THE IHARA ZETA FUNCTION

TONGYU YANG

Abstract. In this paper, we introduce the definition of the Ihara zeta function and its determinant formula. We provide a detailed proof of the determinant formula by using the edge zeta function. This paper is based on an article [1] written by Audrey A. Terras with the addition of details and a focus on the proof.

1. Introduction

The famous Riemann zeta funtion $\zeta(s)$, which plays a critical role in number theory and complex analysis, is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} (1 - p^{-s})^{-1},$$

where s is a complex number with $\mathrm{Re}(s) > 1$. Greatly used in many areas such as physics and statistics, many other kinds of zeta functions haven been investigated. Yasutaka Ihara first made a connection between the zeta function and graphs, and introduced the vertex (Ihara) zeta function on a graph $G = (V, E)$. It is commonly used to relate closed walks to the spectrum of the adjacency matrix. Its definition is an infinite product, and Bass provided a proof of its determinant form, which enables the calculation of the function. In this paper, we will give a definition of the Ihara zeta function and expand the proof of the determinant formula by filling in the details.

2. Background

We start with some background and definitions.

A **graph** is composed of two parts: vertices V and edges E. We usually denote it by the pair $X = (V, E)$. A graph is called **directed** if the edges connecting two vertices are oriented.

A **path** P in X is a collection of directed edges $\{a_1, a_2, \cdots, a_n\}$ such that $P = a_1a_2\cdots a_n$. A path $P = a_1a_2\cdots a_n$ is **closed** if the starting vertex of a_1 is the ending vertex of a_n. So we can take n steps to traverse P from the starting edge a_1 to a_n. A graph $X = (V, E)$ is **connected** if for any two vertices x, y, there exists a path P from x to y. From now on, we will assume our graph is finite, undirected, and connected.

Definition 1 (Backtrackless)**.** *A path $P = a_1\cdots a_n$ is backtrackless if $a_{i+1} \neq a_i^{-1}$ $\forall i \in \{1, \cdots, n\}$, where a_i^{-1} is the edge a_i with the inverse orientation.*

This is to say, a backtrackless graph cannot walk back from the same edge immediately. Now, we can define tailless in a similar way.

Definition 2 (Tailless)**.** *A path $P = a_1\cdots a_n$ is tailless if it has no such backtrack with $a_n = a_1^{-1}$.*

From here, we can assume the graphs on which we apply the Ihara zeta function have no backtracking or tails. We need several more definitions for the Ihara zeta function.

Definition 3 (Equivalence class)**.** *An equivalence class for a path $P = a_1a_2\cdots a_n$ is*

$$[P] = \{a_1a_2\cdots a_n, a_2a_3\cdots a_1, \cdots, a_na_1\cdots a_{n-1}\}.$$

Definition 4 (Primitive path)**.** *A path P in graph X is primitive if $P \neq D^n$, where D is a closed path in X, $n > 1$, and D^n means traversing path D n times.*

So a primitive path means it is not a repetition of the same path. Now, we can turn back to the main focus of paper, the Ihara zeta function.

3. Discussion on The Ihara Zeta Function

Definition 5. *Let $v(P)$ denote the number of vertices in a path P. The Ihara(vertex) zeta function of a graph X is defined at $u \in \mathbb{C}$ by*

$$\zeta(u, X) = \prod_{[P]\, primitive} (1 - u^{v(P)})^{-1},$$

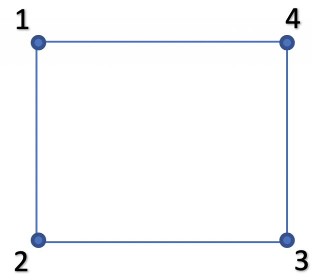

FIGURE 1. The Square Graph X

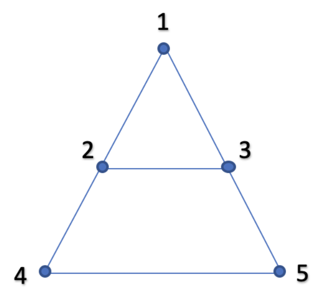

FIGURE 2. The Triangle Graph Y

where $|u|$ is sufficiently small and $[P]$ are the equivalence classes of all closed backtrackless tailless primitive paths.

The product is infinite when we have a nontrivial graph. However, Hashimoto and Bass proved that this function can actually be calculated by the determinant of some matrices.

