Intersection Theory and the Poincaré–Hopf Theorem

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1 Introduction

The Poincaré–Hopf theorem is a beautiful theorem about vector fields on smooth oriented manifolds which relates the zeros of the vector field to the Euler characteristic of the manifold. Our purpose is to present this theorem as clearly as possible to someone with a background in general topology and linear algebra.

2 Manifolds and Tangent Spaces

First, we must define some basic terms.

Definition 2.1. *A function $f : U \to \mathbf{R}^m$, where U is open in $\mathbf{R}^n$, is called* ***smooth*** *if it has continuous partial derivatives of all orders. A function f defined on an arbitrary subset X of $\mathbf{R}^n$ is called smooth if for all $x \in X$ there is a neighborhood U of x and a smooth map $F : U \to \mathbf{R}^m$ such that F equal f on $U \cap X$.*

The term "local" will be used frequently, and it is usually referring to open sets around a point. That is, if a space X locally has a property at x, then there is an open neighborhood of x which has that property. So, the above definition can be written more simply as "f is smooth if it can be locally extended to a smooth map on open sets."

Now, just as in general topology we have a homeomorphism and in linear algebra we have an isomorphism, here we define a concept with which we can view two sets as essentially equivalent.

Definition 2.2. *A smooth map $X \to Y$, where $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ is called a* ***diffeomorphism*** *if f is bijective and $f^{-1} : Y \to X$ is also smooth.*

Notice that a diffeomorphism is a homeomorphism where the continuity condition is replaced with smoothness. Now we are ready to define smooth manifolds.

Definition 2.3. *A set $X \subset \mathbf{R}^N$ is a k-dimensional* ***smooth manifold*** *if it is locally diffeomorphic to $\mathbf{R}^k$. That is, for all $x \in X$, there exists a diffeomorphism (called a parametrization) $\phi : U \to V$, where U is open in $\mathbf{R}^k$ and V is an open neighborhood of x in X. For convenience, we may assume $\phi(0) = x$. The inverse $\phi^{-1} : V \to U$ is called a coordinate system on V.*

Every manifold that we are working with is smooth, so we will often refer to them simply as "manifolds." Numerous examples of manifolds should readily come to mind, including the space $\mathbf{R}^k$ itself, along with the familiar $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$. Using only the definition, it is not very easy to prove that a given space is a manifold because you need to find a parametrization around every point on the manifold. Luckily, there are often easier ways of showing something is a manifold.

Theorem 2.4. *If X and Y are manifolds, so is $X \times Y$, and $\dim X \times Y = \dim X + \dim Y$.*

Proof. Let $(x, y) \in X \times Y$. Then there exist parametrizations $\phi : U \to X$ and $\psi : W \to Y$ where U and W are open sets in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, such that $\phi(0) = x$ and $\psi(0) = y$. Then the map $\phi \times \psi : U \times W \to X \times Y$ given by $(\phi \times \psi)(u, w) = (\phi(u), \psi(w))$ is a parametrization from an open subset of $\mathbf{R}^{m+n}$ to $X \times Y$ around (x, y), and the result follows. ■

If X and Z are both manifolds in $\mathbf{R}^N$ and $Z \subset X$, then Z is called a submanifold of X. In order to do any sort of useful analysis on manifolds, we need to introduce a concept of derivatives of maps from one manifold to another. First, recall the usual definition of a derivative. If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is smooth, $x \in U$ and $h \in \mathbf{R}^n$, then we have

$$Df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

The derivative is a linear map represented by the Jacobian matrix

$$Df_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

The derivative has a number of useful properties. The first is the Chain Rule, which says that if $f : U \to V$ and $g : V \to \mathbf{R}^\ell$ are smooth, then for all $x \in U$,

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x.$$

Also, if L is a linear map, then $DL_x = L$ for all x in the domain of L. Finally, the derivative of a function is its best linear approximation. Let X be a smooth manifold and $\phi : U \to X$ a local parametrization around x, where U is open in $\mathbf{R}^k$. We can assume for convenience that $\phi(0) = x$. Then the best linear approximation to ϕ at 0 is the map

$$f(u) = \phi(0) + D\phi_0(u).$$

This leads us naturally to the definition of a tangent space.

