Honor Thesis, Mathematics B.S. Honors

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Abstract

In this paper, a discrete version of the Hausdorff dimension for families of finite point sets is defined. In the first section, the following topics are reviewed: some preliminary results deriving from the discrete Hausdorff dimension definition proved in the TRIPODS NSF REU-GradForAll 2021, B.Hunt's work on the Hausdorff dimension of the graph of Weierstrass functions with random phases, and some relevant theorems from probability theory on the distribution and density of the sum of independent random variables. In the second section, it is shown that, given dimension $d \in \mathbb{Z}$ and $s \in [d-1, d)$, it is possible to construct a function f such that the family of finite point sets of the form $P_n = \left\{(j/q, f(j/q)) : j \in \mathbb{Z}^{d-1} \cap [0, q)^{d-1}\right\}$ with $n = q^{d-1}$ and $q \in \mathbb{Z}$ has discrete Hausdorff dimension s. Such construction involves the mixture of the topics exposed in the first section.

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1 Introductory background

1.1 Discrete Hausdorff dimension for families of finite sets

In the literature[9], one of the equivalent ways of defining the Hausdorff dimension of a compact set relies on the computation of the following quantity: the energy integral.

Definition 1. Given $0 \leq \alpha \leq d$, the α-dimensional energy of a positive measure with compact support contained in $\mathbb{R}^d$ is:

$$I_\alpha(\mu) = \iint |x-y|^{-\alpha} \, d\mu(x) d\mu(y) \tag{1}$$

With this in mind, we can then define the the Hausdorff dimension.

Definition 2. Given a compact set E and the set of all probability measures compactly supported on E $P(E)$, its Hausdorff dimension is:

$$\sup \{\alpha \in [0,d] : \exists \, \mu \in P(E) : I_\alpha(\mu) < \infty\} \tag{2}$$

In this paper, for $d \geq 2$, we consider families of finite point sets of the form

$$\mathcal{P} = \{\mathcal{P}_n\}_{n \in M \subseteq \mathbb{Z}}, \mathcal{P}_n \subseteq [0,1]^d \tag{3}$$

Particularly, we take the finite sets to be points of the graph of a function $f : [0,1]^{d-1} \to [0,1]$:

$$\mathcal{P}_n = \left\{ (j/q, f(j/q)) : j \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1} \right\}, \qquad n = q^{d-1} \tag{4}$$

where $n \in \left\{ q^{d-1} : q \in \mathbb{Z} \right\}$.
We can formulate a definition for the Hausdorff dimension of such families by adopting an approximation by Riemann sums to the previously given definitions. In fact, we define the discrete energy of the sets $\mathcal{P}_n$ and the discrete Hausdorff dimension of the family of sets $\mathcal{P}$ as follows:

Definition 3. Given $0 \leq \alpha \leq d$, the α-dimensional discrete energy of a finite set E of size n is:

$$I_\alpha(E) = n^{-2} \sum_{\substack{p,p' \in E \\ p \neq p'}} \left| p - p' \right|^{-\alpha} \tag{5}$$

Definition 4. Given a family of finite sets $\mathcal{P} = \{\mathcal{P}_n\}_{n \in M \subseteq \mathbb{Z}}$, its Hausdorff dimension is:

$$dim_{\mathcal{H}}(\mathcal{P}) = \sup \left\{ \alpha \in [0,d] : \sup_{n \in M \subseteq \mathbb{Z}} I_\alpha(\mathcal{P}_n) < \infty \right\} \tag{6}$$

1.2 Motivations

In this subsection, we present some motivations behind the choice of defining a notion of discrete Hausdorff dimension and study it on families of points sets of the form as described in (3) and (4) as a potentially useful tool in data science.
Fractal-based analysis of time series, has found extensive applications in almost all scientific disciplines, including data science[4]. To understand better the reasons why this happened, we can consider the following simple and practical example[1]. Suppose that we have a time series representing sales of a retail store going back 40 years. Suppose that one wished to look at the times when the sales were in the top % of all sales in a given year, and it turned out that this happened every July, December and April, and that during those months it happened during the first week, and during that week it happened on Fridays and Saturdays, and that on those days, the sales peaked in the mornings. It is then noticeable that, continuing to zoom into smaller and smaller time scales, the arising structure becomes a fine one, characterized by self-similar details infinitely nested in time which exhibit an authentic fractal behaviour. These considerations, then, call for an adequate definition of dimension able to capture such fractal property. As explained by Falconer[3], due to the fact of being defined for any compact set

(compactness is not even a strictly necessary condition) and relying on measures (which are easy to manipulate), the Hausdorff dimension has a central role in fractal geometry indicating, roughly speaking, how much space a set occupies near to each of its points . Its most fundamental definition (which is equivalent to Definition 2[3] and also explains why it can capture fractal properties of a set and the way it distributes in space) is the following:

Definition 5. Given a set E, its Hausdorff dimension is

$$\inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\} \tag{7}$$

where $\mathcal{H}^s(E)$ is the Hausdorff measure of E and it is defined as

$$\lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subseteq \cup_{i=1}^{\infty} U_i, |U_i| < \delta \ \forall i \in \mathbb{Z} \right\} \tag{8}$$

The Hausdorff dimension also quantifies the roughness or smoothness of a time series in the limit as the observational scale becomes infinitesimally fine[4]. With respect to this, referring to the previous example, fractal analysis can be applied meaningfully not only on the time axis of the considered time series (mentioned before), but also on the values of the sales. Such analyisis, in fact, would reflect interestingly the volatility of the recorded values. For example, it has been shown that the fractional Brownian path (used to model stock market prices) has Hausdorff dimension > 1, reflecting the volatility of the data. After noticing this, then, we arrive at a very interesting situation where we have a set of effective dimension < 1 (as it is embedded in a uni-dimensional space) on the time axis, combined with the volatile data modeled by a function whose graph has dimension > 1. Thus, exploring discrete families of point sets resembling finer and finer time series finds its motivation. In this paper, we will focus on the volatile part of the time series and, for this reason, the set of points on the time axis is taken to be equidistributed as in (4) to simplify things. Finally, by looking at Definition 5, it is noticeable that the Haussdorf dimension of any discrete set is 0. Since the time series are discrete objects, then, it becomes crucial to restate the definition of a discrete Hausdorff dimension as done in Definition 4 in order to obtain meaningful results.

