# The Erdos Distinct Distance Problem on Compact Riemannian 2-Manifolds

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**Abstract**

We examine the Erdos distinct distance problem on compact Riemannian 2-manifolds. We prove an analogue of Erdos's original result in the Riemannian setting. In the process, we prove several results about geodesic existence and regularity, along with results about perturbations of Euclidean metrics. We finally describe a possible application of the Erdos problem to spectral sets on Riemannian manfiolds similar to that in the Euclidean case.

## Contents

# 1 Introduction and History

## 1.1 History of Problem

For a finite set $P \subseteq \mathbb{R}^2$, let

$$\Delta(P) = \{|x - y| : x, y \in P\}$$

We call $\Delta(P)$ the distance set associated with $P$, as $\Delta(P)$ is the set of all distinct distances associated with the points in $P$. In 1946, Paul Erdos studied how $|\Delta(P)|$ depends on $|P|$. More precisely, he proved that, for large enough $n$, there exists a constant $C$ independent of $n$ such that for any $P \subseteq \mathbb{R}^2$ with $|P| = n$ we have the bound

$$|\Delta(P)| \geq Cn^{1/2}$$

In this same paper, he proved that we have an upper bound

$$|\Delta(P)| \leq \tilde{C}\frac{n}{\sqrt{\log n}}$$

by considering a square grid with size length $\sqrt{n}$. The study of the size of $\Delta(P)$ for finite sets in the plane, known as the Erdos distance problem, has been primarily focused on improving the lower bound for $|\Delta(P)|$. It is currently conjectured that

$$|\Delta(P)| \geq C\frac{n}{\sqrt{\log n}}$$

for large enough $n$. Since 1946, there have been significant improvements in Erdos's original lower bound. In 1952, Leo Moser proved in [8] that

$$|\Delta(P)| \geq Cn^{2/3}$$

The most recent bound is due to Larry Guth and Netz Katz in 2015 ([4]), and says that

$$|\Delta(P)| \geq C\frac{n}{\log n}$$

There of course have been several successive improvements in the lower bound between 1952 and 2015, however, in this paper, we will concern ourselves only with the $n^{1/2}$ in the body of the paper, and with a brief discussion of the $n^{2/3}$ bound in the final section.

The Erdos distance problem has also been extended to finite subsets of $\mathbb{R}^d$ for $d \geq 2$. Using the same notation as above, it is known that for any subset $P \subseteq \mathbb{R}^d$ with $|P| = n$

$$|\Delta(P)| \geq C_d n^{\frac{1}{d}+\varepsilon}$$

for some fixed $\varepsilon > 0$. In fact, there is a stronger result stating that

$$|\Delta(P)| \geq C_d n^{\frac{3}{3d-2}}$$

(Theorem 12.13 in [9]).

## 1.2 Applications to Harmonic Analysis

The Erdos distance problem, while being an interesting geometric problem, has also been useful in harmonic analysis, and we will briefly discuss its application here. If $D \subseteq \mathbb{R}^d$ has finite non-zero Lebesgue measure, $D$ is said to be spectral if $L^2(D)$ has an orthogonal basis of exponentials. The authors of [5] used the result

$$|\Delta(P)| \geq C_d n^{\frac{1}{d}+\varepsilon}$$

to show that any affine image of $\overline{B}_d = \{x \in \mathbb{R}^d : x| \leq 1\}$ is not spectral.

## 1.3 Notation, Results, and Outline of Paper

If $(M, g)$ is a Riemannian manifold, let $d_g$ denote the usual distance function on $M$ associated to the metric $g$. If $P \subseteq M$ is finite, define

$$\Delta_g(P) = \{d_g(x, y) : x, y \in P\}$$

We will study lower bounds for $|\Delta(P)|$ specifically in the case when $M$ is compact and $\dim M = 2$. In this paper, we transport the original Erdos bound to this setting, and discuss ways to transport Moser's bound. We prove the following result:

**Theorem** *Let $(M, g)$ be a compact connected Riemannian 2-manifold. Then, there exists constants $C_M$ and $n_0$ such that, if $P \subseteq M$ is a finite subset with $n = |P| \geq n_0$, then*

$$|\Delta_g(P)| \geq C_M n^{1/2}$$

We begin by presenting proofs of both the Erdos and Moser bounds in the plane. We then develop preliminary machinery to prove our principal result. We prove the Banach fixed point theorem, discuss spaces of $C^1$ functions, and Riemannian geometry. We then spend time getting specific regularity results for solutions of the geodesic equation. In Section 4, we then present a brief outline of our argument, give a preliminary perturbation argument, and then prove our main theorem. We then discuss further work that could be done on this topic, namely work in higher dimensions, Moser's bound, and application to spectral sets on manifolds.

## 2 $n^{1/2}$ Bound in the Plane

In this section, we present a proof of the $n^{1/2}$ bound for the Erdos distace problem in the plane. While several proofs of both bounds are known ([3] for other methods of proof), the proofs we present are likely the easiest to transport to the manifold setting.

**Theorem** *There exists a constant $C > 0$ and an $n_0 \in \mathbb{Z}^+$ such that, for each $n \geq n_0$ and each $P \subseteq \mathbb{R}^2$ with $|P| = n$, we have*

$$|\Delta(P)| \geq Cn^{1/2}$$

PROOF: Our proof will be from [3].Let $n \geq 2$ and let $P \subseteq \mathbb{R}^2$ be a set with $n$ points. Fix a $p_0 \in P$, and let

$$A_{p_0} = \{|p - p_0| : p \in P\}$$

Let $k = |A_{p_0}|$. Let $C_r$ denote the circle centered at $p_0$ of radius $r$ (for $r > 0$). Consider the collection of circles $C_t$ for $t \in A_{p_0}$. Then, by the pigeonhole principle, there must exist a $t_0 \in A_{p_0}$ such that

$$|P \cap C_{t_0}| = \frac{n-1}{k} \geq \frac{n}{2k}$$

Bisect $C_{t_0}$ via the line passing through $p_0$ which is parallel to the $x$-coordinate axis. Then, we see that we must have at least $n/4k$ points on either the northern half or on the southern half. Without loss of generality, assume that there are at least $n/4k$ points on the northern half. Picking the right-most point, we see that there must then be at most $n/4k$ distinct distances on the northern half. Therefore, we have that

$$|\Delta(P)| \geq \max\{k, n/4k\} \geq \sqrt{k}\sqrt{\frac{n}{4k}} = \frac{n^{1/2}}{2}$$

◭

*Remark.* Our argument relies on the fact that Euclidean distances increase as one moves along a hemisphere of a circle. More precisely, fix a circle centered at any point of radius $d$, and parameterize this circle by $\theta$. Fixing a point $\theta_0$ on this circle, we consider its distance between another point $\theta$ on this circle which can easily be computed to be

$$d\sqrt{2}\sqrt{1 - \cos(\theta - \theta_0)}$$

In particular, if we consider this as a function of $\theta$ on the interval $[\theta_0, \theta_0 + \pi]$, this function is strictly increasing. Transporting this type of argument to the Riemannian setting will turn out to be the biggest hurdle. Indeed, we will see later that it is easier to work on $[\theta_0, \theta_0 + \pi/2]$ instead, to ensure distances increase.

# 3 Technical Preliminaries

## 3.1 Banach Fixed Point Theorem

We will begin by proving a classical fixed point theorem for contractions on complete metric spaces.

**Definition** *Let $(X, d)$ be a metric space. A contraction on $(X, d)$ is a function $f : X \to X$ such that there exists $C \in [0, 1)$ such that, for each $x, y \in X$*

$$d(f(x), f(y)) \leq Cd(x, y)$$

*Remark.* It is clear that any contraction is continuous. Indeed, suppose $f : X \to X$ is a contraction with constant $C \in [0, 1)$, and let $\varepsilon > 0$. Fixing $a \in X$, we see that

$$d(f(x), f(a)) \leq Cd(x, a) < d(x, a) < \varepsilon$$

whenever $d(x, a) < \varepsilon$. This gives continuity at $a \in X$. Since $a \in X$ was arbitrary, we obtain global continuity.