Definition 2.5. *The* ***tangent space*** $T_x(X)$ *of* X *at* x *is the image of the map* $D\phi_0 : \mathbf{R}^k \to \mathbf{R}^N$*. It is a vector subspace of* $\mathbf{R}^N$ *such that* $x + T_x(X)$ *is the best linear approximation to* X *through* x*.*

To show that the tangent space is well-defined, we have to show that it does not depend on the choice of parametrization. So, suppose $\phi : U \to X$ and $\psi : V \to X$ are both parametrizations around $x \in X$, with $\phi(0) = \psi(0) = x$. Let $W = \phi(U) \cap \psi(V)$, which is a nonempty open subset of X. Define $U' = \phi^{-1}(W)$ and $V' = \psi^{-1}(W)$. This is done so that $\phi(U') = \psi(V')$. Define $h = \psi^{-1} \circ \phi : U' \to V'$. Then, by the Chain Rule, $D\phi_0 = D\psi_0 \circ Dh_0$. This implies that the image of $D\psi_0$ contains the image of $D\phi_0$. By symmetry, the image of $D\phi_0$ contains the image of $D\psi_0$ as well, so their images are the same. Therefore, the tangent space is well-defined.

Proposition 2.6. $\dim T_x(X) = \dim X$*.*

Proof. If $\phi : U \to V$ is a local parametrization about $x \in X$, then $\phi^{-1} : V \to U$ can be locally extended to a smooth map $\Phi' : W \to \mathbf{R}^k$, where W is open in $\mathbf{R}^N$. Then $\Phi' \circ \phi$ is the identity map on U. By the Chain Rule, $D\Phi'_x \circ D\phi_0$ is the identity map on $\mathbf{R}^k$, so $D\phi_0$ is an isomorphism from $\mathbf{R}^k$ to $T_x(X)$. Therefore, $\dim T_x(X) = k = \dim X$. ■

Now, we'll extend the notion of derivative to smooth functions between manifolds, which in general are not defined on open subsets of $\mathbf{R}^N$. It has the characteristics one would expect from a derivative map. Namely, if $f : X \to Y$ is smooth, then Df_x is a linear map from $T_x(X)$ to $T_{f(x)}(Y)$, which is the best linear approximation to f at x. It also satisfies the Chain Rule. This new derivative is defined as follows:

Definition 2.7. *Let* $f : X \to Y$ *be a smooth map of manifolds, with* $f(x) = y$*. Suppose* $\phi : U \to X$ *parametrizes* X *about* x *and* $\psi : V \to Y$ *parametrizes* Y *about* y*, where* $U \subset \mathbf{R}^k$ *and* $V \subset \mathbf{R}^l$*, and say* $\phi(0) = x$*,* $\psi(0) = y$*. Define* $h : U \to V$ *by* $h = \psi^{-1} \circ f \circ \phi : U \to V$*. Then the* ***derivative*** *of* f *at* x *is defined to be* $Df_x = D\psi_0 \circ Dh_0 \circ D\phi_0^{-1}$*.*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h=\psi^{-1}\circ f\circ\phi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Figure 5

Theorem 2.8. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are smooth maps of manifolds, then*

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x.$$

3 The Derivative and Local Behavior of Maps

The purpose of looking at derivatives is to describe the local behavior of a function near a point. We can often determine everything about the local behavior of a map just from its derivative.

Let $f : X \to Y$ be a smooth map of manifolds. Let $x \in X$ and $y = f(x) \in Y$. Then, if Df_x is an isomorphism, f is a local diffeomorphism at x. This is known as the **Inverse Function Theorem**. We will not use it in proving the Poincaré–Hopf Theorem, though, so we will leave it unproved. Of greater interest are the related Local Immersion Theorem and Local Submersion Theorem.

First, we must introduce some terminology. If Df_x is injective, then f is called an **immersion** at x. If f is an immersion everywhere, then it is simply called an immersion. On the other hand, if Df_x is surjective, then f is called a **submersion** at x.

Theorem 3.1. *If $f : X \to Y$ is an immersion at x and $y = f(x)$, then there exist local coordinates around x and y such that*

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

In other words, f is locally equivalent to the canonical immersion near x.

Our sole purpose for immersions is to give us a criterion for when the image of a smooth map $f : X \to Y$ is a submanifold of Y. Unfortunately, being an immersion is not quite enough, we need a few more conditions. A map $f : X \to Y$ is called *proper* if the preimage of any compact subset of Y is a compact subset of X. Now, we define an **embedding** as a proper, injective, immersion. It turns out this is just the condition we need in order for the image of X to be a submanifold of Y.

Theorem 3.2. *An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.*

Proof. h ∎

Theorem 3.3. *If $f : X \to Y$ is a submersion at x and $y = f(x)$, then there exist local coordinates around x and y such that*

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell).$$

In other words, f is locally equivalent to the canonical submersion near x.