1.3 Statement of the main theorem and preliminary results

The main theorem proved in this paper is the following:

Theorem 6. *Given $f : [0,1]^{d-1} \to [0,1]$, let $\mathcal{P} = \{\mathcal{P}_n\}$ be the time series with*

$$\mathcal{P}_n = \left\{ (j/q, f(j/q)) : j \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1} \right\}, \qquad n = q^{d-1} \tag{9}$$

For any $s \in [d-1, d)$, there exists f such that $dim_{\mathcal{H}}(\mathcal{P}) = s$.

Before proving the just stated theorem, we mention several results proved in the paper written during the TRIPODS NSF REU-GradForAll 2021[1] (in preparation).
First, we provide an elementary Lemma that gives a trivial lower bound for the discrete Hausdorff dimension of a family of finite point sets of the form as presented in (9).

Lemma 7. *Let $\mathcal{P} = \{P_n\}$ be a family of point set graphs as in (9) above. Then $dim_{\mathcal{H}}(\mathcal{P}) \geq d-1$.*

Proof. We show that the s-discrete energy of $\mathcal{P}_n$ is uniformly bounded if $0 \leq s < d-1$. As usual, we let $n = q^{d-1}$ for a positive integer q. Let $p = (j/q, f(j/q)) \in \mathcal{P}_n$, where here $j \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1}$. Note, for each such pair p, p',

$$|p - p'|^{-s} \leq |j/q - j'/q|^{-s} = q^s |j - j'|^{-s}. \tag{10}$$

Hence, we have

$$I_s(\mathcal{P}_n) = n^{-2} \sum_{\substack{p,p' \in E \\ p \neq p'}} |p - p'|^{-s} \leq q^{-2(d-1)+s} \sum_{j \neq j'} |j - j'|^{-s}, \tag{11}$$

where here j and j' range over $\mathbb{Z}^{d-1} \cap [0,q)^{d-1}$. By a change of variables $j - j'$ to j, the above become:

$$\begin{aligned}
&q^{-2(d-1)+s} \sum_{j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1}} \sum_{\substack{j+j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1} \\ j \neq 0}} |j|^{-s} \\
&= q^{-2(d-1)+s} \sum_{j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1}} \sum_{\substack{j \in \mathbb{Z}^{d-1} \cap \prod_{i=1}^{d-1} [-j'_i, q-j'_i) \\ j \neq 0}} |j|^{-s} \\
&\leq q^{-2(d-1)+s} q^{d-1} \sum_{\substack{j \in \mathbb{Z}^{d-1} \cap (-q,q)^{d-1} \\ j \neq 0}} |j|^{-s} \leq C q^{-(d-1)+s} \int_0^q r^{-s+d-2} \, dr \\
&= C q^{-(d-1)+s} q^{d-1-s} = C
\end{aligned} \tag{12}$$

by the integral test, where here C is a constant depending on d and s only. Note that the integral converged for values $s < d-1$. The conclusion of Lemma 7 follows by (6). □

Next, we present a Lemma that gives us a lower bound on the amount of distances achieved by points in a set of given upper Minkowski dimension inside a ball of fixed radius.

Lemma 8. *Given a set E with upper Minkowski dimension α, there exists a positive constant c depending only on E such that*

$$|\{(p,p') : |p-p'| \leq r\}| \geq c r^{\alpha} n^2 \tag{13}$$

for $0 < r \leq 1$ and where $p, p' \in E$.

Proof. Take a minimal cover of E by N_r balls of radius r. Since E has upper Minkowski dimension α, $N_r \leq C_E r^{-\alpha}$. Select a partition of E into N_r pairwise disjoint sets

$$E = E_1 \cup \cdots \cup E_{N_r} \tag{14}$$

where each part E_i is contained in a ball in the cover. Note,

$$\begin{aligned}
|\{(p,p') : |p-p'| \leq r\}| &\geq \sum_{j=1}^{N_r} |E_j \cap P_n|^2 \\
&= \sum_{j=1}^{N_r} |E_j \cap P_n|^2 \sum_{j=1}^{N_r} \left(\frac{1}{\sqrt{N_r}} \right)^2 \\
&\geq N_r^{-1} \left(\sum_j |E_j \cap P_n| \right)^2 \geq C_E^{-1} r^{\alpha} n^2
\end{aligned} \tag{15}$$

where the second inequality is an application of Cauchy-Schwarz. This concludes the proof of the lemma. □

The just provided lemma allows us to prove the following theorem, which states that the discrete Hausdorff dimension of a generic family of finite point sets is bounded above by the upper Minkowski dimension of a set that contains all the point sets constituting the family.

Theorem 9. *Let $P = \{P_n\}_{n \in M \subseteq \mathbb{Z}}$ be a family of point sets contained in a subset $E \subset [0,1]^d$ of upper Minkowski dimension α. Then, $\dim_{\mathcal{H}} \mathcal{P} \leq \alpha$. Moreover, if $s > \alpha$, we have the following quantitative bound:*

$$I_s(\mathcal{P}_n) \geq \frac{\alpha}{s-\alpha} (C_E^{-1})^{\frac{s}{\alpha}} n^{\frac{s}{\alpha}-1} - \frac{C_E^{-1} s}{s-\alpha} + \frac{1}{n}, \tag{16}$$

whenever $\mathcal{P}_n \subset E$, where C_E is a constant such that the number of balls of radius r (small) needed to cover E is $\leq C_E r^{\alpha}$.

Proof. The goal is to show that $I_s(P_n)$ is unbounded for $s > \alpha$, and provide the quantitative lower bound (16) provided $\mathcal{P}_n$ is confined to E, and where the constant implicit in the notation depends on s and α but is independent of n. We write

$$\begin{aligned}|p-p'|^{-s} &= s\int_{|p-p'|}^{\infty} r^{-s-1}\,dr \\ &= \int_0^{\infty} \mathbf{1}_{[|p-p'|,\infty)}(r) r^{-s-1}\,dr \\ &= s\int_0^{\infty} \mathbf{1}_{[0,\infty)}(r-|p-p'|) r^{-s-1}\,dr,\end{aligned} \tag{17}$$

so that

$$\begin{aligned}I_s(\mathcal{P}_n) &= n^{-2}\sum_{\substack{p,p'\in\mathcal{P}_n\\ p\neq p'}} |p-p'|^{-s} \\ &= sn^{-2}\sum_{\substack{p,p'\in\mathcal{P}_n\\ p\neq p'}} \int_0^{\infty} \mathbf{1}_{[0,\infty)}(r-|p-p'|) r^{-s-1}\,dr \\ &= sn^{-2}\int_0^{\infty} \left(|\{(p,p')\in\mathcal{P}_n^2 : |p-p'|\le r\}|-n\right) r^{-s-1}\,dr.\end{aligned} \tag{18}$$