**Theorem** *(Banach Fixed Point Theorem): Let $(X, d)$ be a non-empty complete metric space, and suppose $f : X \to X$ is a contraction. Then $f$ has a unique fixed point, i.e, there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$*

PROOF: Uniqueness is trivial, for assume that $x_0, x_0'$ are fixed points for $f$. We then have

$$d(x_0, x_0') = d(f(x_0), f(x_0')) \leq Cd(x_0, x_0')$$

If $x_0 \neq x_0'$, then this says that

$$d(x_0, x_0') < d(x_0, x_0')$$

as $C \in [0, 1)$ and $d(x_0, x_0') \neq 0$, which is absurd. Therefore, if $f$ has a fixed point, it must be unique.

We now must prove that $f$ has a fixed point. Fix $x \in X$. Define the sequence

$$x_n = f^n(x)$$

where $f^n$ denotes the $n$-fold composition of $f$ with itself. We note that if $k$ is a positive integer, induction easily shows that

$$d(f^k(x), x) \leq \sum_{i=1}^{k} C^k d(f(x), x) \leq d(f(x), x) \sum_{i=0}^{\infty} C^k = \frac{d(f(x), x)}{1 - C}$$

since $C \in [0, 1)$. Let $\varepsilon > 0$. Choose $N \in \mathbb{Z}^+$ such that

$$\frac{C^N d(f(x), x)}{1 - C} < \varepsilon$$

(which we may do since $0 \leq C < 1$). We then see that, if $N \leq m \leq n$, we have

$$d(f^n(x), f^m(x)) \leq C^m d(f^{n-m}(x), x) \leq C^m d(f^{n-m}(x), x) \leq \frac{C^m d(f(x), x)}{1-C} \leq \frac{C^N d(f(x), x)}{1-C} < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\{x_n\}$ is Cauchy, and thus converges to some $x_0 \in X$. Since $f$ is continuous as it is a contraction, we have that

$$f(x_0) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$$

showing that $x_0$ is a fixed point of $f$. ◭

## 3.2 Spaces of $C^1$ Functions

We recall the definition of a Banach Space:

**Definition** *Let $(V, ||\cdot||)$ be a normed vector space. We say that $(V, ||\cdot||)$ is a banach space if $V$ is a complete metric space under the metric induced by $||\cdot||$, i.e, if $V$ is a complete metric space with the metric given by*

$$d(x, y) = ||x - y||$$

We will primarily concern ourselves with a specific class of Banach spaces, namely those consisting of $C^1$ functions defined on a compact interval. More specifically, let

$$C^1([a, b], \mathbb{R}^k)$$

be the set of all continuously differentiable functions defined on $[a, b]$ with values in $\mathbb{R}^k$. We define a norm on this space as follows

$$||f||_{C^1} = \sup |f| + \sup |f'|$$

for $f \in C^1([a, b], \mathbb{R}^k)$. We call this the $C^1$ norm. Note that this is well defined and always finite by continuity of $f$ and $f'$, along with the compactness of $[a, b]$. We must show that this is indeed a norm. Clearly $||f||_{C^1} \geq 0$. Moreover, since $\sup |f|$ and $\sup |f'|$ are both non-negative, we see that $||f||_{C^1} = 0$ if and only if both $\sup |f| = 0$ and $\sup |f'| = 0$. If $f = 0$, then clearly $\sup |f| = 0$ and $\sup |f'|0$. Conversely, if $\sup |f| = 0$ and $\sup |f'| = 0$, then, by definition, we have that $|f(x)| \leq 0$ for each $x \in [a, b]$, and therefore $f = 0$. Thus, positive definiteness has been proven. To show homogeneity, note that if $a \geq 0$, and $g : [a, b] \to \mathbb{R}$ is continuous, we have

$$\sup(ag) = a \sup g$$

This is trivial for $a = 0$, so assume $a > 0$. To show for this case, note that we have for each $x \in [a, b]$

$$(ag)(x) = ag(x) \leq a \sup g$$

Thus

$$\sup(ag) \leq a \sup g$$

Suppose that

$$\sup(ag) < a \sup g$$

Then, for each $x \in [a, b]$, we have

$$ag(x) \leq \sup(ag) < a \sup g$$

Since $a > 0$, this implies that

$$g(x) < \sup g$$

for each $x \in [a, b]$. In particular, we see that $\sup g$ is not attained by $g$, contradicting the extreme value theorem. Thus, we must have

$$\sup(ag) = a \sup g$$

Applying this result to $|a||f| = |af|$ and $|a||f'| = |af'| = |(af)'|$ for $f \in C^1([a, b], \mathbb{R}^k)$, we see that

$$||af||_{C^1} = \sup |af| + \sup |af'| = |a| \sup |f| + |a| \sup |f'| = |a| \cdot ||f||_{C^1}$$

Finally to show the triangle inequality, suppose that $g, h : [a, b] \to \mathbb{R}$ are bounded functions. Then, for each $x \in [a, b]$, we have

$$(g + h)(x) = g(x) + h(x) \leq \sup g + \sup h$$

and thus

$$\sup(g + h) \leq \sup g + \sup h$$

Therefore, if $f_1, f_2 \in C^1([a, b], \mathbb{R}^k)$, we have

$$\sup |f_1 + f_2| \leq \sup(|f_1| + |f_2|) \leq \sup |f_1| + \sup |f_2|$$

$$\sup |(f_1 + f_2)'| = \sup |f_1' + f_2'| \leq \sup(|f_1'| + |f_2'|) \leq \sup |f_1'| + \sup |f_2'|$$

where we have leveraged the triangle inequality along with the monotonicity of the supremum. Therefore

$$||f_1 + f_2||_{C^1} \leq ||f_1||_{C^1} + ||f_2||_{C^1}$$

as desired. Therefore $C^1([a, b], \mathbb{R}^k)$ along with the above norm is indeed a normed vector space. However, we have a much stronger result, namely that this space is a Banach space.

**Theorem** *The space $C^1([a, b], \mathbb{R}^k)$ along with the $C^1$ norm is a Banach space*

PROOF: Note that $C^1([a,b],\mathbb{R}^k)$ is indeed a vector space and the $C^1$ norm is indeed a norm. We will thus show completeness. Suppose $\{f_n\}$ is a Cauchy sequence of functions in $C^1([a,b],\mathbb{R}^k)$. Let $\varepsilon > 0$. Then, by definition, there exists $N$ such that, if $n,m \geq N$

$$||f_m - f_n||_{C^1} = \sup|f_n - f_m| + \sup|f'_n - f'_m| < \varepsilon$$

In particular, for $n,m \geq N$

$$\sup|f_n - f_m| < \varepsilon$$

$$\sup|f'_n - f'_m| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, it follows that the $f_n$ converge uniformly in absolute value to some function $f$ on $[a,b]$ and the $f'_n$ converge uniformly in absolute value to some function $g$ on $[a,b]$. A theorem from undergraduate analysis shows that $g$ is continuous on $[a,b]$, and a similar theorem shows that $f$ is differentiable on $[a,b]$ with $f' = g$ (see [10]). Thus, we see that $f \in C^1([a,b],\mathbb{R}^k)$. We now verify that the $f_n$ converge to $f$ in the $C^1$ norm. Let $\varepsilon > 0$. By uniform convergence of $f_n \to f$ and $f'_n \to f'$, there exists $N$ such that if $n \geq N$

$$\sup|f_n - f| < \frac{\varepsilon}{2}$$

$$\sup|f'_n - f| < \frac{\varepsilon}{2}$$

Thus, if $n \geq N$, we have

$$||f_n - f||_{C^1} = \sup|f_n - f| + \sup|f_n - f'| < \varepsilon$$

and thus $f_n \to f$ in $C^1$. ◭

## 3.3 Basics of Riemannian Geometry

In this section, we will review much of the Riemannian geometry we will need for our problem.

### 3.3.1 Riemannian Metrics

**Definition** *Let $M$ be a smooth manifold. A Riemannian metric $g$ on $M$ is a smooth, symmetric, positive definite section of $T^*M \otimes T^*M$. If $g$ is a Riemannian metric on $M$, the pair $(M,g)$ is said to be a Riemannian manifold*

Unpacking this definition, we see that a Riemannian metric $g$ on $M$ defines an inner product $g_p$ on $T_pM$ at each point $p \in M$, and this choice of inner product varies smoothly. For brevity, we will often call $g$ a metric on $M$. Such a metric is not to be confused with metrics in the sense of metric spaces. When referring to a metric on a metric space, we will instead use the term distance function.