4 Manifolds with Boundary

Some topological spaces, like the closed disk $D = \{x \in \mathbf{R}^2 \mid |x| \leq 1\}$, are almost manifolds but not quite, because there is no parametrization for points on the boundary of the disk. For this reason, we define manifolds with boundary.

Definition 4.1. *Let H^k be the k-dimensional "upper half space" given by $H^k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \mid x_k \geq 0\}$. A set $X \subset \mathbf{R}^N$ is a k-dimensional **manifold with boundary** if it is locally diffeomorphic to H^k.*

The boundary of a manifold with boundary X, denoted ∂X, is the set of points in x whose final coordinate is 0 under some local coordinate system. In fact, if the final coordinate of x is 0 under one local coordinate system, then the same must be true for any other local coordinate system. This implies that ∂X is a manifold of dimension $k - 1$.

5 Transversality

The Poincaré–Hopf theorem makes use of oriented intersection theory. The intersection of two manifolds can, in general, be very pathological, so we will usually impose a useful condition on their intersection, which we call transversality.

Definition 5.1. *Let X and Z be submanifolds of a manifold Y. We say X and Z are **transverse**, or intersect transversely, if*

$$T_y(X) + T_y(Z) = T_y(Y)$$

for all $y \in X \cap Z$. This relation is denoted $X \pitchfork Y$.

Since $T_y(X)$ and $T_y(Z)$ are both vector subspaces of $T_y(Y)$, their sum is the set

$$T_y(X) + T_y(Z) = \{x + z \mid x \in T_y(X), z \in T_y(Z)\}.$$

Notice that if X and Z do not intersect at all, then they intersect transversely.

Example 5.2. *Let $Y = \mathbf{R}^2$, X be the x-axis, and Z be the graph of the function x^3. These only intersect at $(0,0)$, and their tangent spaces at that point are the real axis. Therefore, X and Z are not transverse in this case.*

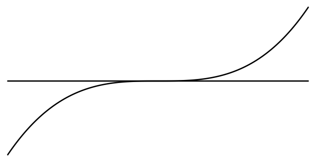

Figure 1

Example 5.3. *Let Y be the sphere S^2, and let X and Z be great circles as pictured in Figure 2. Then $X \pitchfork Z$.*

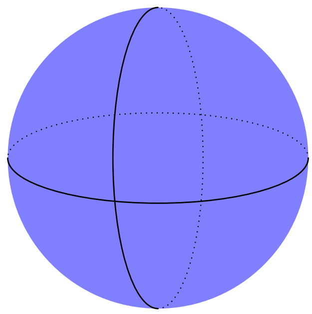

Figure 2

6 Orientation

Definition 6.1. *Let V be a finite-dimensional real vector space. If β_1 and β_2 are bases of V, then there exists a unique linear isomorphism $A : V \to V$ such that $A\beta_1 = \beta_2$. We say β_1 and β_2 are equivalently oriented if* $\det A > 0$*. An orientation of V is an assignment of $+1$ to one equivalence class and -1 to the other. Thus every basis is either positively or negatively oriented.*

Definition 6.2. *Let X be a smooth manifold with boundary. An orientation of X is a smooth choice of orientation for all the tangent spaces $T_x(X)$.*

Proposition 6.3. *A connected, oriented manifold with boundary admits exactly two orientations.*

Definition 6.4. *If X and Y are oriented and one of them is boundaryless, then $X \times Y$ acquires a product orientation as follows. At each point $(x, y) \in X \times Y$,*

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

Let $\alpha = \{v_1, \ldots, v_k\}$ and $\beta = \{w_1, \ldots, w_\ell\}$ be ordered bases for $T_x(X)$ and for $T_y(Y)$, respectively, and denote by $(\alpha \times 0, 0 \times \beta)$ the ordered basis $\{(v_1, 0), \ldots, (v_k, 0), (0, w_1), \ldots, (0, w_\ell)\}$ of $T_x(X) \times T_y(Y)$. Define the orientation on $T_x(X) \times T_y(Y)$ by setting

$$\mathit{sign}(\alpha \times 0, 0 \times \beta) = \mathit{sign}(\alpha)\mathit{sign}(\beta).$$

Definition 6.5. *We orient $T_x(\partial X)$ by declaring the sign of any ordered basis $\beta = \{v_1, \ldots, v_{k-1}\}$ to be the sign of the ordered basis $\{n_x, \beta\} = \{n_x, v_1, \ldots, v_{k-1}\}$ for $T_x(X)$, where n_x is the outward unit normal at x.*

Example 6.6. *As an oriented manifold,* $\partial(I \times X) = X_1 \cup (-X_0) = X_1 - X_0$.