Theorem 1. *The determinant formula of the Ihara zeta function in a graph X is*

$$\zeta(u, X)^{-1} = (1-u^2)^{r-1} det(I - A_X u + Q_X u^2),$$

where $r-1 = |E| - |V|$, A_X is the adjacency matrix of X, and Q_X is the diagonal matrix whose i^{th} entry is $deg(v_i) - 1$.

Let us illustrate the formula with an application in the simple square graph in Fig.1.

Note that in this graph, there are only two prime paths $P_1 = 1234$ counterclockwise and $P_2 = 1432$ clockwise with length 4. Any other closed backtrackless tailless primitive paths are in the equivalence classes of these two paths. So by Ihara zeta function, $v(P) = 4$ for these two paths. We have

$$\begin{aligned}\zeta(u,X) &= (1-u^4)^{-1}(1-u^4)^{-1}\\ &= (1-u^4)^{-2}.\end{aligned}$$

On the other hand, the A_X and Q_X matrices for X are

$$A_X = \begin{pmatrix} 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 \end{pmatrix},\ Q_X = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$I_4 - A_X u + Q_X u^2 = \begin{pmatrix} 1+u^2 & -u & 0 & -u\\ -u & 1+u^2 & -u & 0\\ 0 & -u & 1+u^2 & -u\\ -u & 0 & -u & 1+u^2 \end{pmatrix}.$$

Since $r-1 = |E|-|V| = 4-4 = 0$ in graph X, the formula in Theorem 1 gives us

$$\begin{aligned}\zeta(u,X)^{-1} &= (1-u^2)^0 det(I_4 - A_X u + Q_X u^2)\\ &= -(u^2+1)(-u^2-1)(u^4-2u^2+1)\\ &= (u^2+1)^2(u^2-1)^2\\ &= (u^4-1)^2.\end{aligned}$$

Thus both formula show the same answer for the square graph X as we expected. This graph is small enough so that we could use Ihara zeta function directly to compute the result. However, in most cases the number of equivalent classes of prime paths is infinite, so the determinant formula of Ihara zeta function provides the way to calculate the function. Consider the triangle graph Y with a line connecting the mid-points of two sides in Fig. 2.

Note that vertices 2 and 3 have degree 3 while other vertices have degree 2. So the A_Y and Q_Y matrices related to this graph are

$$A_Y = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \; Q_Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$I_5 - A_Y u + Q_Y u^2 = \begin{pmatrix} 1+u^2 & -u & -u & 0 & 0 \\ -u & 1+2u^2 & -u & -u & 0 \\ -u & -u & 1+2u^2 & 0 & -u \\ 0 & -u & 0 & 1+u^2 & -u \\ 0 & 0 & -u & -u & 1+u^2 \end{pmatrix}.$$

By the determinant formula,

$$\begin{aligned} \zeta(u,Y)^{-1} &= (1-u^2)^{r-1} det(I_5 - A_Y u + Q_Y u^2) \\ &= (1-u^2)^{6-5}(4u^{10} + 3u^8 - 2u^7 - 4u^5 - u^4 - 2u^3 + u^2 + 1) \\ &= -4u^{12} + u^{10} + 2u^9 + 3u^8 + 2u^7 + u^6 - 2u^5 - 2u^4 - 2u^3 + 1. \end{aligned}$$

We can use SageMath online to calculate this graph of the zeta function, and it gives

$$\zeta(u,Y)^{-1} = -4u^{12} + u^{10} + 2u^9 + 3u^8 + 2u^7 + u^6 - 2u^5 - 2u^4 - 2u^3 + 1.$$

which is the same as our calculation.

4. Proof of the Determinant Formula

4.1. **The Edge Zeta Function.**

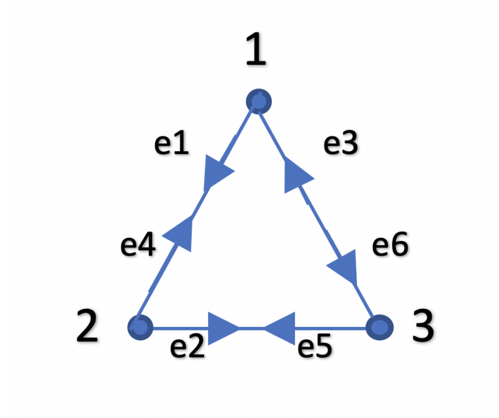

FIGURE 3. The Oriented Graph Z

Definition 6. *For a connected undirected graph X, orient all edges in both directions to obtain $2|E|$ edges and label them by $e_1, e_2, \cdots, e_m, e_{m+1} = e_1^{-1}, \cdots, e_{2m} = e_m^{-1}$. The 0,1 edge matrix W_1 is a $2m \times 2m$ matrix with entry $ij = 1$ if the end vertex of edge e_i is the same as the start vertex of edge e_j with $j \neq i^{-1}$, and otherwise entry $ij = 0$. The edge matrix W is defined the same as W_1 except the entry is denoted by variable w_{ij} instead of 1 if the end vertex of edge i is the same as the start vertex of edge j.*