If $(U,x^i)$ is a chart at $p$, then we have the following expression for $g$ in this chart

$$g = g_{ij}\,dx^i \otimes dx^j$$

where

$$g_{ij} = g(\partial_i, \partial_j)$$

We note that the $g_{ij}$ are smooth functions on $U$.

Given a smooth curve $\gamma : I \to M$, if $[a, b] \subseteq I$, we define

$$\ell_a^b(\gamma) = \int_a^b ||\gamma'(t)||dt$$

where

$$||\gamma'(t)|| = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}$$

If $[a, b] = I$, we will often write $\ell(\gamma)$ instead of $\ell_a^b(\gamma)$.

If $(M, g)$ is connected (and therefore path connected), then it is well known that any two points on $M$ can be connected by a smooth curve. For $p, q \in M$, we define $d_g : M \times M \to \mathbb{R}$

$$d_g(p, q) = \inf_{\gamma \in C} \ell(\gamma)$$

where

$$C = \{\gamma : \gamma \text{ is a smooth curve from } p \text{ to } q\}$$

We have the following theorem

**Theorem** *If $(M, g)$ is a Riemannian manifold, then $d_g$ is a distance function on $M$ whose induced metric topology agrees with the topology on $M$*

### 3.3.2 Levi-Civita Connection and the Covariant Derivative

The main goal of this section is to describe the Levi-Civita connection on a Riemannian manifold. We will begin by briefly describing the notion of an affine connection, which gives us a notion of how to differentiate a vector field with respect to another vector field. We will also discuss how such a connection allows us to differentiate vector fields along curves. We will then restrict ourselves to the Riemannian setting, describing a unique affine connection associated with a Riemannian manifold, which is known as the Levi-Civita connection. We denote the set of smooth vector fields on a manifold by $\mathcal{X}(M)$.

On an open subset $U$ of $\mathbb{R}^n$, one may naturally view a smooth vector field as simply a smooth function from $\mathbb{R}^n \to \mathbb{R}^n$. Doing so, we see that if $X = (X^1, ..., X^n)$ and $Y = (Y^1, ..., Y^n)$ are smooth vector fields on $U$, we may define the following vector field

$$D_X Y = (X \cdot \nabla Y^1, ...., X \cdot \nabla Y^n)$$

which we view as differentiating $Y$ along $X$. We have thus defined an operator $D : \mathcal{X}(\mathbb{R}^n) \times \mathcal{X}(\mathbb{R}^n) \to \mathcal{X}(\mathbb{R}^n)$. It is clear that this operator satisfies the following properties:

1. If $X, Y, Z \in \mathcal{X}(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$ then

$$D_{fX+Y}Z = fD_XZ + D_YZ$$

2. If $X, Y, Z \in \mathcal{X}(\mathbb{R}^n)$ and if $f \in C^\infty(\mathbb{R}^n)$ then

$$D_X(fY + Z) = fD_XY + (X \cdot \nabla f)Y + D_XZ$$

In other words, $D$ is an $\mathbb{R}$ bi-linear map which is $C^\infty(\mathbb{R}^n)$ linear in the first coordinate and satisfies a Leibniz rule in the second coordinate. This motivates the following definition:

**Definition** *Let $M$ be a smooth manifold. An affine connection on $M$ is a bi-linear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that*
*1): For each $f \in C^\infty(M)$ and $X, Y \in \mathcal{X}(M)$*

$$\nabla_{fX}Y = f\nabla_XY$$

*2): For each $f \in C^\infty(M)$ and $X, Y \in \mathcal{X}(M)$*

$$\nabla_X(fY) = f\nabla_XY + (Xf)Y$$

*where we write $\nabla_XY$ for $\nabla(X, Y)$*

As an example, our function $D$ is an affine connection on $\mathbb{R}^n$ (and we will soon see that it is an example of an important type of affine connection).

Given an affine connection $\nabla$, one can show that $\nabla_XY\Big|_p$ depends only on $X_p$ and a the behavior of $Y$ in an arbitrarily small neighborhood of $p$. One can then use this fact as follows: given a chart $(U, x^i)$ at $p$, one can consider the vector field $\nabla_{\partial_i}\partial_j$ on $U$. Since the $\partial_k$ form a basis for the set of all smooth vector fields on $U$, we may write

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

The functions $\Gamma^k_{ij}$ on $U$ are called Christoffel symbols.

Affine connections also give us a way to differentiate vector fields defined along smooth curves. Given a smooth curve $\gamma : I \to M$, a smooth vector field along $\gamma$ is a smooth mapping $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for each $t \in I$. We have the following theorem:

**Theorem** *Let $M$ be a smooth manifold with affine connection $\nabla$. Then, given a smooth curve $\gamma : I \to M$, there exists a unique correspondance which associates to a smooth vector field $V$ along $\gamma$ to a smooth vector field $D_tV$ along $\gamma$ with the following properties:*
*1): $D_t(V + W) = D_tV + D_tW$*
*2): $D_t(fV) = (f')V + fD_tV$ for any smooth function $f : I \to \mathbb{R}$*
*3): If $V$ is induced by a vector field $Y$ on $M$ (in other words, there is some vector field $Y$ on $M$ with $V(t) = Y_{\gamma(t)}$ for each $t$), then*

$$D_tV = \nabla_{\gamma'}Y$$

*(where the term on the right hand side is the vector field given by the affine connection)*

To prove this, we define $D_t V$ in a chart. If $V = v^i \partial_i$ and $\gamma = (x^1, ..., x^n)$ then we define

$$D_t V = (v^k)' \partial_k + (x^i)'(v^j)' \Gamma^k_{ij} \partial_k$$

It is then quick to show that $D_t V$ is well defined as the expression is invariant under change of coordinates. This gives existence. By locally extending $V$ in a chart, one can also show that such a correspondence with properties 1-3 must have the above local coordinate definition, which gives uniqueness. We call this assignment the covariant derivative, and it will be of great importance.

There are two special types of affine connections, one which can be defined on general manifolds and one which can be defined only on Riemannian manifolds. We define these below:

**Definition** *Suppose $\nabla$ is an affine connection on a smooth manifold $M$. We say that $\nabla$ is symmetric if for each $X, Y \in \mathcal{X}(M)$ we have*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

**Definition** *Suppose $\nabla$ is an affine connection on a Riemannian manifold $(M, g)$. We say that $\nabla$ is compatible with the metric $g$ if for each $X, Y, Z \in \mathcal{X}(M)$ we have*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

We view symmetry as a measure on how well differentiating a vector fields with respect to each other commutes, and we view compatibility as a type of product rule. With these definitions, we may state the following important theorem, often referred to as the Fundamental Theorem of Riemannian Geometry:

**Theorem** *Let $(M, g)$ be a Riemannian manifold. Then there exists a unique affine connection $\nabla$ on $M$ which is symmetric and compatible with the metric $g$*

We call the connection $\nabla$ the Levi-Civita (or Riemannian) connection on $(M, g)$. We for brevity, we will not prove the above theorem. For proofs, see [2] or [7]. As an example, it is a standard exercise to show that our above operator $D$ is the Levi-Civita connection on $\mathbb{R}^n$ with the usual metric given by

$$g = \delta_{ij} dx^i \otimes dx^j$$

defined in standard coordinates.

### 3.3.3 Geodesics

Let $(M, g)$ be a Riemannian manifold, and let $\nabla$ be the Levi-Civita connection associated to $(M, g)$. A smooth curve $\gamma : I \to M$ is said to be a geodesic if

$$D_t \gamma' = 0$$

It is important to gain some intuition on this definition, we note that lines $t \mapsto at + b$ have second derivative zero. Noting that the covariant derivative in Euclidean space is the usual derivative, we see that a geodesic on Euclidean space is exactly a straight line. Furthermore, noting that in elementary differential geometry, a second derivative is a measurement of curvature, one may interpret the equation $D_t\gamma' = 0$ (which is known as the geodesics equation) as saying that one cannot sense the curvature of the curve $\gamma$ when one moves along $\gamma$. In other words, $\gamma$ resembles a straight line when one moves along it.

