}^{3} P_n$.

$L \sim L'$ implies K is equivalent to K' and $L' \sim L''$ implies K' is equivalent to K''. Then there exists a cutting-and-pasting of K to K' and then to K'', so K is equivalent to K''. Thus $L \sim L''$.

□

3.1 Structure of the CW-Complex

To understand the space that is $P = \bigcup_{n=1}^{3} P_n$ with the equivalence relation given in Proposition 3.2, consider the equivalence classes for the linkages on the boundary of P. These are given in Figure 11. By the equivalence relation of Proposition 3.2, it appears that P consists of 2 verticies, 8 edges, and 10 faces coming from 3 polyhedron. However, the linkages which form edges in P in the equivalence class E_{2_P} are equivalent to themselves except for the linkage which is the midpoint of the edge. That is if L is a linkage on some edge E in the equivalence class E_{2_P}, then there exists a linkage L' also on E with $L \sim L'$. There is one linkage W on E such that if $W \sim W'$ for W' on E, then W' is a relabeling of W and so W and W' are the same linkage, up to a relabeling. It turns out that W will be the midpoint of E in the configuration space and can be regarded as a third vertex of P.

It follows that P is a CW-complex with the n-cell breakdown as given in Figure 12. An n-cell is defined as being homeomorphic to a n-dimensional ball. The n-skeleton is the union of all of the m-cells for $m \leq n$. The 1-skeleton of P is given in Figure 12.

Equiv. Class	Members
F_{1_P}	F_{5_0}
F_{2_P}	$F_{1_0}, F_{1_2}, F_{2_2}$
F_{3_P}	F_{9_0}, F_{3_2}, F_{4_2}, F_{5_2}
F_{4_P}	F_{6_2}, F_{2_3}, F_{3_3}, F_{4_3}, F_{5_3}
F_{5_P}	F_{7_2}, F_{8_2}, F_{1_3}, F_{6_3}
F_{6_P}	faces in P_1 formed by: $\{V_{8_0}, V_{18_0}, V_{3_P}\}$, $\{V_{5_0}, V_{17_0}, V_{3_P}\}$
F_{7_P}	faces in P_1 formed by: $\{V_{1_0}, V_{16_0}, V_{3_P}\}$, $\{V_{4_0}, V_{13_0}, V_{3_P}\}$
F_{8_P}	faces in P_1 formed by: $\{V_{1_0}, V_{18_0}, V_{3_P}\}$, $\{V_{2_0}, V_{17_0}, V_{3_P}\}$
F_{9_P}	faces in P_1 formed by: $\{V_{8_0}, V_{13_0}, V_{3_P}\}$, $\{V_{5_0}, V_{16_0}, V_{3_P}\}$
F_{10_P}	faces in P_1 formed by: $\{V_{2_0}, V_{4_0}, V_{3_P}\}$
E_{1_P}	$E_{1_0} - E_{8_0}$, $E_{11_0} - E_{14_0}$, $E_{23_0} - E_{30_0}$, E_{3_2}, E_{4_2}, $E_{7_2} - E_{10_2}$
E_{2_P}	E_{9_0}, E_{10_0}, E_{5_2}, E_{6_2}, $E_{5_3} - E_{8_3}$
E_{3_P}	$E_{15_0} - E_{18_0}$, E_{2_2}, E_{11_2}, E_{14_2}, E_{15_2}, E_{16_2}, E_{1_3}, E_{2_3}, E_{10_3}, E_{12_3}
E_{4_P}	$E_{19_0} - E_{22_0}$, E_{1_2}, E_{12_2}, E_{13_2}, E_{17_2}, E_{18_2}, E_{3_3}, E_{4_3}, E_{9_3}, E_{11_3}
E_{5_P}	edges in P_1 between: $\{V_{1_0}, V_{3_P}\}$, $\{V_{2_0}, V_{3_P}\}$, $\{V_{4_0}, V_{3_P}\}$
E_{6_P}	edges in P_1 between: $\{V_{13_0}, V_{3_P}\}$, $\{V_{16_0}, V_{3_P}\}$
E_{7_P}	edges in P_1 between: $\{V_{17_0}, V_{3_P}\}$, $\{V_{18_0}, V_{3_P}\}$
E_{8_P}	edges in P_1 between: $\{V_{5_0}, V_{3_P}\}$, $\{V_{8_0}, V_{3_P}\}$
V_{1_P}	$V_{1_0} - V_{20_0}$, $V_{1_2} - V_{12_2}$, $V_{1_3} - V_{8_3}$
V_{2_P}	midpoint of edges in the equivalence class E_{2_P}
V_{3_P}	point $(\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4}) \in P_1$= regular octagon

Figure 11: These are the equivalence classes for the linkages on the boundary of P given for each n-cell ($n = 0, 1, 2$) and the linkages which belong to them.