The subtracted n is here to eliminate the contribution to $|\{(p,p')\in\mathcal{P}_n^2 : |p-p'|\le r\}|$ of the pairs (p,p') with $p=p'$. Using the previous Lemma 8, then:

$$|\{(p,p')\in\mathcal{P}_n^2 : |p-p'|\le r\}|-n \ge \begin{cases}0 & r\le (C_E^{-1}n)^{-1/\alpha}\\ C_E^{-1}r^{\alpha}n^2-n & r\ge (C_E^{-1}n)^{-1/\alpha}.\end{cases} \tag{19}$$

Hence,

$$\begin{aligned}I_s(\mathcal{P}_n) &\ge sn^{-2}\int_{(C_E^{-1}n)^{-1/\alpha}}^{1} C_E^{-1}r^{\alpha-s-1}n^2 - r^{-s-1}n\,dr \\ &= \frac{s}{\alpha-s}(C_E^{-1}-(C_E^{-1})^{\frac{s}{\alpha}}n^{\frac{s}{\alpha}-1}) + (n^{-1}-(C_E^{-1})^{\frac{s}{\alpha}}n^{\frac{s}{\alpha}-1}) \\ &\ge \frac{\alpha}{s-\alpha}(C_E^{-1})^{\frac{s}{\alpha}}n^{\frac{s}{\alpha}-1} - \frac{C_E^{-1}s}{s-\alpha}+\frac{1}{n},\end{aligned} \tag{20}$$

This quantity is unbounded as $n\to\infty$ when $s>\alpha$. The conclusion of theorem 9 follows by (6). $\square$

As an immediate corollary of the just proved theorem, we have the following result:

Lemma 10. *Let $\mathcal{P}_n$ be as in* (9)*, and consider the corresponding $\mathcal{P}$. Suppose that $f : [0,1]^{d-1}\to[0,1]$ is Hölder continuous of order $\alpha\in(0,1]$. Then*

$$dim_{\mathcal{H}}(\mathcal{P}) \le d-\alpha.$$

where the Hölder continuity condition of order α on f means that there exists a positive constant $\rho\ge 0$ such that, for any $x,y\in[0,1]^{d-1}$, $|f(x)-f(y)|\le\rho\,|x-y|^{-\alpha}$

Proof. We claim the upper Minkowski dimension of the graph of f is at most $d-\alpha$, afterwards Theorem 9 completes the proof.
We will bound the upper box-counting dimension of the graph of f, which is the same as the upper Minkowski dimension. For fixed q, let Q_j denote the cube $\prod_{i=1}^{d-1}[\frac{j_i}{q},\frac{j_i+1}{q}]$ where here j ranges over $\mathbb{Z}^{d-1}\cap[0,q)^{d-1}$. Since f is Hölder continuous of order α, $f(Q_j)$ has diameter at most $\rho\sqrt{d-1}q^{-\alpha}$. Hence, we can cover the graph of f over Q_j by $\rho\sqrt{d-1}q^{1-\alpha}$ d-dimensional cubes of sidelength q^{-1}. Repeating for each of the q^{d-1} cubes Q_j, we cover the whole graph of f by $q^{d-1}\cdot\rho\sqrt{d-1}q^{1-\alpha}=\rho\sqrt{d-1}q^{d-\alpha}$ cubes of length q^{-1}, and our claim is proved. $\square$

1.4 Review of relevant theorems from probability theory

In this section, we review some standard theorems from the literature of probability theory [2] that are later used in the paper.

Recall that, any real valued random variable X defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ induces a probability push-forward measure μ on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by setting $\mu(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(X \in A)$ for any $A \in \mathcal{B}(\mathbb{R})$. Moreover, we define the cumulative distribution function of X to be $F(x) = \mathbb{P}(X \leq x) = \mu(X \in (-\infty, x])$ and, if this one can be written as $F(x) = \int_{-\infty}^{x} f(y)dy$, we say that X has density function f. The first theorem presents a formula to express the expectation of a measurable function of two independent random variables in terms of their respective distributions.

Theorem 11. *Suppose that X and Y are independent real valued random variables with respective distributions μ and ν. If $h : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function with $h \geq 0$ or $\mathbb{E}\,|h(X,Y)| < \infty$, then:*

$$\mathbb{E}\left(h(X,Y)\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\mu(dx)\nu(dy) \tag{21}$$

In particular, if $h(x,y) = f(x)g(y)$ where $f, g : \mathbb{R} \to \mathbb{R}$ are measurable functions with $f, g \geq 0$ or $\mathbb{E}\,|f(X)| < \infty$ and $\mathbb{E}\,|g(Y)| < \infty$, then:

$$\mathbb{E}\left(f(X)g(Y)\right) = \mathbb{E}\left(f(X)\right)\mathbb{E}\left(g(Y)\right) \tag{22}$$

Proof. First, note that thanks to the independence of the variables X and Y, for any measurable rectangle $C_1 \times C_2$

$$\begin{aligned} \mu \times \nu(c_1 \times C_2) &= \mu(C_1)\nu(C_2) \\ &= \mathbb{P}(X \in C_1)\mathbb{P}(Y \in C_2) \\ &= \mathbb{P}(X \in C_1 \cap Y \in C_2) \\ &= \mathbb{P}((X,Y) \in C_1 \times C_2) \end{aligned} \tag{23}$$

Then, by the $\pi - \lambda$ theorem, the product measure $\mu \times \nu$ is the distribution of the random vector (X, Y). Then, by the change of variable formula and Fubini's theorem (we can use it since $h \geq 0$ or $\mathbb{E}\,|h(X,Y)| < \infty$), we obtain the expression:

$$\mathbb{E}(h(X,Y)) = \int_{\mathbb{R}^2} hd(\mu \times \nu) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\mu(dx)\nu(dy). \tag{24}$$

If $h(x,y) = f(x)g(y)$, supposing $f, g \geq 0$, by (24), the fact that $g(y)$ does not depend on x, and the change of variable formula:

$$\begin{aligned} \mathbb{E}(h(X,Y)) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(y)\mu(dx)\nu(dy) = \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x)\mu(dx)\nu(dy) \\ &= \int_{\mathbb{R}} \mathbb{E}(f(X))g(y)\nu(dy) = \mathbb{E}\left(f(X)\right)\mathbb{E}\left(g(Y)\right) \end{aligned} \tag{25}$$

supposing instead that $\mathbb{E}\,|f(X)| < \infty$ and $\mathbb{E}\,|g(Y)| < \infty$, applying (25) on $|f|$ and $|g|$, yields:

$$\mathbb{E}\,|f(x)g(y)| = \mathbb{E}\,|f(x)|\,\mathbb{E}\,|g(x)| < \infty \tag{26}$$

Finally, combining (26) with (24) and repeating the steps of (25), we complete the proof of the theorem. □

Next, by setting h to be the right function, it is possible to find an expression for the cumulative distribution of the sum of the two variables.