Suppose $p$ is a point in $\gamma(I)$, and $(U, x^i)$ is a chart at $p$. Recall the Christoffel symbols associated with a coordinate chart $(U, x^i)$, which are defined to be the unique smooth functions $\Gamma^k_{ij}$ on $U$ such that

$$\nabla_{\partial_i}\partial_j(q) = \Gamma^k_{ij}(q)\partial_k|_q$$

We then have

$$\gamma'(t) = (x^i)'(t)\partial_i|_{\gamma(t)}$$

for $t$ such that $\gamma(t) \in U$, where we have written $x^i(t)$ for the $i$-th coordinate function of $x\circ\gamma(t)$. Using our local coordinate formula for the covariant derivative, we see that

$$D_t\gamma' = (x^k)''\partial_k + (x^i)'(x^j)'\Gamma^k_{ij}\partial_k$$

We thus obtain the geodesic equation in local coordinates, which is the following system of ordinary differential equations

$$(x^k)'' + \Gamma^k_{ij}(x^i)'(x^j)' = 0$$

### 3.3.4 The Exponential Map and Normal Coordinates

In coordinates, the geodesic equation is a second order ordinary differential equation. If we specify the initial condition and initial velocity, we can apply existence and uniqueness theory to show that there is a unique solution defined on some interval $(-\delta, \delta)$, where this interval depends on the initial conditions. If $\gamma : (-\delta, \delta) \to M$ is a geodesic starting at $p$ with initial velocity $v$ and $s > 0$, then it is easily shown that the map $\tilde{\gamma} : (-\delta/s, \delta/s) \to M$ defined by

$$\tilde{\gamma}(t) = \gamma(st)$$

is the geodesic starting at $p$ with initial velocity $sv$. In particular, we see that, given $p \in M$, there exists a $\delta > 0$ such that for all $v \in T_pM$ with $|v| < \delta$, the unique geodesic starting at $p$ with initial velocity $v$, which we denote by $\gamma_{p,v}$, is defined on an interval containing 1. We may define the map $\exp_p : B_\delta(T_pM) \to M$ by

$$\exp_p(v) = \gamma_{p,v}(1)$$

By constructing flows on the tangent bundle $TM$, it can be shown that the exponential map is smooth (see [2] or [7] for details). Moreover, we see that

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0}\exp_p(tv) = \frac{d}{dt}\Big|_{t=0}\gamma_{p,tv}(1) = \frac{d}{dt}\Big|_{t=0}\gamma_{p,v}(t) = \gamma'_{p,v}(0) = v$$

Thus, the inverse function theorem implies that $\exp_p$ is a local diffeomorphism at 0. Therefore, there exists $\varepsilon_p > 0$ such that $\exp_p$ is a diffeomorphism from the ball $B_{\varepsilon_p}(T_pM)$ onto some open subset of $M$ containing $p$. From this, we obtain a specialized chart at $p$, which we call normal coordinates at $p$. It can be shown that, in normal coordinates, we have

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2)$$

There are several proofs of this and similar results; it follows from Theorem 5.24 of [7], but stronger versions exist as well, see for example Theorem 2.65 of [1].

#### 3.3.5 Convexity Radius

Throughout this section, we have discussed the interpretation of geodesics as generalizations of straight lines. Keeping this, we may define notions of convexity on Riemannian manifolds. In particular, if $p \in M$, and $U$ be an open set containing $p$, we say that $U$ is geodesically convex if for each $q_1, q_2 \in U$, there is a unique distance minimizing geodesic connecting $q_1$ and $q_2$ whose image is contained within $U$. We have the following theorem from [7]:

**Theorem** *Let $(M, g)$ be a Riemannian manifold. Then for each $p \in M$, there is some $\varepsilon > 0$ such that the normal ball $B_\varepsilon$ centered at $p$ is defined and geodesically convex*

The above theorem says that each point on a Riemannian manifold has a geodesically convex normal ball centered at that point. Therefore, we may define $conv : M \to (0, \infty]$ by

$$conv(p) = \sup\{\varepsilon > 0 : \text{There is a geodesically convex normal ball of radius } \varepsilon \text{ centered at } p\}$$

It is a fact (see problem 6-6 of [7]) that $conv$ is a continuous function on $M$. Thus, if $M$ is compact, there exists an $\varepsilon > 0$ such that for each $p \in M$, there is a geodesically convex normal ball of radius $\varepsilon$ centered at $p$.

### 3.4 Smoothness of Geodesics Depending on Initial Conditions

Let us re-examine the geodesic equation in coordinates. Recall that the geodesic equation is given by

$$(y'')^k = -\Gamma^k_{ij}(y^i)'(y^j)'$$

Let us define the function $\Gamma : B_\varepsilon(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ by the function whose $k$-th coordinate is

$$\Gamma^k(x, x') = \Gamma^k_{ij}(x)(x^i)'(x^j)'$$

where we have denoted the coordinate on $B_\varepsilon(\mathbb{R}^n) \times \mathbb{R}^n$ by $(x, x')$. The geodesic equation then becomes

$$y''(s) = -\Gamma(y(s), y'(s))$$

If $y(0) = c$ and $y'(0) = v$, we may re-write this as the equivalent integral equation

$$y(t) = c + tv - \int_0^t \int_0^s \Gamma(y(s'), y'(s'))ds'ds$$

Thus, we may study geodesics by studying fixed points of the operator

$$t \mapsto c + tv - \int_0^t \int_0^s \Gamma(y(s'), y'(s'))ds'ds$$

To begin our study of this operator, we must recall the Frobenius norm of a matrix.

**Definition** *Let $A = (a_{ij})$ be an $n \times m$ matrix. The Frobenius norm of A, denoted by $|A|$ is defined to be*

$$|A| = \sqrt{\sum_{i,j} a_{ij}^2}$$

**Lemma** *Suppose $A$ is an $m \times n$ matrix and $v \in \mathbb{R}^n$. Then*

$$|Av| \leq |A||v|$$

PROOF: If

$$v = (v_1, ..., v_n)$$

and

$$A = (a_{ij})$$

Then the $k$-th component of $Av$ is

$$(Av)_k = \sum_{j=1}^n a_{kj} v_j$$

Cauchy-Schwarz then yields

$$|Av|^2 = \sum_{k=1}^n (Av)_k^2 = \sum_{k=1}^n \left(\sum_{j=1}^n a_{kj} v_j\right)^2 \leq \sum_{k=1}^n \left(\left(\sum_{j=1}^n a_{kj}^2\right)\left(\sum_{j=1}^n v_j^2\right)\right) = \left(\sum_{k=1}^n \sum_{j=1}^n a_{kj}^2\right)\left(\sum_{j=1}^n v_j^2\right) = |A|^2|v|^2$$

Taking square roots gives our result. ◭

We use this result in the following lemma, which is a useful lemma about $C^1$ functions:

**Lemma** *Suppose that $F : U \to \mathbb{R}^m$ is $C^1$ where $U \subseteq \mathbb{R}^n$ is convex, and suppose that $|DF|$ is bounded above on $U$, where $DF$ is the Jacobian matrix of $F$ and $|\cdot|$ denotes the Frobenius norm on matrices. Then $F$ is Lipschitz on $U$ and the Lipschitz constant can be taken to be any upper bound for $|DF|$ on $U$*

PROOF: For each $t \in [0, 1]$, $ty + (1 - t)x \in U$ by convexity. Let $M$ be an upper bound for $|DF|$ on $U$. We have by the fundamental theorem of calculus and the chain rule

$$F(y) - F(x) = \int_0^1 \frac{d}{dt} F(ty + (1-t)x)dt = \int_0^1 DF(ty + (1-t)x)(y-x)dt$$

Therefore

$$|F(y) - F(x)| = \left| \int_0^1 DF(ty + (1-t)x)(y-x)dt \right| \leq \int_0^1 |DF(ty + (1-t)x)||y-x|dt \leq M \int_0^1 |y-x|dt = M|y-x|$$

◭

The following results about switching integrals and limits will also be useful.