4 Properties of the CW-Complex

To understand P beyond how it is constructed, some of its topological properties are calculated. The next three subsections explore the Euler characteristic, homology groups, and fundamental group of P.

4.1 Euler Characteristic

For a finite CW complex X, the Euler characteristic $\chi(X)$ is defined to be the alternating sum $\sum_n (-1)^n c_n$ where c_n is the number of n-cells of X [4]. For a surface S, this is the familiar formula: $\chi(S) = F - E + V = 2 - 2g$, where F is the number of faces, E the number of edges, V the number of vertices, and g the genus of S. For the space P,

$$\chi(P) = \sum_{n=0}^{3} (-1)^n c_n = 3 - 8 + 10 - 3 = 2.$$

A motivation for calculating the Euler characteristic is that it is a topological invariant, meaning that it is preserved under homeomorphism and thus can be used to show that other spaces are not related to P.

n-cell	P
0	3
1	8
2	10
3	3

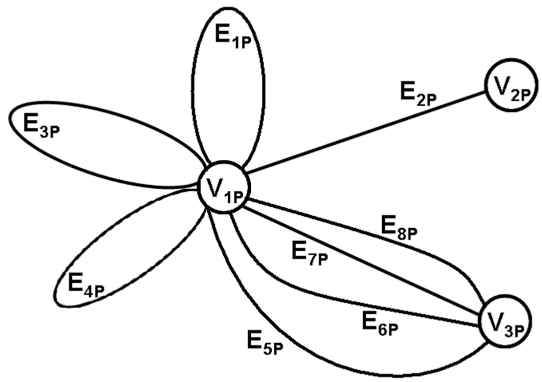


Figure 12: This figure shows the number of n-cells the CW-complex P has for each n. The 1-skeleton for P is also given.

4.2 Homology Groups

The homology groups of a space are also topological invariants. The n^{th} homology group of X (for $n \in \mathbb{Z}$) is defined to be the abelian group

$$H_n(X) = ker(\partial_n)/Im(\partial_{n+1}),$$

where $\partial_n : C_n(X) \to C_{n-1}(X)$ is a homomorphism on the n-chains of X. The n-chains are the abelian groups whose elements are integer linear combinations of oriented n-cells. It follows that the set of n-cells will generate the n-chains and so $rank(C_n) = c_n$. ∂_n is the boundary operator acting on the n-chains.

The chain complex for P is defined as:

$$(0) \xrightarrow{\partial_4} C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} (0)$$

The matrix representation of ∂_n, for each n, can be found by determining the map of each unique n-cell onto the $(n-1)$-cells which comprise its boundary. The boundary of each n-cell of P was found using the following process:

∂_0: The boundary of a 0-cell is trivial.

∂_1: The boundary of an oriented 1-cell, E_{i_P}, is $V_{j_P} - V_{k_P} \in C_0(P)$, where E_{i_P} is oriented toward V_{j_P} and away from V_{k_P}. The orientation of each 1-cell with respect to P was determined by the cutting-and-pasting mapping discussed in Section 3. For example, $E_{2_P} = V_{2_P} - V_{1_P}$.

∂_2: The boundary of an oriented 2-cell, F_{i_P}, is $a_j E_{j_P} + a_k E_{k_P} + ... + a_m E_{m_P} \in C_1(P)$, where F_{i_P} is oriented in either the clockwise or counter-clockwise direction and the sign of the coefficient of each 1-cell is determined by whether it is oriented to go with the orientation of F_{i_P} $(+)$ or if it goes against the orientation of F_{i_P} $(-)$. For example, $F_{2_P} = E_{1_P} + E_{4_P} - E_{1_P} - E_{3_P} + E_{1_P} + E_{2_P} - E_{2_P} = E_{1_P} - E_{3_P} + E_{4_P}$.