Theorem 12. *If X and Y are independent with cumulative distribution functions $F(x) = \mathbb{P}(X \leq x)$ and $G(x) = \mathbb{P}(Y \leq y)$, then:*

$$\mathbb{P}(X + Y \leq z) = \int F(z-y)dG(y) \tag{27}$$

Proof. Let $h(x,y) = \mathbb{1}_{\{x+y\leq z\}}$. Then, by Theorem 11:

$$\begin{aligned}\mathbb{P}(X+Y\leq z) = \mathbb{E}\left(\mathbb{1}_{\{X+Y\leq z\}}\right) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{x+y\leq z\}}\mu(dx)\nu(dy) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{x\leq z-y\}}\mu(dx)\nu(dy) = \int_{\mathbb{R}} \mathbb{E}\left(\mathbb{1}_{\{X\leq z-y\}}\right)\nu(dy) \\ &= \int_{\mathbb{R}} \mathbb{P}(X\leq z-y)\nu(dy) = \int_{\mathbb{R}} F(z-y)\nu(dy) \\ &= \int_{\mathbb{R}} F(z-y)dG(y)\end{aligned} \tag{28}$$

□

By applying the definition of probability density function, the next result follows.

Theorem 13. *Suppose that X with density f and Y with cumulative distribution G are independent. Then, $X+Y$ has density*

$$h(x) = \int f(x-y)dG(y) \tag{29}$$

When Y has density g, the last formula

$$h(x) = f * g(x) \tag{30}$$

where $f * g(x) = \int f(x-y)g(y)dy$ is the convolution of the densities f and g.

Proof. Using Theorem 12, the definition of density ,and Fubini's theorem:

$$\begin{aligned}\mathbb{P}(X+Y\leq z) = \int_{\mathbb{R}} F(z-y)dG(y) &= \int_{\mathbb{R}}\int_{-\infty}^{z} f(x-y)dxdG(y) \\ &= \int_{-\infty}^{z}\int_{\mathbb{R}} f(x-y)dG(y)dx\end{aligned} \tag{31}$$

Then, by definition of density again together with (31), $X+Y$ has density:

$$\begin{aligned}h(x) &= \int_{\mathbb{R}} f(x-y)dG(y) = \int_{\mathbb{R}} f(x-y)\nu(dy) \\ &= \int_{\mathbb{R}} f(x-y)g(y)dy = f * g(x)\end{aligned} \tag{32}$$

where the last step follows in case Y has density. □

1.5 Hausdorff dimension of the 2-dimensional graph of Weierstrass functions with random phases (Brian Hunt)

In the recent years, the Hausdorff dimension of the 2-dimensional graph of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n cos\,(2\pi b^n)$ has been proven to be $2-\alpha$ where $\alpha = -\frac{\log(a)}{\log(b)}$ [8].

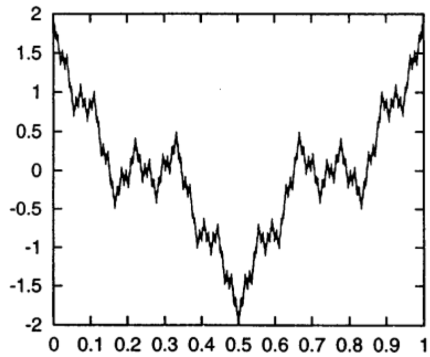

Figure 1: Graph of $f(x)$ with $a = 0.5$ and $b = 3$.

In his paper, Brian Hunt[5] considers Weierstrass functions with random phases $f_\theta : [0,1] \to [0,1]$ of the form:

$$f_\theta(x) = \sum_{n=0}^{\infty} a^n cos\,(2\pi(b^n x + \theta_n)) \tag{33}$$

where $0 < a < 1 < b$, $ab < 1$, and $\theta = (\theta_1, \theta_2, ...) \in [0,1]^\infty = H$ is randomly chosen by sampling each of its entries according to the uniform distribution on $[0,1]$.
With a rather simple proof, he shows the following theorem:

Theorem 14. *If each θ_n is chosen independently with respect to the uniform probability measure on $[0,1]$, then, almost surely, the Hausdorff dimension of the graph of f_θ is $2-\alpha$.*

In his proof, he lets μ_θ be the measure supported on the graph of f_θ that is induced by the Lebesgue measure λ on a given interval J. More precisely, for $S \in \mathbb{R}^2$:

$$\mu_\theta(S) = \lambda\left(\{x \in J : (x, f_\theta(x)) \in S)\}\right) \tag{34}$$

According to (1), the s-energy of μ_θ is:

$$I_s(\mu_\theta) = \int_J \int_J |(x, f_\theta(x)) - (y, f_\theta(y))|^{-s} \; dx\, dy \tag{35}$$

Considering the Weierstrass functions simplifies considerably the proof due to the following observation. In light of (2) and Lemma 7, to show the result of Theorem 14, it is enough to show that the quantity in (35) is finite for almost every $\theta \in H$ and for $s \in (1, 2-\alpha)$. And for showing this, it is sufficient to show that the expectation $\mathbb{E}_H\left(I_s\left(\mu_\theta\right)\right)$ is finite. By the Tonelli theorem, we can rewrite such expectation as:

$$\mathbb{E}_H\left(I_s\left(\mu_\theta\right)\right) = \int_J \int_J \mathbb{E}_H\left(|(x, f_\theta(x)) - (y, f_\theta(y))|^{-s}\right) \; dx\, dy \tag{36}$$

This proposition is then crucial in Hunt's proof and, in fact, the result of Theorem 14 follows with no particular effort from it.