**Lemma** *Suppose that $a \in \mathbb{R}^n$. Suppose for some $r > 0$, the function*

$$f : [a,b] \times D_{a,r} \to \mathbb{R}$$

*is defined and continuous, where*

$$D_{a,r} = \{x \in \mathbb{R}^n : 0 < |x-a| < r\}$$

*Suppose further that the function $g : [a,b] \to \mathbb{R}$ defined by*

$$g(t) = \lim_{x \to a} f(t,x)$$

*is defined and continuous. Then, for each $t \in [a,b]$*

$$\lim_{x \to a} \int_a^t f(s,x)ds = \int_a^t \lim_{x \to a} f(s,x)ds$$

PROOF: First define $F : [a,b] \times B_r(a) \to \mathbb{R}$ by

$$F(t,x) = \begin{cases} f(t,x) & x \neq a \\ g(t) & x = a \end{cases}$$

We will show that $F$ is continuous. Suppose $(t,x) \in [a,b] \times B_r(a)$. If $x \neq a$, then there is a neighborhood containing $(t,x)$ such that any $(t',x')$ inside said neighborhood is such that $x' \neq a$. Thus, it is clear that $F$ is continuous at $(t,x)$ by continuity of $f$. Now suppose $x = a$. Let $\varepsilon > 0$. By continuity of $g$, let $\delta_1 > 0$ be such that

$$|g(t') - g(t)| < \frac{\varepsilon}{2}$$

whenever $|t - t'| < \delta_1$. Let $\delta_2 > 0$ be such that

$$|f(t',x') - g(t')| < \frac{\varepsilon}{2}$$

whenever $0 < |a - x'| < \delta_2$. Take $\delta = \min(\delta_1, \delta_2)$. Now suppose that $(t', x') \in [a, b] \times B_r(a)$ is such that

$$|(t', x') - (t, x)| = |(t', x') - (t, a)| < \delta$$

Then we must have that $|t' - t| < \delta$ and $|x - a| < \delta$. If $x' = a$, we have

$$|F(t', x') - F(t, a)| = |g(t') - g(t)| < \frac{\varepsilon}{2} < \varepsilon$$

If $x' \neq a$ then we have

$$|F(t', x') - F(t, a)| = |f(t', x') - g(t)| \leq |f(t', x') - g(t')| + |g(t') - g(t)| < \varepsilon$$

Thus, $F$ is continuous at $(t, a)$. Therefore, $F$ is continuous. In particular, we see that $F$ is continuous on $[a, b] \times \overline{B}_{r/2}(a)$. Since this set is compact, we see that $F$ is uniformly continuous on this set.

Let $t \in [a, b]$. We must show that

$$\lim_{x \to a} \int_a^t f(s, x) ds = \int_a^t g(s) ds$$

Let $\varepsilon > 0$. By uniform continuity, choose $\frac{r}{2} > \delta > 0$ such that

$$|F(s, x) - F(s', x')| < \frac{\varepsilon}{b - a}$$

whenever $|(s, x) - (s', x')| < \delta$ where $(s, x), (s', x') \in [a, b] \times \overline{B}_{r/2}(a)$. Let $x$ be such that $0 < |x - a| < \delta$. Then, for each $s \in [a, b]$, we see that $(s, x), (s, a) \in [a, b] \times \overline{B}_{r/2}(a)$ and

$$0 < |(s, x) - (s, a)| = |x - a| < \delta$$

Therefore, if $0 < |x - a| < \delta$ we have

$$|f(s, x) - g(s)| = |F(s, x) - F(s, a)| < \frac{\varepsilon}{b - a}$$

Thus, we have, if $0 < |x - a| < \delta$

$$\left| \int_a^t f(s, x) ds - \int_a^t g(s) ds \right| \leq \int_a^t |f(s, x) - g(s)| ds < (t - a) \frac{\varepsilon}{b - a} \leq \varepsilon$$

as desired. ◭

We obtain the following corollary

**Corollary** *Suppose that $a \in \mathbb{R}^n$. Suppose for some $r > 0$, the function*

$$f : [a, b] \times D_{a,r} \to \mathbb{R}^m$$

*is defined and continuous, where*

$$D_{a,r} = \{x \in \mathbb{R}^n : 0 < |x-a| < r\}$$

*Suppose further that the function $g : [a,b] \to \mathbb{R}^m$ defined by*

$$g(t) = \lim_{x \to a} f(t,x)$$

*is defined and continuous. Then, for each $t \in [a,b]$*

$$\lim_{x\to a} \int_a^t f(s,x)ds = \int_a^t \lim_{x\to a} f(s,x)ds$$

PROOF: This follows by applying the above lemma to the coordinate functions of $f$ and $g$. ◭

Let $\varepsilon > 0$ be such that $\varepsilon < 1/6$. Define the following sets

$$X = \{\Gamma \in C^1(\overline{B}_5(0,\mathbb{R}^n) \times \mathbb{R}^n, \mathbb{R}^n) : ||\Gamma||_{C^1} < \varepsilon\}$$

$$Z = C^1([0,2], \overline{B}_5(0,\mathbb{R}^n))$$

Note that $X$ is an open subset of a locally convex topological vector space. Define

$$\Phi : B_1(0,\mathbb{R}^n) \times \overline{B}_1(\mathbb{R}^n) \times X \times Z \to Z$$

by

$$\Phi(c,v,\Gamma,y)(t) = c + tv - \int_0^t \int_0^s \Gamma(y(s'), y'(s'))ds'ds$$

We will first show that the codomain of $\Phi$ is well defined. We have that

$$\sup_t |\Phi(y)(t)| = \sup_t \left| c + tv - \int_0^t \int_0^s \Gamma(y(s'), y'(s'))ds' \right| \le \sup_t |c| + \sup_t |tv| + \sup_t \left| \int_0^t \int_0^s \Gamma(y(s'), y'(s'))ds'ds \right|$$

$$\le 1 + 2 + \sup_t \int_0^t \int_0^s |\Gamma(y(s'), y'(s'))|ds' < 1 + 2 + \varepsilon \sup_t \int_0^t \int_0^s ds'ds \le 1 + 2 + 4\varepsilon < 1 + 2 + 1 = 3$$

$$\sup_t |\Phi(y)'(t)| = \sup_t \left| v + \int_0^t \Gamma(y(s'), y'(s'))ds' \right| \le 1 + \sup_t \int_0^t |\Gamma(y(s'), y'(s'))|ds' < 1 + 2\varepsilon < 1 + 1 = 2$$

Thus, we see that

$$||\Phi(y)||_{C^1} < 5$$

and thus the codomain of $\Phi$ is well defined.

We note that $Z$ is a closed subset of a Banach space and is therefore a complete metric space. We would like to show that $\Phi$ has a fixed point in $Z$ for fixed $\Gamma$, $c$, and $v$. We note that by our above lemma, $\Gamma$ is Lipschitz with Lipschitz constant less than $\varepsilon$ since $\overline{B}_5(0) \times \mathbb{R}^n$

is convex and $\sup_x |D\Gamma(x)| < \varepsilon$ on the domain of $\Gamma$ by hypothesis. We then have that, if $y_1, y_2 \in Z$

$$||\Phi(y_2) - \Phi(y_2)||_{C^1} = \sup_t \left| \int_0^t \int_0^s \Gamma(y_1(s'), y_1'(s')) - \Gamma(y_2(s'), y_2'(s')) ds' ds \right| + \sup_t \left| \int_0^t \Gamma(y_1(s'), y_1'(s')) - \Gamma(y_2(s'), y_2'(s')) ds' ds \right|$$

$$\leq 6\varepsilon \sup_t |y_1(s') - y_2(s')| + 6\varepsilon \sup_t |y_1'(s') - y_2'(s')| = 6\varepsilon ||y_2 - y_1||_{C^1} < ||y_2 - y_1||_{C^1}$$