∂_3: The boundary of an oriented 3-cell, P_i, is $b_j F_{j_P} + b_k F_{k_P} + ... + b_m F_{m_P} \in C_2(P)$, where the sign of the coefficient of each 2-cell is determined by the 2-cells orientation. The orientation of each 2-cell can be described by the set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of the canonical basis, where $\vec{e}_3$ is the outward facing normal vector of the 2-cell and so $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ denotes a 2-cell oriented counter-clockwise while $(\vec{e}_2, \vec{e}_1, \vec{e}_3)$ denotes a 2-cell oriented clockwise. Taking the determinant of the matrix representation of $(\vec{e}_i, \vec{e}_j, \vec{e}_3)$ results in 2-cells oriented counter-clockwise having a positive sign and those oriented clockwise have a negative sign. For example, $P_3 = F_{5_P} - F_{5_P} - F_{4_P} + F_{4_P} - F_{4_P} - F_{5_P} = -2F_{5_P}$. Notice that a 2-cell is not required to always have the same orientation with respect to the 3-cell since its orientation is determined by how the 1-cells around it are placed.

For the boundary operators of P, let $M(\partial_n)$ denote the matrix representation of ∂_n. Then:

$$M(\partial_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad M(\partial_1) = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}$$

$$M(\partial_2) = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad M(\partial_0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

Since $H_n(P)$ is an abelian group, we can apply the Fundamental Theorem for Finitely Generated Abelian Groups.

Theorem 4.1 *[6] Let G be a finitely generated abelian group. Let T be its torsion subgroup.*
(a) There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.
(b) There are finite cyclic groups $T_1, ..., T_k$ where T_i has order $t_i > 1$, such that $t_i|t_2|...|t_k$ and $T = T_1 \oplus ... \oplus T_k$.
(c) The numbers β and $t_1, ..., t_k$ are uniquely determined by G.

In particular, for each n,

$$H_n(P) \cong \mathbb{Z}^{\beta_n} \oplus \mathbb{Z}/t_{1_n} \oplus ... \oplus \mathbb{Z}/t_{k_n}.$$

The number β_n is called the n^{th} *Betti* number of P. The Smith Normal form of the matrix representation of the boundary operators of P gives all of these numbers. The torsion coefficients of $H_n(P)$ are given by the entries which are greater than one in $M(\partial_{n+1})$. Furthermore, $rank(ker(\partial_{n+1}))$ is the number of zero columns of $M(\partial_{n+1})$. Additionally, the number of non-zero rows of $M(\partial_{n+1})$ is equal to $rank(Im(\partial_{n+1}))$. It follows that

$$\beta_n = rank(ker(\partial_n)) - rank(Im(\partial_{n+1})) = rank(H_n).$$

Let $SNF(\partial_n)$ denote the Smith Normal form of $M(\partial_n)$, then:

$$SNF(\partial_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad SNF(\partial_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$SNF(\partial_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad SNF(\partial_0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

Figure 13 shows the homology groups of P found from the Smith Normal Form of the $M(\partial_n)$.

n	β_n	t_{i_n}	$H_n(P)$
0	3-2=1	-	$\mathbb{Z}$
1	6-6=0	5	$\mathbb{Z}/5\mathbb{Z}$
2	4-3=1	2	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
3	0-0=0	-	$\{0\}$

Figure 13: This table shows the homology groups and their rank for P.

The homology groups of a space keep track of the number of n-dimensional holes in the space. For example, $rank(H_0(P)) = 1$, so P has one 0-dimensional hole (an open point) and so P is path-connected since it will have a single path. Similarly, P has no 1-dimensional holes (a circle, S^1) nor 3-dimensional holes (S^3), but P does have a 2-dimensional hole (S^2).

The homology groups are related to the Euler characteristic in that for a finite CW-complex X, $\chi(X) = \sum_n (-1)^n rank(H_n(X))$ [4]. This is just the alternating sum of the Betti numbers for each homology group; so

$$\chi(P) = 1 - 0 + 1 - 0 = 2.$$

This agrees with the calculation of $\chi(P)$ given in Section 4.1.