Proposition 6.7. *The sum of the orientation numbers at the boundary points of any compact oriented one-dimensional manifold with boundary is zero.*

Proposition 6.8. *Suppose that* $V = V_1 \oplus V_2$ *is a direct sum. Then orientations on any two of these automatically induce a direct sum orientation on the third, as follows. Choose ordered bases* β_1 *and* β_2 *for* V_1 *and* V_2*, respectively, and let* $\beta = (\beta_1, \beta_2)$ *be the combined ordered basis for* V*. Now simply demand that* $sign(\beta) = sign(\beta_1) \cdot sign(\beta_2)$.

Definition 6.9. *Let* $f : X \to Y$ *be a smooth map with* $f \pitchfork Z$ *and* $\partial f \pitchfork Z$*, where* X*,* Y*, and* Z *are all oriented and the last two are boundaryless. We define a preimage orientation on the manifold-with-boundary* $S = f^{-1}(Z)$*. If* $f(x) = z \in Z$*, then* $T_x(S)$ *is the preimage of* $T_z(Z)$ *under the derivative map* $Df_x : T_x(X) \to T_z(Y)$*. Let* $N_x(S; X)$ *be the orthogonal complement to* $T_x(S)$ *in* $T_x(X)$*. Then*

$$N_x(S; X) \oplus T_x(S) = T_x(X),$$

so that we only need to choose an orientation on $N_x(S; X)$ *to obtain a direct sum orientation on* $T_x(S)$*. Because*

$$Df_x T_x(X) + T_z(Z) = T_z(Y),$$

and $T_x(S)$ *is the entire preimage of* $T_z(Z)$*, we get a direct sum*

$$Df_x N_x(S; X) \oplus T_z(Z) = T_z(Y).$$

Thus the orientations on Z *and* Y *induce a direct image orientation on* $Df_x N_x(S; X)$*. But* $T_x(S)$ *contains the entire kernel of the linear map* Df_x*, so* Df_x *must map* $N_x(S; X)$ *isomorphically onto its image. Therefore the induced orientation on* $Df_x N_x(S; X)$ *defines an orientation on* $N_x(S; X)$ *via the isomorphism* Df_x.

Proposition 6.10. $\partial[f^{-1}(Z)] = (-1)^{codim Z}(\partial f)^{-1}(Z)$

7 Oriented Intersection Number

Definition 7.1. *If* X*,* Y*, and* Z *are boundaryless oriented manifolds,* X *is compact,* Z *is a closed submanifold of* Y*, and* $\dim X + \dim Z = \dim Y$*, then we say* $f : X \to Y$ *and* Z *are appropriate for intersection theory.*

Definition 7.2. *If* $f : X \to Y$ *is transversal to* Z*, then* $f^{-1}(Z)$ *is a finite number of points, each with an orientation number* ± 1 *provided by the preimage orientation. Define the intersection number* $I(f, Z)$ *to be the sum of these orientation numbers.*

If $x \in f^{-1}(Z)$, then the contribution of x to $I(f, Z)$ is $+1$ if the direct sum orientation on $Df_x T_x(X) \oplus T_z(Z)$ is the same as the given orientation on $T_z(Y)$ and -1 otherwise.

Proposition 7.3. *If* $X = \partial W$*, where* W *is compact, and* $f : X \to Y$ *extends to* W*, then* $I(f, Z) = 0$.

Proposition 7.4. *Homotopic maps always have the same intersection numbers.*

Definition 7.5. *Let* Y *be a connected manifold of the same dimension as* X*. The degree of a smooth function* $f : X \to Y$ *is the intersection number of* f *with any point* y*,* $\deg(f) = I(f, \{y\})$.

Proposition 7.6. *If* $f : X \to Y$ *is a smooth map of compact oriented manifolds of the same dimension, with* $X = \partial W$ *for some compact* W*, and* f *can be extended to all of* W*, then* $\deg(f) = 0$.

Definition 7.7. *If* X *is also a submanifold of* Y*, then we define* $I(X, Z)$ *to be the intersection number of the inclusion map of* X *with* Z.