Thus, the function

$$\Phi_{c,v,\Gamma} : Z \to Z$$

given by

$$\Phi_{c,v,\Gamma}(y) = \Phi(c, v, \Gamma, y)$$

is a contraction on $Z$, and therefore has a unique fixed point $y(c, v, \Gamma)$. Let

$$y : B_1(0, \mathbb{R}^n) \times \overline{B}_1(\mathbb{R}^n) \times X \to Z$$

denote the function which maps $(c, v, \Gamma)$ to the function $y \in Z$ such that

$$y(c, v, \Gamma) = \Phi(c, v, \Gamma, y(c, v, \Gamma))$$

We will now show smoothness of $\Phi$ in its parameters. We have that

$$\frac{\Phi_{c,v,y}(\Gamma + \xi\tilde{\Gamma}) - \Phi_{c,v,y}(\Gamma)}{\xi}(t) = -\int_0^t \int_0^s \tilde{\Gamma}(y(s'), y'(s')) ds' ds$$

Thus, the Gateux derivative

$$d(\Phi_{c,v,y})(\Gamma, \tilde{\Gamma})$$

exists and is the mapping

$$t \mapsto -\int_0^t \int_0^s \tilde{\Gamma}(y(s'), y'(s')) ds' ds$$

Let us denote this mapping by $\varphi_y(\Gamma)$ for fixed $y$ (noting that it is independent of $c$ and $v$, and our original choice of $\Gamma$). We will show that $\varphi_y$ is a continuous function of $\Gamma$. We have that

$$||\varphi_y(\Gamma + \tilde{\Gamma}) - \varphi_y(\Gamma)||_{C^1} = \sup_t \left| \int_0^t \int_0^s \tilde{\Gamma}(y(s'), y'(s') ds' ds \right| + \sup_t \left| \int_0^t \tilde{\Gamma}(y(s'), y'(s')) ds' ds \right| \leq 6 \sup_t |\tilde{\Gamma}|$$

which tends to zero as $\tilde{\Gamma} \to 0$ in $C^1$. Therefore, $\Phi$ is continuously Gateaux differentiable in $\Gamma$. Similarly, we find that the derivatives $d(\Phi_{v,y,\Gamma})(c, \tilde{c})$ and $d(\Phi_{c,y,\Gamma})(v, \tilde{v})$ are given by the respective mappings

$$t \mapsto \tilde{c}$$

$$t \mapsto t\tilde{v}$$

It is clear that these maps are continuous in $\tilde{c}$ and $\tilde{v}$ respectively. We now must compute $d(\Phi_{c,v,\Gamma})(y, \tilde{y})$. We have

$$\lim_{\xi\to 0}\left(\frac{\Phi(y+\xi\tilde{y}) - \Phi(y)}{\xi}\right)(t) = \lim_{\xi\to 0}\left(-\frac{1}{\xi}\int_0^t\int_0^s \Gamma(y(s')+\xi\tilde{y}(s'), y'(s')+\xi\tilde{y}'(s')) - \Gamma(y(s'), y'(s'))ds'ds\right)$$

We note that, for $0 < \xi < 1$, the function defined by

$$\frac{\Gamma(y(s')+\xi\tilde{y}(s'), y'(s')+\xi\tilde{y}'(s')) - \Gamma(y(s'), y'(s'))}{\xi}$$

is $C^1$ in both $s'$ and $\xi$, and therefore is differentiable and hence continuous. Furthermore, we see that

$$\lim_{\xi\to 0}\frac{\Gamma(y(s')+\xi\tilde{y}(s'), y'(s')+\xi\tilde{y}'(s')) - \Gamma(y(s'), y'(s'))}{\xi} = D\Gamma(y(s'), y'(s'))\cdot(\tilde{y}(s'), \tilde{y}'(s'))$$

is a continuous function of $s'$. By a double application of an above corollary, we may pull the limit inside the integral. Thus we have that the derivative $d(\Phi_{c,v,\Gamma)}(y, \tilde{y})$ exists and equals the mapping

$$t \mapsto -\int_0^t\int_0^s D\Gamma(y(s'), y'(s'))\cdot(\tilde{y}(s'), \tilde{y}'(s'))ds'ds$$

Moreover, this is continuous by continuity of $D\Gamma$.

We have thus showed that the operator $\Phi$ is $C^1$ in each of its parameters, and is therefore continuously Frechet differentiable (see proposition A.3 in [1]). From this, we can show using the implicit function theorem that the fixed points of $\Phi$, and the derivative of said fixed points, are $C^1$ in $c, \Gamma$, and $v \neq 0$. We have thus shown that geodesics whose speed is in the unit ball on a Riemannian manifold depend in $C^1$ on the initial point, velocity, and on the Christoffel symbols. Furthermore, we have that, if $\ell_{c,v}$ denotes the line $t \mapsto c + vt$, then

$$|y_{c,v,\Gamma} - \ell_{c,v}|(t) = \left|\int_0^t\int_0^s \Gamma(y_{c,v,\Gamma}(s'), y'_{c,v,\Gamma}(s'))ds'ds\right| \leq 4\varepsilon$$

and similarly

$$|y'_{c,v,\Gamma} - \ell'_{c,v}|(t) \leq 2\varepsilon$$

Thus, we see that, for any $c, v$, if $||\Gamma||_{C^1} < \varepsilon$ then

$$||y_{c,v,\Gamma} - \ell_{c,v}||_{C^1} \leq 6\varepsilon$$

In other words, the geodesics $y_{c,v,\Gamma}$ can be made uniformly close to the lines $\ell_{c,v}$ in $(c, v)$ by taking the Christoffel symbols $\Gamma$ small in $C^1$.

# 4 The Erdos Distance Problem on Manifolds

In this section, we will discuss and prove the following result:

**Theorem** *Let $(M, g)$ be a compact connected Riemannian 2-manifold. Then, there exists constants $C_M$ and $n_0$ such that, if $P \subseteq M$ is a finite subset with $n = |P| \geq n_0$, then*

$$|\Delta_g(P)| \geq C_M n^{1/2}$$

The proof of this result is quite technical, and thus we will first give an outline of the proof idea before presenting the formal proof.

## 4.1 Outline of Argument

The main idea behind the proof technique is a perturbation argument that will allow us to transport the arguments used in the plane to those on manifolds, with technical modifications where needed. We begin by noticing that, using normal coordinates and a pigeonhole argument, we may choose coordinates such that we work in a ball of radius 1, with a metric of the form

$$g_{ij} = \delta_{ij} + \varepsilon^2 \mathcal{O}(|x|^2)$$

where $\varepsilon > 0$ is to be determined. We then fix a point in $p \in P$, and draw geodesic circles at this point to each other point in $P$. If there are $t$ circles, we again pigeonhole to obtain a circle with $n/t$ points contained on it. We cut this circle into fourths using standard Euclidean lines. We then use a perturbation argument to argue that distances increase as we move along this geodesic circle, in the same fashion that we did in the Euclidean case. The rest of the proof of the Euclidean case follows through as usual, as we have

$$|\Delta_g(P)| \geq \max\{t, n/t\} \geq \sqrt{n}$$

where we have ignored constants in the above inequality.

## 4.2 The Perturbation Argument

Let us work on a ball of radius 5 in $\mathbb{R}^2$ with metric

$$g_{ij} = \delta_{ij} + \varepsilon^2 \mathcal{O}(|x|^2)$$

Let $E_{ij}$ be the error term, and let $C$ be a constant such that

$$|E_{ij}(x)| \leq C|x|^2$$

By choosing $\varepsilon$ small enough, our computations in the previous section show that geodesics starting at points in $B_1(0)$ with velocities $|v| \leq 1$ always exist. Furthermore, we will assume that $\varepsilon$ is chosen such that our ball is geodesically convex. Fix a point $a \in B_{1/8}(0)$, and let $0 < d < 1/4$. Then, by taking $\varepsilon$ small enough, our computations from the last section allow us to ensure that the circles of radius $d$ about $a$ is contained in $B_1(0)$. More precisely, we have

$$G_d = \{q \in B_5(0) : d_g(q, a) = d\} \subseteq B_1(0)$$

$$C_d = \{q \in B_5(0) : |a - q| = d\} \subseteq B_1(0)$$

Note that each point on either of these circles can be described as a unit speed geodesic starting at $p$. Parameterizing $S^1$ by $\theta$, we see that each point on these circles can be smoothly parameterized by $\theta \in [0, 2\pi]$. As notation, given $\varphi \in [0, 2\pi]$, let $c_\varphi$ denote the corresponding point on $C_d$ and $g_\varphi$ denote the corresponding point on $G_d$. Fix $\theta_0 \in [0, 2\pi)$. Then, by our remarks above, for any $\theta \in [0, 2\pi)$, the unit speed geodesic $\gamma_\theta$ from the point $g_{\theta_0}$ to $g_\theta$ exists, and can be made close in $C^1$ to the line

$$\ell_\theta = td(\cos\theta, \sin\theta) + (1-t)d(\cos\theta_0, \sin\theta_0)$$