4.3 The Fundamental Group

The group of loops in a space X starting and ending at a basepoint $x_0 \in X$ is the fundamental group of X, denoted $\pi_1(X)$. Two loops are considered to be the same if one can be deformed into the other within X [4]. The space P is path-connected; allowing for the use of Van Kampen's Theorem to calculate $\pi_1(P)$.

Theorem 4.2 (Van Kampen) *[4] If X is the union of path-connected open sets A_α, A_β each containing the basepoint $x_0 \in X$ and if the intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\phi : \pi_1(A_\alpha) * \pi_1(A_\beta) \to \pi_1(X)$ is surjective. If in addition the intersection*

$A_\alpha \cap A_\beta$ is path-connected, then the kernel of ϕ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi(A_\alpha \cap A_\beta)$, and hence ϕ induces an isomorphism $\pi_1(X) \cong (\pi_1(A_\alpha) * \pi_1(A_\beta))/N$.

For calculating $\pi_1(P)$, let

$$P = \mathbb{R}^2 \cup \left(\bigcup_{k=1}^{10} U_k \right)$$

where U_k is an open set around the 1-skeleton for each face of P. $\pi_1(\mathbb{R}^n) = \{0\}$ for all n, so $\pi_1(P) = *_k \pi_1(U_k)/ker(\phi)$. (Note: It is sufficient to consider only the 2-skeletons of P. Taking into account the n-skeletons of P for $n > 2$ will add no new information to $\pi_1(P)$ [4].) Let p_0 denote the basepoint for finding $\pi_1(P)$ where p_0 is the vertex in P which corresponds to the vertex that was used to plot the configuration space for each linkage type. The loops around each F_k will generate $ker(\phi)$ since each will start and end at p_0 and are distinct from one another since the U_k are distinct (due to the faces of P being distinct). The generators of $*_k \pi_1(U_k)$ are the distinct paths taken along the edges of P from p_0 to p_0. Figure 14 shows each face of of P with their corresponding edges and vertices.

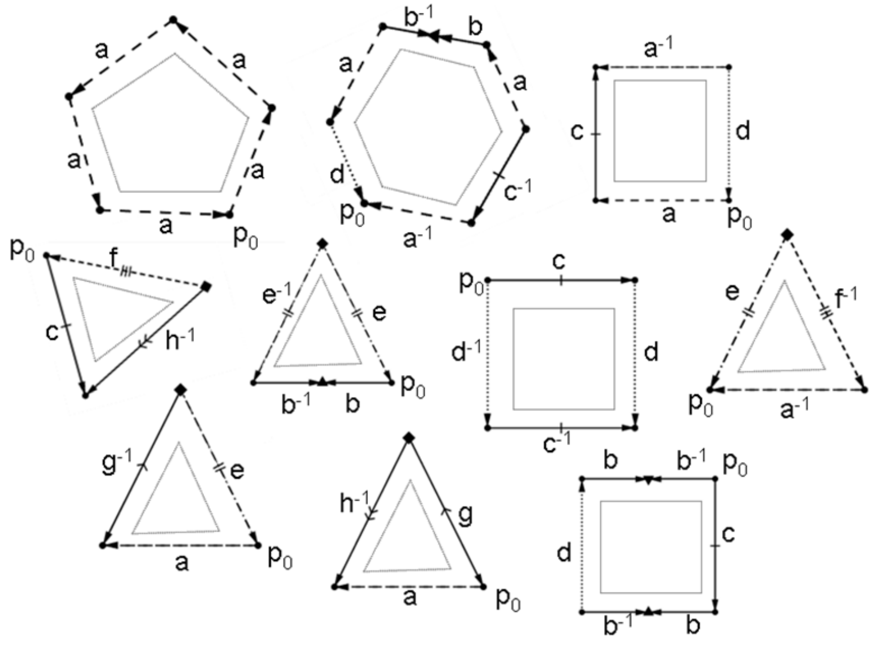


Figure 14: The labels and orientations used for each edge in P are shown with respect to the faces which they lie on. The orientation of each face is implied by the power of each edge. The interior line indicates the boundary of the open set around the 1-skeleton, U_k, defining each face. For example, the 2-cell which has five edges (F_{1_P}) is generated by edge a and $aaaaa \in ker(\phi)$. The basepoint p_0 is also indicated.