Consider the triangle graph Z in Fig. 3 as an example. The edges e_1, e_2, e_3 are oriented counterclockwise and $e_4 = e_1^{-1}, e_5 = e_2^{-1}, e_6 = e_3^{-1}$ are oriented clockwise. Then we have W_1 and W matrices:

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ W = \begin{pmatrix} 0 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{23} & 0 & 0 & 0 \\ w_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_{46} \\ 0 & 0 & 0 & w_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & 0 \end{pmatrix}.$$

Definition 7. *For a path $P = a_1 a_2 \cdots a_n$, the path weight of P is calculated by $N(P) = w_{a_1a_2} w_{a_2a_3} \cdots w_{a_{n-1}a_n}$.*

From the above definitions, we define the **edge zeta function** by

$$\zeta_E(W, X) = \prod_{[P]} (1 - N(P))^{-1},$$

where $N(P)$ is the path weight over primitive paths in X and W is the edge matrix of X.

To prove Theorem 1, we need to connect the infinite product in the formula with the determinant of some matrix. Note that by letting all weights in W equal to u, i.e. $W = W_1 u$, we get $N(P) = u^{v(P)}$. Then the edge zeta function leads to the Ihara zeta function. We will show the connection via edge zeta function and specialize it to the Ihara zeta function.

Theorem 2 (Determinant Formula for the Edge Zeta Function)**.**

$$\zeta_E(W, X) = det(I - W)^{-1}.$$

To prove the theorem, we need the help of Euler's theorem about homogeneous functions and a theorem of entries in adjacency matrices.

Lemma 1 (Euler's Homogeneous Function Theorem [3])**.** *A multivariable polynomial function f is homogeneous of degree k if and only if $x \cdot \nabla f(x) = kf(x)$.*

Lemma 2. *Let W be any square matrix. Then*

$$det(1 - W) = exp(-\sum_{n=1}^{\infty} Tr(W^n)\frac{1}{n}),$$

where $Tr(W^n)$ is the trace of matrix W^n.

Let's omit the proof of above two lemmas here, and provide a proof of the third lemma in the following.

Lemma 3 (Entries of Adjacency Matrix [2])**.** *Let A be the adjacency matrix of a graph X. For any integer $k > 0$, the ij^{th} entry of matrix A^k is the number of walks of length k from vertex i to vertex j.*

Proof of lemma 3. We will prove this by induction on k. In the base case when $k = 1$, entry $ij = 1$ if and only if vertex i connects vertex j by some edge, i.e. there is a walk from i to j of length 1.

Assume this is true for all positive integers up to $k-1$, we need to show that $(A^k)_{ij}$ is the number of walks of length k from i to j. Note that

$$(A^k)_{ij} = \sum_{v=1}^{n} (A^{k-1})_{iv} A_{vj},$$

where $v \in V$ runs over all vertices. For the entry A_{vj}, it is 1 if vertex v connects to j and 0 otherwise. By the induction hypothesis the entry in $(A^{k-1})_{iv}$ is equal to the number of walks of length $k-1$ from i to v, and so for each $v \in V$ the product is 1 if there is a walk of length k which connects i to v and v to j, and is 0 if there is no such walk. Thus, the summation over all vertices gives us the total number of walks of length k from vertex i to vertex j.

□

Proof of theorem 2. Firstly, applying $-log$ to both sides of the edge zeta function, we get

$$\begin{aligned} -log\zeta_E(W, X) &= -log(\prod_{[P]} (1 - N(P))^{-1}) \\ &= log(\prod_{[P]} (1 - N(P))) \\ &= \sum_{[P]} log(1 - N(P)). \end{aligned}$$