Furthermore, both $\gamma_\theta$ and $\ell_\theta$ vary smoothly in $\theta$ by our results from last section. Elementary trigonometry shows that on $C_{1/2}$ we have

$$|c_\theta - c_{\theta_0}| = \frac{1}{2}\sqrt{2}\sqrt{1 - \cos(\theta - \theta_0)}$$

which increases for $\theta \in [\theta_0, \theta_0 + \pi/2]$. Furthermore, on $[\theta_0, \theta_0 + \pi/2]$, this function is $C^1$ (where at $\theta_0$ we consider the right-sided derivative). We would like to show that $d_g(g_\theta, g_{\theta_0})$ increases for $\theta$ in some interval. We have have on $[\theta_0, \theta_0 + \pi/2]$

$$\frac{d}{d\theta} d_g(g_\theta, g_{\theta_0}) = \frac{d}{d\theta}\int_0^1 g(\gamma_\theta'(t), \gamma_\theta'(t))^{1/2} dt = \int_0^1 \partial_\theta g(\gamma_\theta'(t), \gamma_\theta'(t))^{1/2} dt = \int_0^1 \frac{1}{g(\gamma_\theta'(t), \gamma_\theta'(t))^{1/2}} g(\partial_\theta \gamma_\theta'(t), \gamma_\theta'(t)) dt$$

where the second equality is a fancy application of a previous lemma, and we have used symmetry of the connection in the last equality. Similarly, we have

$$\frac{d}{d\theta}|c_\theta - c_{\theta_0}| = \int_0^1 \frac{1}{\sqrt{|\ell_\theta'(t)|^2}} (\partial_\theta \ell_\theta'(t)) \cdot \ell_\theta'(t) dt$$

We thus have

$$\sup_\theta \left| \frac{d}{d\theta} d_g(g_\theta, g_{\theta_0}) - \frac{d}{d\theta}|c_\theta - c_{\theta_0}| \right| \leq \sup_\theta \int_0^1 \left| \frac{1}{g(\gamma_\theta'(t), \gamma_\theta'(t))^{1/2}} g(\partial_\theta \gamma_\theta'(t), \gamma_\theta'(t)) - \frac{1}{\sqrt{|\ell_\theta'(t)|^2}} (\partial_\theta \ell_\theta'(t)) \cdot \ell_\theta'(t) \right| dt$$

Note that $\partial_\theta \gamma_\theta' \to \partial_\theta \ell_\theta'$ and $\gamma_\theta' \to \ell_\theta'$ as $\varepsilon \to 0$, and this is uniform in $t$, and independent of $\theta$, $\theta_0$. Thus, given $\eta > 0$, we may choose $\varepsilon$ small enough such that

$$\sup_\theta \left| \frac{d}{d\theta} d_g(g_\theta, g_{\theta_0}) - \frac{d}{d\theta}|c_\theta - c_{\theta_0}| \right| \leq \sup_\theta \int_0^1 \left| \frac{1}{g(\gamma_\theta'(t), \gamma_\theta'(t))^{1/2}} g(\partial_\theta \gamma_\theta'(t), \gamma_\theta'(t)) - \frac{1}{\sqrt{|\ell_\theta'(t)|^2}} (\partial_\theta \ell_\theta'(t)) \cdot \ell_\theta'(t) \right| dt < \eta$$

When choosing such an $\varepsilon$, it will be assumed that it is chosen such that the above holds with the metric $\tilde{g}_{ij} = \delta_{ij} + 5C\varepsilon^2$ (and thus our choice of $\varepsilon$ will allow our desired inequality to hold for any metric of the form $\delta_{ij} + \varepsilon^2\mathcal{O}(|x|^2)$ if the constant hidden by the big-O term is $C$). Thus, given $\eta > 0$, we may choose $\varepsilon$ as above such that, on $[\theta_0, \theta_0 + \pi/2]$, we have

$$\frac{d}{d\theta} d_g(g_\theta, g_{\theta_0}) = \frac{d}{d\theta}|c_\theta - c_{\theta_0}| + g(\theta)$$

where $\sup_\theta |g(\theta)| < \eta$, and where the choice of $\varepsilon$ is independent of $\theta_0$. In other words, given $\eta > 0$, there exists an $\varepsilon > 0$ such that for any $\theta_0$ as above, we have on $[\theta_0, \theta + \pi/2]$ we have

$$\frac{d}{d\theta} d_g(g_\theta, g_{\theta_0}) = \frac{d}{d\theta} |c_\theta - c_{\theta_0}| + g(\theta)$$

where $\sup_\theta |g(\theta)| < \eta$. In particular, noting that $\frac{d}{d\theta}|c_\theta - c_{\theta_0}|$ is continuous and positive on $[\theta_0, \theta_0 + \pi/2]$, we may choose $\varepsilon$ as above such that for $\theta_0$ as above, we have

$$\frac{d}{d\theta} d_g(g_{\theta_0}, g_\theta) > 0$$

on $[\theta_0, \theta_0 + \pi/2]$, and thus $d_g(g_{\theta_0}, g_\theta)$ increases on this interval. We will now show that given $0 < d < 1/2$, our choice of $\varepsilon$ as above also allows us to guarantee that $d_g(g_\theta, g_{\theta_0})$ increases on $G_d$. Consider the mapping $\varphi : B_5(0) \to B_{10d}(0)$ given by $\varphi(x) = 2dx$. We then have

$$(\varphi^* g)_{ij}(x) = 4d^2(\delta_{ij}(x) + \varepsilon^2 E_{ij}(2dx))$$

where $E_{ij}$ is the error term in the big-O for $g$. Let $h$ be the metric

$$h_{ij}(x) = \delta_{ij}(x) + \varepsilon^2 E_{ij}(2dx)$$

since we would have

$$|E_{ij}(2dx)| \leq C|2dx|^2 = 4d^2 C|x|^2 < C|x|^2$$

as $4d^2 < 1$ (where $C$ is the constant such that $|E_{ij}(x)| \leq C|x|^2$). Thus, by our above discussion, our choice of $\varepsilon$ will allow for

$$d_g(\varphi^{-1}(g_\theta), \varphi^{-1}(g_{\theta_0}))$$

to increase on $[\theta_0, \theta_0 + \pi/2]$. Since the distances in $h$ are the same as the distances in $\varphi^* g$ (and thus the distances in $g$ on $G_d$) up to scaling, we see that $d_g(g_\theta, g_{\theta_0})$ increases on $[\theta_0, \theta_0 + \pi/2]$ for any $\theta_0$. In other words, w may choose $\varepsilon$ small enough such that given $0 < d \leq 1/2$, and given $\theta_0$, the distances on $G_d$ increase on $[\theta_0, \theta_0 + \pi/2]$. This will form the basis of our perturbation argument.

## 4.3 Formal Proof

We finally have built up the machinery to prove our desired result.

**Theorem** *Let $(M, g)$ be a compact connected Riemannian 2-manifold. Then, there exists constants $C_M$ and $n_0$ such that, if $P \subseteq M$ is a finite subset with $n = |P| \geq n_0$, then*

$$|\Delta_g(P)| \geq C_M n^{1/2}$$

PROOF: Let $P \subseteq M$ be finite with $|P| = n$. Let $C = \inf_M conv$. Then, at each point, the normal ball of radius $C$ has metric

$$g_{ij} = \delta_{ij} + \mathcal{O}(|\tilde{x}|^2)$$

Let $N_p$ denote the supremum of the error term given by the big-O term (this will be finite since the $g_{ij}$ are smooth and $M$ is compact). Then $N_p$ will be a continuous function of $p$. By compactness, let $N = \sup_{p\in M} N_p$. Choose $\eta > 0$ such that:

1): For each $p \in M$, $\exp_p : B_\eta(0) \subseteq T_pM \to M$ is a diffeomorphism onto an open subset of $M$ containing $p$, and the open subset geodesically convex, which we will do by taking $\eta \leq C$. Call this open subset $V_p$.