Proposition 15. *For any x, y and any such that $|x-y| < 1/2b^2$:*

$$\mathbb{E}_H \left(|(x, f_\theta(x)) - (y, f_\theta(y))|^{-s} \right) \leq C\, |x-y|^{(1-\alpha-s)} \tag{37}$$

for any $s \in (1, 2-\alpha)$

Proof. Fix x, y such that $0 < |x-y| \leq \frac{1}{2b^2}$. Now, let $z = f_\theta(x) - f_\theta(y)$. Note that, for fixed values of x and y, $z : H \to \mathbb{R}$ is a real-valued random variable (i.e. measurable function). Let $h(z)$ be the density function of the just mentioned random variable. Then, performing two changes of variable (one to rewrite the expectation in terms of the density and the other one consisting of the simple substitution $z = |x-y|\,\omega$):

$$\begin{aligned}
\mathbb{E}_H \left(|(x, f_\theta(x)) - (y, f_\theta(y))|^{-s} \right) &= \int_{-\infty}^{\infty} \frac{h(z)}{((x-y)^2 + z^2)^{\frac{s}{2}}}\, dz \\
&= \int_{-\infty}^{\infty} \frac{h(|x-y|\,\omega)\, |x-y|}{|x-y|^s\, (1+\omega^2)^{\frac{s}{2}}}\, d\omega \\
&\leq \sup_z h(z)\, |x-y|^{1-s} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^{\frac{s}{2}}}\, d\omega \\
&\lesssim \sup_z h(z)\, |x-y|^{1-s}
\end{aligned} \tag{38}$$

The last step followed since the above integral is convergent for $s > 1$ and we are taking $s \in (1, 2-\alpha)$. After obtaining this bound, to complete the proof, it is then sufficient to show that $h(z) \leq C\, |x-y|^{-\alpha}$ for any z.
In order to prove such result, first, note that the random variable z can be rewritten as:

$$\begin{aligned}
z &= f_\theta(x) - f_\theta(y) \\
&= \sum_{n=0}^{\infty} b^{-\alpha n} \left(\cos(2\pi\, (b^n x + \theta_n)) - \cos(2\pi\, (b^n y + \theta_n)) \right) \\
&= \sum_{n=0}^{\infty} 2 b^{-\alpha n} \sin\left(\frac{2\pi b^n (y-x)}{2} \right) \sin\left(2\pi \left(\frac{b^n (y-x)}{2} + \theta_n \right) \right) \\
&= \sum_{n=0}^{\infty} q_n \sin(r_n + 2\pi\theta_n) \\
&= \sum_{n=0}^{\infty} z_n
\end{aligned} \tag{39}$$

where $q_n = 2b^{-\alpha n} \sin(\pi b^n (y-x))$ and $r_n = 2b^{-\alpha n} \sin(\pi b^n (y+x))$ are independent of any θ_n (i.e. independent of the infinite sequence θ).“ Next, for any n, the cumulative distribution of the random variables z_n is:

$$\begin{aligned}
\mathbb{P}\left(q_n \sin\left(r_n + 2\pi\theta_n \right) \leq y \right) &= \frac{1}{2\pi} \left(1 - |\{ x \in [r_n, r_n + 2\pi] : q_n \sin\left(r_n + 2\pi\theta_n \right) > y \}| \right) \\
&= \frac{1}{2\pi} \left(1 - 2 \arccos\left(\frac{y}{q_n} \right) \right)
\end{aligned} \tag{40}$$

for $y \in [-q_n, q_n]$. This is because $r_n + 2\pi\theta_n$ in the argument of the sin function is uniformly distributed on an interval of length 2π and also since

$$\begin{aligned}
y &= q_n \sin\left(\frac{\pi}{2} - \frac{|\{ x \in [r_n, r_n + 2\pi] : q_n \sin\left(r_n + 2\pi\theta_n \right) > y \}|}{2} \right) \\
&= q_n \cos\left(\frac{|\{ x \in [r_n, r_n + 2\pi] : q_n \sin\left(r_n + 2\pi\theta_n \right) > y \}|}{2} \right)
\end{aligned} \tag{41}$$

Since the z_n are continuous random variables, we can then obtain their probability density distributions:

$$
\begin{aligned}
h_n(z_n) &= \frac{d}{dz_n}\mathbb{P}\left(q_n \sin\left(r_n + 2\pi\theta_n\right) \leq z_n\right) \\
&= \frac{d}{dz_n}\left(\frac{1}{2\pi}\left(1 - 2\arccos\left(\frac{y}{q_n}\right)\right)\right) \\
&= \begin{cases} \frac{1}{\pi\sqrt{q_n^2 - z_n^2}} & z_n \in [-q_n, q_n] \\ 0 & z_n \notin [-q_n, q_n]. \end{cases}
\end{aligned}
\tag{42}
$$

Then, using the result of theorem (13), since $z = \sum_{n=0}^{\infty} z_n$, we have that its density is the infinite convolution $h = h_0 * h_1 ...$. Furthermore, noting that the maximum value of a probability density cannot increase under convolution with another probability density, in order to bound $h(z)$, it is sufficient to find a bound on a finite convolution $h_j * ... * h_k$ for some values of j and k. Recall that, by hypothesis, $0 < |x - y| < \frac{1}{2b^2}$. Choose an integer $k \geq 2$ such that $\frac{1}{2}b^{-k-1} < |x - y| < \frac{1}{2}b^{-k}$. Thus:

$$
\frac{\pi}{2b^3} < \left|2\pi b^{k-2}\left(\frac{y - x}{2}\right)\right| < \left|2\pi b^{k}\left(\frac{y - x}{2}\right)\right| \leq \frac{\pi}{2},
\tag{43}
$$

and hence

$$
|q_n| > 2\sin\left(\frac{\pi}{2b^3}\right) b^{-\alpha k} > 2\sin\left(\frac{\pi}{2b^3}\right) b^{-\alpha}
\tag{44}
$$

for $n = k - 2, k - 1, k$. Now,notice that with a change of variable z_n to $q_n z_n$:

$$
\begin{aligned}
\|h_n\|_p^s &= 2\,|q_n|^{1-s}\int_0^1 \frac{1}{\left|\pi\sqrt{1 - z_n^2}\right|^p}\,dz_n \\
&\leq 2\,|q_n|^{1-s}\int_0^1 \frac{1}{\left|\pi\sqrt{1 - z_n^2}\right|^p}\,dz_n \\
&= 2\,|q_n|^{1-s}\int_0^1 \frac{1}{|\pi z_n|^{\frac{p}{2}}}\,dz_n
\end{aligned}
\tag{45}
$$

where the above integral converges for values $p < 2$. Then, combining (44) and (45), for $n = k - 2, k - 1, k$, we have that:

$$
\|h_n\|_{\frac{3}{2}} = K\,|q_n|^{-\frac{1}{2}} \leq K'\,|x - y|^{-\frac{\alpha}{3}}
\tag{46}
$$

where K is an absolute constant and K' depends only on b. Then, by an application of Young's inequality,

$$
\|h_{k-1} * h_k\|_3 \leq \|h_{k-1}\|_{\frac{3}{2}}\,\|h_k\|_{\frac{3}{2}}
\tag{47}
$$

and combining Hölder's inequality with (47) and (46), we obtain:

$$
\begin{aligned}
h(z) &= h_0 * h_1 ... \leq h_{k-2} * h_{k-1} * h_k \\
&\leq \|h_{k-2}\|_{\frac{3}{2}}\,\|h_{k-1} * h_k\|_3 \leq \|h_{k-2}\|_{\frac{3}{2}}\,\|h_{k-1}\|_{\frac{3}{2}}\,\|h_k\|_{\frac{3}{2}} \\
&\leq K'^3\,|x - y|^{-\alpha}
\end{aligned}
\tag{48}
$$

for any z, completing the proof. □

In this paper, in order to prove the 2-dimensional case of Theorem 6 in the context of the discrete Hausdorff dimension defined in (6), we will use the Weierstrass functions f_θ and an approach similar to the one used by Hunt in the proof of Theorem 14. In doing so, Proposition 15 is used and it has a key-role. We show that for almost every f_θ, $dim_{\mathcal{H}}(\mathcal{P}) = 2 - \alpha$ ($\mathcal{P}$ being the family of $\mathcal{P}_n$ sets as defined in (9)).The conclusion of the 2-dimensional case of Theorem 6 then follows from the fact that for any given value $x \in (0, 1]$ we can find values of a and b such that $\alpha = x$. This is a consequence of the fact that $\alpha = -\frac{\log(a)}{\log b}$ and $0 < a < 1 < b$ with $ab \geq 1$. In fact, fixing a positive $a < 1$, $b = \frac{1}{a^x} \geq \frac{1}{a}$ for any value of $x \in (0, 1]$ and $\alpha = x$.

1.6 Extending the result to higher dimensions

To generalize to higher dimensions (i.e. $d \geq 3$), we consider the following functions $g_\theta : [0,1]^{d-1} \to [0,1]$ of the form:

$$g_\theta(x) = g_\theta(x_1, ..., x_{d-1}) = f_\theta(x_1) \tag{49}$$

where the functions f_θ are defined as in (33).
The reason to define such functions in order to obtain the multidimensional result is because of the following theorem [7]:

Theorem 16. *If A and B are two Borel sets and the upper box counting dimension of A is equal to its Hausdorff dimension, then:*

$$dim_{\mathcal{H}}(A \times B) = dim_{\mathcal{H}}(A) + dim_{\mathcal{H}}(B) \tag{50}$$

In fact, since the graph of the Weierstrass function, with or without the random phases, has its upper box counting dimension equal to its Hausdorff dimension $(2-\alpha)$ [6], noting that the graph of g_θ consists of the Cartesian product of the graph of f_θ and a $(d-2)$-dimensional rectangle, $dim_{\mathcal{H}}(\mathcal{P}) = (2-\alpha) + (d-2) = d - \alpha$ by (50). In the proof of Theorem 6, using the functions g_θ, the multidimensional result follows also in the context of the discrete Hausdorff dimension of the family $\mathcal{P}$ of sets $\mathcal{P}_n$ as defined in (9).

2 Proof of the main theorem

2.1 2-dimensional case

Proof. In order to prove the case for $d = 2$, the construction used by B.Hunt is followed. In such construction, we consider Weierstrass functions with random phases $f_\theta : [0,1] \to [0,1]$ of the form:

$$f_\theta(x) = \sum_{n=0}^{\infty} a^n cos\left(2\pi(b^n x + \theta_n)\right) \tag{51}$$

where $0 < a < 1 < b$, $ab < 1$, and $\theta = (\theta_1, \theta_2, ...) \in [0,1]^\infty = H$ is randomly chosen by sampling each of its entries according to the uniform distribution on $[0,1]$. Note that, for any $\theta \in H$, f_θ is an Holder continuous function with exponent $\alpha = -\frac{\log(a)}{\log(b)}$. In fact, for any $x, y \in [0,1]$, substituting $a = b^{-\alpha}$:

$$\begin{aligned} |f_\theta(x) - f_\theta(y)| &= \sum_{n=0}^{\infty} b^{-n\alpha} \cos(2\pi(b^n x + \theta_n)) - \sum_{n=0}^{\infty} b^{-n\alpha} \cos(2\pi(b^n y + \theta_n)) \\ &\leq \sum_{n=0}^{\infty} b^{-n\alpha} \min(2, 2\pi b^n |x-y|) \\ &\leq \sum_{n=0}^{m-1} 2\pi b^{n(1-\alpha)} |x-y| + \sum_{n=m}^{\infty} 2b^{-n\alpha} \\ &= 2\pi \frac{b^{(1-\alpha)m} - 1}{b^{(1-\alpha)} - 1} |x-y| + 2\frac{b^{-\alpha m}}{1 - b^{-\alpha}} \end{aligned} \tag{52}$$

for any integer $m > 0$. The first inequality followed thanks to the fact that the cosine function is Lipschitz (immediate consequence of the mean value theorem). Set m to be the positive integer such that $b^{-m} < |x-y| \leq b^{-(m-1)}$. Then, resuming the estimate:

$$\begin{aligned} |f_\theta(x) - f_\theta(y)| &\leq 2\pi \frac{\left(\frac{b}{|x-y|}\right)^{(1-\alpha)}}{b^{(1-\alpha)} - 1} |x-y| + 2\frac{|x-y|^\alpha}{1 - b^{-\alpha}} \\ &\leq \left(\frac{2\pi b^{(1-\alpha)}}{b^{(1-\alpha)} - 1} + \frac{2}{1 - b^{-\alpha}}\right) |x-y|^\alpha . \end{aligned} \tag{53}$$