The Taylor expansion of $log(1-x)$ is $-\sum_{n=1}^{\infty} \frac{x^n}{n}$. Then, we have

$$log\zeta_E(W, X) = \sum_{[P]} \sum_{j=1}^{\infty} \frac{1}{j} N(P)^j.$$

Since there are $v(P)$ elements in every equivalent class of $[P]$, we can write

$$\sum_{[P]} N(P)^j = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{P \\ v(P)=n}} N(P)^j,$$

where P under the second summation runs over all prime paths. It is clear that $N(P)$ is a homogeneous function of degree n. Let $F = \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}$ be the Euler operator in lemma 1 and apply to both sides of the equation, we have

$$\begin{aligned}
F(log\zeta_E(W,X)) &= F(\sum_{[P]}\sum_{j=1}^{\infty}\frac{1}{j}N(P)^j) \\
&= F(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{jn}\sum_{\substack{P \\ v(P)=n}} N(P)^j) \\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{jn}\sum_{\substack{P \\ v(P)=n}} F(N(P)^j).
\end{aligned}$$

By lemma 1, the function $N(P)$ is homogeneous of degree n, so $F(N(P)^j) = jnN(P)^j$. It follows that

$$\begin{aligned}
Flog\zeta_E(W,X) &= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{jn}\sum_{\substack{P \\ v(P)=n}} jnN(P)^j \\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\substack{P \\ v(P)=n}} N(P)^j \\
&= \sum_{C} N(C),
\end{aligned}$$

where C denotes all closed backtrackless tailless paths in X, and C need not to be a prime path. This is because the summation over n gives the length of closed prime paths and the summation over j cancels the primitive identity. By lemma 3, the summation of all closed paths $N(C)$ is exactly the summation of the product of diagonals in the adjacency matrix by closeness. Then we have

$$\begin{aligned}
Flog\zeta_E(W,X) &= \sum_{m\geq 1}\sum_{\substack{C \\ v(C)=m}} N(C) \\
&= \sum_{m\geq 1} \mathrm{Tr}(W^m).
\end{aligned}$$

Note that $\mathrm{Tr}(W^m)$ is homogeneous of degree m, and so applying the Euler operator and we have

$$F(\mathrm{Tr}(W^m)) = m\mathrm{Tr}(W^m).$$

By lemma 2,

$$\begin{aligned} Flog(det(I-W)^{-1}) &= -(-\sum_{m\geq 1} m\frac{Tr(W^m)}{m}) \\ &= \sum_{m\geq 1} Tr(W^m). \end{aligned}$$

This completes the proof.

□

Now, if we let all non-zero weights in W be u, then the edge zeta function equals the Ihara zeta function.

Corollary 1.

$$\zeta(u, X) = det(I - W_1 u)^{-1}.$$

4.2. **Bass's Proof of the Determinant Formula.** Given a graph X, set $|V(X)| = n$ and $|E(X)| = m$. Orient the edges and construct two matrices with size $n \times 2m$, the starting matrix S and the terminal matrix T, as follows:

$$S_{ve} = \{v \in V, e \in E | S_{ve} = 1 \text{ if } e \text{ starts from } v, S_{ve} = 0 \text{ otherwise}\}$$

$$T_{ve} = \{v \in V, e \in E | T_{ve} = 1 \text{ if } e \text{ ends in } v, T_{ve} = 0 \text{ otherwise}\}$$

Let the superscript H^T denote the transpose of a matrix H. A and Q are the same matrix defined in the beginning. Define the matrix $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ of size $2m \times 2m$. Then we have several observations:

[1] $ST^T = A$.

[2] $T^T S = W_1 + J$.

[3] $SS^T = TT^T = Q + I =$ diagonal matrix with whose i^{th} entry is $\deg(v^i)$.

[4] $SJ = T$ and $TJ = S$.

The first observation can be shown by considering the $n \times n$ matrix ST^T. The ij^{th} entry of ST^T is calculated by the summation $\sum_{e \in E} S_{ie} T_{je}$, which is the number of edges starting at vertex i and ending at vertex j. This is 1 if and only if an edge connects i and j, which is the same as adjacency matrix A.

Similarly, consider the $2m \times 2m$ matrix $T^T S$. The ab^{th} entry of $T^T S$ is calculated by the summation $\sum_{v \in V} T_{va} S_{vb}$, which is the number of vertices x such that x is the starting vertex of edge b and is the ending vertex of edge a. By definition of W_1, this is the same as W_1 except when $b = a^{-1}$. Since edge $e_b = e_a^{-1}$ if $e_b = e_{a+m}$, by adding the matrix J, we have $T^T S = W_1 + J$.