Calculating $\pi_1(P)$ yields:

$$\pi_1(P) = \langle a, bb^{-1}, c, d, h^{-1}g, f^{-1}e, g^{-1}e, h^{-1}f \rangle /$$
$$\langle a^5, a^{-1}c^{-1}abb^{-1}ad, aca^{-1}d, cbb^{-1}dbb^{-1}, cdc^{-1}d^{-1}, ah^{-1}g, a^{-1}f^{-1}e, ag^{-1}e, ch^{-1}f, bb^{-1}e^{-1}e \rangle.$$

For notation simplicity, a, b, c, d, e, f, g, h was used to replace E_{1_P}, E_{2_P}, E_{3_P}, E_{4_P}, E_{5_P}, E_{6_P}, E_{7_P}, E_{8_P}, respectively.

This can be reduced slightly since the loop $bb^{-1}e^{-1}e$ is trivial, as well as bb^{-1}. Additionally, the generator $h^{-1}f$ can be generated by three other generators of $*_k\pi_1(F_k)$ and thus is unnecessary. Then:

$$\pi_1(P) = \langle a, c, d, h^{-1}g, f^{-1}e, g^{-1}e\rangle/$$
$$\langle a^5, a^{-1}c^{-1}a^2d, aca^{-1}d, cd, cdc^{-1}d^{-1}, ah^{-1}g, a^{-1}f^{-1}e, ag^{-1}e, ch^{-1}f\rangle.$$

To simplify the notation of the generators, let $h^{-1}g = w$, $f^{-1}e = u$, and $g^{-1}e = v$, yielding:

$$\pi_1(P) = \langle a, c, d, w, u, v\rangle/$$
$$\langle a^5, a^{-1}c^{-1}a^2d, aca^{-1}d, cd, cdc^{-1}d^{-1}, aw, a^{-1}u, av, cwvu^{-1}\rangle.$$

The fundamental group is related to the first homology group by the following theorem.

Theorem 4.3 *[4] By regarding loops as singular 1 cycles, we obtain a homomorphism h : $\pi_1(X, x_0) \to H_1(X)$. If X is path-connected, then h is surjective and has kernel the commutator subgroup of $\pi_1(X)$, so h induces an isomorphism from the abelianization of $\pi_1(X)$ onto $H_1(X)$.*

Calculating the abelianization of $\pi_1(P)$ does indeed yield an outcome isomorphic to $H_1(P)$.

$$Abel(\pi_1(P)) = \pi_1(P)/[\pi_1(P), \pi_1(P)]$$
$$= \langle a, c, d, w, u, v\rangle/\langle a^5, ac^{-1}d, cd, aw, a^{-1}u, av, cwvu^{-1}\rangle.$$

Since $ker(\phi) = \{a^5, ac^{-1}d, cd, aw, a^{-1}u, av, cwvu^{-1}\}$, it follows that each element of the set is equal to the identity element and so $a = w^{-1} = u = v^{-1}$ and $c = d^{-1}$. This leads to $c = uv^{-1}w^{-1} = a^3$ and $c = ad = ac^{-1}$ so $c^2 = a$. Then $c^5 = id$. $Abel(\pi_1(P))$ can then be reduced to:

$$Abel(\pi_1(P)) = \langle a\rangle/\langle a^5\rangle \cong \mathbb{Z}/5\mathbb{Z} \cong H_1(P).$$

Acknowledgments

It is a pleasure to thank my fellow researcher Kathryn Marsh of Purdue University with whom I collaborated on this project. I would also like to thank my mentor, Prof. Christopher Judge of Indiana University, the NSF for their funding and the REU program at Indiana University.

References

[1] M. A. Armstrong: *Basic Topology.* Springer-Verlag.New York, 1983.

[2] M. Genduphe and C. Judge, *Well-rounded holomorphic 1-forms* 2014 (in preparation)

[3] R. Ghrist: *Elementary Applied Topology*, 2014 (in preparation)

[4] A. Hatcher: *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[5] H. Masur, *Ergodic Theory of Translation Surfaces.*

[6] J. R. Munkres: *Elements of Algebraic Topology.*Benjamin/Cummings Publishing, Menlo Park, 1984.

[7] J. R. Munkres: *Topology.* 2nd Ed., Princeton Hall, Upper Saddle River, 2000.

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