Thus, combining (53) and Lemma 6, we have that $dim_{\mathcal{H}}(\mathcal{P}) \geq 2 - \alpha$. To prove the other inequality, the random phases are very useful as they allow us to estimate the energy integral indirectly. In fact, the key point of the proof is to show that, for any $s < 2 - \alpha$, $\mathbb{E}_H(I_s(G_n)) < \infty$ independently from n (i.e. independently from q). Consequently, this implies that for almost every random sequence of phases $\theta \in H$, f_θ satisfies $dim_{\mathcal{H}}(\mathcal{P}) \leq 2 - \alpha$ (completing the proof for $d = 2$). To perform the estimate, by linearity of expectation:

$$\mathbb{E}_H(I_s(G_n)) = q^{-2} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z}}} \mathbb{E}_H\left(\left|\left(\frac{j}{q}, f_\theta\left(\frac{j}{q}\right)\right) - \left(\frac{j'}{q}, f_\theta\left(\frac{j'}{q}\right)\right)\right|^{-s}\right) \tag{54}$$

Next, the following lemma provides an upper bound for the expectations inside the sum when $\frac{j}{q}$ and $\frac{j}{q}$ are close enough.
Then, resuming the proof, we can decompose the sum in two terms:

$$\mathbb{E}_H(I_s(G_n)) = I + II \tag{55}$$

where

$$I = q^{-2} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z} \\ |j - j'| \leq \frac{q}{2b^2}}} \mathbb{E}_H\left(\left|\left(\frac{j}{q}, f_\theta\left(\frac{j}{q}\right)\right) - \left(\frac{j'}{q}, f_\theta\left(\frac{j'}{q}\right)\right)\right|^{-s}\right) \tag{56}$$

$$II = q^{-2} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z} \\ |j-j'| > \frac{q}{2b^2}}} \mathbb{E}_H \left(\left| \left(\frac{j}{q}, f_\theta\left(\frac{j}{q}\right) \right) - \left(\frac{j'}{q}, f_\theta\left(\frac{j'}{q}\right) \right) \right|^{-s} \right) \tag{57}$$

The second term is easy to bound as:

$$II \leq q^{-2} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z} \\ |j-j'| > \frac{q}{2b^2}}} \left| \frac{j}{q} - \frac{j'}{q} \right|^{-s} \leq q^{-2} q^2 (2b^2)^s = (2b^2)^s \tag{58}$$

For the first term, using (37):

$$\begin{aligned} I &\lesssim q^{-2} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z} \\ |j-j'| \leq \frac{q}{2b^2}}} \left| \frac{j}{q} - \frac{j'}{q} \right|^{1-\alpha-s} \lesssim q^{-3+\alpha+s} \sum_{\substack{j \neq j' \\ j,j' \in [0,q) \cap \mathbb{Z}}} |j-j'|^{1-\alpha-s} \\ &\lesssim q^{-2+\alpha+s} \sum_{\substack{j \in (-q,q) \cap \mathbb{Z} \\ j \neq 0}} |j|^{1-\alpha-s} \lesssim q^{-2+\alpha+s} q^{2-\alpha-s} \lesssim 1 \end{aligned} \tag{59}$$

completing the proof. Note that the third and fourth inequality are, respectively, due to the change of variable $j - j'$ to j and due the to integral test (where the integral converges for values of s strictly smaller than $2 - \alpha$, as desired).

□

2.2 Higher dimensional case for $d \geq 3$

Proof. For any $\theta \in H$, g_θ is again Holder continuous with exponent α. This is because, using (53) and (49), for any $x, y \in [0,1]^{d-1}$:

$$\begin{aligned} |g_\theta(x) - g_\theta(y)| = |f_\theta(x_1) - f_\theta(y_1)| &\leq \left(\frac{2\pi b^{(1-\alpha)}}{b^{(1-\alpha)} - 1} + \frac{2}{1 - b^{-\alpha}} \right) |x_1 - y_1|^\alpha \\ &\leq \left(\frac{2\pi b^{(1-\alpha)}}{b^{(1-\alpha)} - 1} + \frac{2}{1 - b^{-\alpha}} \right) |x - y|^\alpha . \end{aligned} \tag{60}$$

Thus, combining (60) and Lemma 6, $dim_{\mathcal{H}}(\mathcal{P}) \geq d - \alpha$.
To prove the other inequality, looking at the expectation over all random phases as in the case $d = 2$, we can perform the following decomposition:

$$\begin{aligned} \mathbb{E}_H(I_s(G_n)) &= q^{-2(d-1)} \mathbb{E}_H \left(\sum_{\substack{j \neq j' \\ j,j' \in [0,q)^{d-1} \cap \mathbb{Z}^{d-1}}} \left| \left(\frac{j}{q}, g_\theta\left(\frac{j}{q}\right) \right) - \left(\frac{j'}{q}, g_\theta\left(\frac{j'}{q}\right) \right) \right|^{-s} \right) \\ &= q^{-2(d-1)} \mathbb{E}_H \left(\sum_{\substack{j \neq j' \\ j,j' \in [0,q)^{d-1} \cap \mathbb{Z}^{d-1}}} \left| \left(\frac{j}{q}, f_\theta\left(\frac{j_1}{q}\right) \right) - \left(\frac{j'}{q}, f_\theta\left(\frac{j'_1}{q}\right) \right) \right|^{-s} \right) \\ &= I + II \end{aligned} \tag{61}$$

where

$$I = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z}}} \mathbb{E}_H \left(\sum_{\bar{j}, \bar{j}' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta\left(\frac{j_1}{q}\right) - f_\theta\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{\bar{j}}{q} - \frac{\bar{j}'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{62}$$

$$II = q^{-2(d-1)} \sum_{\substack{j_1 = j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z}}} \mathbb{E}_H \left(\sum_{\substack{\bar{j} \neq \bar{j}' \\ \bar{j}, \bar{j}' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta\left(\frac{j_1}{q}\right) - f_\theta\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{\bar{j}}{q} - \frac{\bar{j}'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{63}$$

where $\bar{j} = (j_2, ..., j_{d-1})$. We just need to worry about bounding term I since it is greater than term II. To see this, note that any addend in the inner sum of II is comparable to an addend of the inner sum of II but, at the same time, the outer sum of I contains way more terms than the outer sum of II (due to the difference between the conditions $j_1 = j_1'$ and $j_1 \neq j_1'$). To bound the first term, we can further decompose it in two pieces $I = III + IV$ where:

$$III = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| \leq \frac{q}{2b^2}}} \mathbb{E}_H \left(\sum_{\bar{j}, \bar{j}' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta\left(\frac{j_1}{q}\right) - f_\theta\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{\bar{j}}{q} - \frac{\bar{j}'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{64}$$

$$IV = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| > \frac{q}{2b^2}}} \mathbb{E}_H \left(\sum_{\bar{j}, \bar{j}' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta\left(\frac{j_1}{q}\right) - f_\theta\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{\bar{j}}{q} - \frac{\bar{j}'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{65}$$