Let us illustrate [2] with the square graph X in Fig. 1. Define edges e_1, e_2, e_3, e_4 by starting from vertex 1, traversing in counterclockwise order. Then $e_5 = e_1^{-1}, \cdots, e_8 = e_4^{-1}$. We have adjacency matrix A_X in the example, and let $J_X = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$. Note that for matrix X,

$$S_X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, by transposing T_X, we have $T_X^T S_X = W_1 + J_X$.

For [3], consider any entry ij of the matrix SS^T. $SS_{ij}^T = \sum_{e \in E} S_{ie} S_{je}$, which is the number of the edges that starts in both vertices i and j. Since every edge can only have one starting vertex, the summation is 0 if $i \neq j$ and equals $\deg(i)$ if $i = j$. The same argument shows that TT^T is a diagonal matrix with entry k equals to the number of edges that ends in vertex k, i.e. the degree of vertex k.

Finally, [4] is trivial since by multiplying matrix J, we interchange the starting and terminal vertices of edge e_i by the definition of edge numbering system, i.e. $e_{m+1} = e_1^{-1}, \cdots, e_{2m} = e_m^{-1}$.

Based on these useful observations, Bass provides a proof of the Ihara zeta function and matches the infinite product with the determinant of some matrix.

Proof. For a graph X with n vertices and $2m$ oriented edges, consider the matrix product of size $(n + 2m) \times (n + 2m)$:

$$\begin{pmatrix} I_n & 0 \\ T^T & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix}. \tag{1}$$

Using the normal product rule of matrix, we get the $(n + 2m) \times (n + 2m)$ matrix

$$\begin{pmatrix} I_n(1-u^2) & Su \\ T^T(1-u^2) & T^T Su + I_{2m} - W_1 u \end{pmatrix}. \tag{2}$$

Now, consider another matrix product of the same size:

$$\begin{pmatrix} I_n - A_u + Qu^2 & Su \\ 0 & I_{2m} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^T - S^T u & I_{2m} \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} I_n - A_u + Qu^2 + SuT^T - SuS^T u & Su \\ T^T + JuT^T - S^T u - JuS^T u & I_{2m} + Ju \end{pmatrix}. \tag{4}$$

Note that by [1]&[3],

$$I_n - A_u + Qu^2 + SuT^T - SuS^T u = I_n - A_u + Qu^2 + Au - (Q + I_n)u^2$$
$$= I_n(1 - u^2).$$

Apply [4] and transpose both sides we can get

$$T^T + JuT^T - S^T u - JuS^T u = T^T + S^T u - S^T u - T^T u^2$$
$$= T^T(1 - u^2).$$

And lastly observation [2] gives us

$$T^T Su + I_{2m} - W_1 u = (W_1 + J)u + I_{2m} - W_1 u$$
$$= I_{2m} + Ju.$$

We have shown that matrix (2) is exactly the same as matrix (4), which indicates the equivalence of the matrix product (1)&(3). Take determinants of both equation we have the equality

$$(1 - u^2)^n \det(I - W_1 u) = \det(I_n - Au + Qu^2) \det(I + Ju). \tag{5}$$

We know that the determinant of a product is the product of determinants, so write

$$\det(I + Ju) = \det \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

and multiply the determinant of matrix $M = \begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix}$ we obtain

$$\begin{aligned}
&\det \begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix} \det \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix} \\
&= \det \begin{pmatrix} I & Iu \\ 0 & I(-u^2+1) \end{pmatrix} \\
&= (1-u^2)^m.
\end{aligned}$$

since the size of M is $2m \times 2m$. By observing that $\det(M) = 1$, equation (5) becomes:

$$\begin{aligned}
(1-u^2)^n \det(I - W_1 u) &= \det(I_n - Au + Qu^2)(1-u^2)^m \\
\det(I - W_1 u) &= \det(I_n - Au + Qu^2)(1-u^2)^{m-n}
\end{aligned}$$

By Corollary 1 and subsitute $m - n$ with $r - 1$ from definition, we finally get the Ihara determinant formula

$$\zeta(u, X)^{-1} = (1-u^2)^{r-1} \det(I_n - Au + Qu^2).$$

□

5. Conclusion

In this paper, we start off with definitions of graphs, and define the Ihara zeta function both in terms of a product on the equivalence classes of primitive paths and in terms of the determinant of a matrix obtained from the original graph X. After this, we provide two simple examples to calculate the Ihara zeta function and compare the results from both formula. Then, the edge zeta function is introduced and we provide a proof of the determinant formula for the edge zeta function. Finally, with the help of a special case in the edge zeta function, the Ihara determinant formula is proved in detail.

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