2): The metric on $B_5(0) \subseteq \mathbb{R}^2$ given in the usual coordinates by

$$g_{ij} = \delta_{ij} + \frac{\eta^2}{25}N$$

is well defined, and such that geodesics with origin in $\overline{B}_1(0)$ with velocities $|v| \leq 1$ exist in this ball. Furthermore, choose $\eta$ such that if $q \in B_{1/4}(0)$, and $q_{\theta_0}$ is on the geodesic circle of radius at most $1/2$ centered at $q$, the function $d_g(q_\theta, q_{\theta_0})$ is increasing on $[\theta_0, \theta_0 + \pi/2]$.

Note that $\eta$ is independent of $P$. Let $\varepsilon = \eta/20$. For $p \in M$, let $U_p$ be the diffeomorphic image of $\exp_p : B_\varepsilon(0) \subseteq T_pM \to M$. Note that $U_p \subseteq V_p$. Consider the covering of $M$ by $U_p$. By compactness, take $k \in \mathbb{Z}^+$ to be the minimum number of $U_p$ required to cover $M$. Note that $k$ depends only on $\eta$, which is independent of $P$. By the pigeonhole principle, there must be some $q \in M$ such that

$$|P \cap U_q| \geq \frac{n}{k}$$

Let $m = n/k$. Consider such $q$, and pass to the normal coordinates on $V_q$. Let $P_q = P \cap U_q$. We now may work on a ball of radius $\eta$ centered at 0 in $\mathbb{R}^2$, with metric

$$g_{ij} = \delta_{ij} + \mathcal{O}(|\tilde{x}|^2)$$

Note that $P_q$ can be considered as contained in $B_\varepsilon(0)$. Define $\varphi : B_5(0) \to B_\eta(0)$ to be the homothety

$$\varphi(x) = \frac{\eta}{5}x$$

Consider metric $\varphi^* g$ on $B_5(0)$, which has component functions

$$(\varphi^* g)_{ij}(x) = \varphi^* g(\partial/\partial x^i, \partial/\partial x^j) = \eta^2 g(\partial/\partial \tilde{x}^i, \partial/\partial \tilde{x}^j) = \eta^2 g_{ij}(\tilde{x}) = \frac{\eta^2}{25} g_{ij}\left(\frac{\eta}{5}x\right)$$

Thus, we see that

$$(\varphi^* g)_{ij} = \frac{\eta^2}{25}\left(\delta_{ij} + \frac{\eta^2}{25}\mathcal{O}(|x|^2)\right)$$

Let $h = \frac{25}{\eta^2}(\varphi^* g)$. Then

$$h_{ij} = \delta_{ij} + \frac{\eta^2}{25}\mathcal{O}(|x|^2)$$

where the maximum of the error term given by the big-O is less than or equal to $N$ (and thus the properties described in 2) also apply to $h$). Since $h$ is just a rescaling of $\varphi^* g$, the distances in $h$ will be the same as the distances in $\varphi^* g$, which will be the same as the distances in $g$ since $\varphi$ is a diffeomorphism. Having normalized our coordinates, we see that $\tilde{P}_q = \varphi^{-1}(P_q)$ is now contained in a ball of radius $1/8$

at zero. Fix $p_0 \in \tilde{P}_q$. Let

$$t = |\{d_g(p_0, p) : p \in \tilde{P}_q\}|$$

Then $\tilde{P}_0$ has at least $t$ distinct distances. Draw geodesic circles centered at $p_0$, passing through the other points of $\tilde{P}_0$. Note that the radii of the circles is at most $1/2$. Then there must be a circle that contains at most $m - 1/t$ points of $\tilde{P}_q$. Let $\theta \in [0, 2\pi]$ parameterize this circle, and split this circle into $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, and $[3\pi/2, 2\pi]$. Then at least one of these intervals must contain $m - 1/4t$ points of $\tilde{P}_q$. Without loss of generality Let $\theta_0 \in [0, \pi/2]$ be such such that the corresponding point on the geodesic circle, $q_{\theta_0}$, is in $\tilde{P}_q$, and there is no smaller $\theta \in [0, \pi/2]$ with this property. By choice of $\eta$, we see that the function $d_g(q_{\theta_0}, q_\theta)$ is increasing on $[\theta_0, \pi/2]$, and thus there are $m - 1/4t$ distinct distances on this circle. We thus see that

$$|\Delta_g(P)| \geq |\Delta_h(\tilde{P}_q)| \geq \max(t, m - 1/4t) \geq \sqrt{\frac{m-1}{4}} = \sqrt{\frac{\frac{n}{k} - 1}{4}} \geq C_k \sqrt{n}$$

for some $C_k$ (at least when $n$ is large). Since $k$ depends only on $M$, we see that

$$|\Delta_g(P)| \geq C_M \sqrt{n}$$

for some constant $C_M$ depending only on $M$, as desired. ◭

# 5 Further Work

## 5.1 Additional Proof of the $n^{1/2}$ Case

As mentioned briefly above, several proofs of the planar $n^{1/2}$ bound are known. In particular, [3] gives the following quick argument. pick two points $p_1, p_2 \in P$, and draw the circles centered at $p_1$ passing through the other $n - 2$ points in $P$. Do the same for $p_2$. Suppose there are $k_1$ circles centered at $p_1$ and $k_2$ circles at $k_2$. At worst, we have at most $2k_1k_2$ intersections. Therefore, since each of the $n - 2$ other points of $P$ are contained on such an intersection, we have

$$n - 2 \leq 2st$$

Thus

$$\sqrt{\frac{n-2}{2}} \leq s \leq |\Delta(P)|$$

which gives the result. It is reasonable to expect that a similar method of proof can be adapted to the Riemannian setting by studying the intersection properties of geodesic circles in normal coordinates. This could be a fun topic of further exploration.

### 5.2 $n^{2/3}$ Case and Higher Dimensions

Having obtained the $n^{1/2}$ case, the next natural step is to try and replicate the $n^{2/3}$ argument on manifolds. It is likely that much of the analysis setup in this paper will assist in this, however it is likely that a more robust analysis is needed. In particular, if one examines Moser's construction (see [3] or [8]), one sees the main sticking point will be transferring the annuli argument to the manifold setting, as one has to delete the annuli such that distances do not repeat.

One could also study this problem for Riemannian manifolds of higher dimension, possibly proving a bound similar to the one discussed at the beginning of this paper.

### 5.3 Applications to Spectral Sets on Manifolds

As a possible application of the principal result in this paper, one could try and prove a result in line with [5]. In this section we briefly discuss the approach given by [6] to make sense of this problem on manifolds. For this section, $(M, g)$ is a compact Riemannian manifold, and $\mu$ is the measure on the Borel $\sigma$-algebra of $M$ obtained by

$$\mu(D) = \int_D dV_g = \int_M \chi_D dV_g$$

We also let $\Delta_g$ be the Laplace-Beltrami operator given by

$$\Delta_g f = -\mathrm{div}(\mathrm{grad} f)$$

for $f \in C^\infty(M)$. The natural analogue of spectral sets in this setting is the study of what sets $D \subseteq M$ are such that $L^2(D)$ has an orthogonal basis of eigenfunctions of the Laplace-Beltrami operator (in this setting, such sets are called spectral). One could also study less restrictive properties, such as whether or not $L^2(D)$ admits a frame of eigenfunctions of $\Delta_g$ (see [6] for a discussion on frames). The authors of [6] prove the following analogue of Fuglede's theorem in this setting:

**Theorem** *Let $(M, g)$ be a compact Riemannian manifold, and $D \subseteq M$ be a set of positive measure which tiles $M$ under a subgroup $G$ of the isomoetries of $M$. Then $D$ is spectral*

Given the connections between spectral sets in $\mathbb{R}^2$ and the Erdos distance problem, especially the result in [5] described in the beginning of this paper, it is natural to wonder if one could obtain analogues on manifolds in the two dimensional case using the principal result of this paper.

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