The last term is bounded as follows:

$$\begin{aligned} IV &\leq q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| > \frac{q}{2b^2}}} \sum_{\bar{j}, \bar{j}' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^{-s} \\ &\leq q^{-2(d-1)} q^2 q^{2(d-2)} (2b^2)^s = (2b^2)^s \end{aligned} \tag{66}$$

In the inner sum of III, the quantity $\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta(\frac{j_1}{q}) - f_\theta(\frac{j_1'}{q}) \right|^2$ is constant (i.e. does not depend on $\bar{j}, \bar{j}'$). For the sake of clarity, set $\beta^2 = \left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f_\theta(\frac{j_1}{q}) - f_\theta(\frac{j_1'}{q}) \right|^2$. We can then find an upper bound for such inner sum:

$$
\begin{aligned}
\sum_{\bar{j},\bar{j}'\in[0,q)^{d-2}\cap\mathbb{Z}^{d-2}} \left(\beta^2+\left|\frac{\bar{j}}{q}-\frac{\bar{j}'}{q}\right|^2\right)^{-\frac{s}{2}} &= \beta^{-s}+\sum_{\substack{\bar{j}\neq\bar{j}'\\ \bar{j},\bar{j}'\in[0,q)^{d-2}\cap\mathbb{Z}^{d-2}}} \left(\beta^2+\left|\frac{\bar{j}}{q}-\frac{\bar{j}'}{q}\right|^2\right)^{-\frac{s}{2}} \\
&\leq \beta^{-s}+q^{d-2+s}\sum_{\substack{\bar{j}\neq\bar{j}'\\ \bar{j}\in(-q,q)^{d-2}\cap\mathbb{Z}^{d-2}}} \left((q\beta)^2+|\bar{j}|^2\right)^{-\frac{s}{2}} \\
&\lesssim \beta^{-s}+q^{d-2+s}\int_{x\in[0,q)^{d-2}} \left((q\beta)^2+|x|^2\right)^{-\frac{s}{2}}\,dx \\
&\lesssim \beta^{-s}+q^{d-2+s}(q\beta)^{d-2-s}\int_{x\in\mathbb{R}^{d-2}} \left(1+|x|^2\right)^{-\frac{s}{2}}\,dx \\
&\lesssim \beta^{-s}+q^{d-2+s}(q\beta)^{d-2-s} \\
&\lesssim q^{2(d-2)}\beta^{d-2-s}
\end{aligned}
\tag{67}
$$

In evaluating the integral, the change of variable x to $(q\beta)x$ is performed. Note that the above integral is convergent for any value $s>d-2$ (which is fine since we consider s such that $d-1<s<d-\alpha$). The first inequality follows by the change of variable $\bar{j}-\bar{j}'$ to $\bar{j}$ and the second one by integral test. Additionally, since we consider $j_1\neq j_1'$ and then $\beta^2\geq q^{-2}$, the last inequality follows because:

$$
\begin{aligned}
&q^{d-2+s}(q\beta)^{d-2-s}=\beta^{-s}q^{d-2+s}(q)^{d-2-s}(\beta)^{d-2} \\
&=\beta^{-s}q^{2(d-2)}(\beta)^{d-2}\geq\beta^{-s}q^{2(d-2)}(q^{-1})^{d-2} \\
&=\beta^{-s}q^{(d-2)}\geq\beta^{-s}
\end{aligned}
\tag{68}
$$

for $d\geq 2$.
Consequently, using (67) and by substituting back for β:

$$
\begin{aligned}
III &\lesssim q^{-2(d-1)}q^{2(d-2)}\sum_{\substack{j_1\neq j_1'\\ j_1,j_1'\in[0,q)\cap\mathbb{Z}\\ |j_1-j_1'|\leq\frac{q}{2b^2}}} \mathbb{E}_H\left(\left(\left|\frac{j_1}{q}-\frac{j_1'}{q}\right|^2+\left|f_\theta\left(\frac{j_1}{q}\right)-f_\theta\left(\frac{j_1'}{q}\right)\right|^2\right)^{\frac{d-2-s}{2}}\right) \\
&= q^{-2}\sum_{\substack{j_1\neq j_1'\\ j_1,j_1'\in[0,q)\cap\mathbb{Z}\\ |j_1-j_1'|\leq\frac{q}{2b^2}}} \mathbb{E}_H\left(\left|\left(\frac{j_1}{q},f_\theta\left(\frac{j_1}{q}\right)\right)-\left(\frac{j_1'}{q},f_\theta\left(\frac{j_1'}{q}\right)\right)\right|^{d-2-s}\right) \\
&\lesssim q^{-2}q^{-1+\alpha-d+2+s}\sum_{\substack{j_1\neq j_1'\\ j_1,j_1'\in[0,q)\cap\mathbb{Z}}} |j_1-j_1'|^{1-\alpha+d-2-s} \\
&\lesssim q^{-1+\alpha-d+s}q\sum_{j_1\in[0,q)\cap\mathbb{Z}} |j_1|^{\alpha+d-1-s}\lesssim q^{-d+\alpha+s}q^{d-\alpha-s}\lesssim 1
\end{aligned}
\tag{69}
$$

completing the proof also for the case $d\geq 3$.The second inequalities follows thanks to (37), the third one by the change of variable j_1-j_1' to j_1 and the forth one by integral test (where the integral converges for values s strictly smaller than $d-\alpha$, as desired). Note that we could use (67) successfully since $1=-d+(d-1)+2<-d+s+2<-d+(d-\alpha)+2=2-\alpha$. □

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