Definition 7.8. *Let* $f : X \to Y$ *and* $g : Z \to Y$ *be smooth, with* X *and* Z *being compact. Then* $f \pitchfork g$ *if* $Df_x T_x(X) + Dg_z T_z(Z) = T_y(Y)$ *whenever* $f(x) = y = g(z)$*. Define the local intersection number at* (x, z) *to be* $+1$ *if the direct sum orientation of* $Df_x T_x(X) \oplus Dg_z T_z(Z)$ *equals the given orientation on* $T_y(Y)$ *and* -1 *otherwise. The intersection number* $I(f, g)$ *is defined as the sum of the local contributions from all pairs* (x, z) *at which* $f(x) = g(z)$.

Lemma 7.9. *Let U and W be subspaces of the vector space V. then $U \oplus W = V$ if and only if $U \times W \oplus \Delta = V \times V$. Assume, also, that U and W are oriented, and give V the direct sum orientation. Now assign Delta the orientation carried from V by the natural isomorphism $V \to \Delta$. Then the product orientation on $V \times V$ agrees with the direct sum orientation from $U \times W \oplus \Delta$ if and only if W is even dimensional.*

Proposition 7.10. *$f \pitchfork g$ if and only if $f \times g \pitchfork \Delta$, and then $I(f, g) = (-1)^{\dim Z} I(f \times g, \Delta)$.*

Proposition 7.11. *If f_0 and g_0 are respectively homotopic to f_q and g_q, then $I(f_0, g_0) = I(f_1, g_1)$.*

Corollary 7.12. *If Z is a submanifold of Y and $i : Z \to Y$ is the inclusion map, then $I(f, i) = I(f, Z)$ for any map $f : X \to Y$.*

Corollary 7.13. *If $\dim X = \dim Y$ and Y is connected, then $I(f, \{y\})$ is the same for every $y \in Y$. Thus $\deg(f)$ is well defined.*

Proposition 7.14. *$I(f, g) = (-1)^{(\dim X)(\dim Z)} I(g, f)$*

Corollary 7.15. *If X and Y are both compact submanifolds of Y, then $I(X, Z) = (-1)^{(\dim X)(\dim Z)} I(Z, X)$*

Definition 7.16. *If Y is a compact, oriented manifold, the Euler Characteristic $\chi(Y)$ is defined to be the self-intersection number of the diagonal Δ in $Y \times Y$:*

$$\chi(Y) = I(\Delta, \Delta)$$

Proposition 7.17. *The Euler characteristic of an odd-dimensional, compact, oriented manifold is zero.*

8 Lefschetz Fixed-Point Theory

Definition 8.1. *The global Lefschetz number of f is defined to be $L(f) = I(\Delta, graph(f)$.*

Theorem 8.2. *Let $f : X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.*

Proposition 8.3. *$L(f)$ is a homotopy invariant.*

Proposition 8.4. *If f is homotopic to the identity, then $L(f)$ equals the Euler characteristic of X. In particular, if X admits a smooth map $f : X \to X$ that is homotopic to the identity and has no fixed points, then $\chi(X) = 0$.*

Definition 8.5. *A Lefschetz map is one where $graph(f) \pitchfork \Delta$.*

Proposition 8.6. *Every map $f : X \to X$ is homotopic to a Lefschetz map.*

Definition 8.7. *A point $x \in X$ is a Lefschetz fixed point of f if 1 is not an eigenvalue of Df_x. Then f is a Lefschetz map if and only if all of its fixed points are Lefschetz.*

Definition 8.8. *The local Lefschetz number $L_x(f)$ at a Lefschetz fixed point is $+1$ if the isomorphism $Df_x - I$ preserves orientation on $T_x(X)$ and -1 otherwise. That is, the sign of $L_x(f)$ equals the sign of the determinant of $Df_x - I$.*

Proposition 8.9. *The Euler characteristic of S^2 is 2.*

Corollary 8.10. *Every map of S^2 that is homotopic to the identity must possess a fixed point. In particular, the antipodal map $x \to -x$ is not homotopic to the identity.*

Proposition 8.11. *The surface of genus k admits a Lefschetz map homotopic to the identity, with one source, one sink, and $2k$ saddles. Consequently, its Euler characteristic is $2 - 2k$.*

Proposition 8.12. *Let U be a neighborhood of the fixed point x that contains no other fixed points of f. Then there exists a homotoy f_t of f such that f_1 has only Lefschetz fixed points in U, and each f_t equals f outside some compact subset of U.*

Definition 8.13. *Suppose x is an isolated fixed point of f in $\mathbf{R}^k$. If B is a small closed ball centered at x that contains no other fixed point, then the assignment*

$$z \to \frac{f(z)-z}{|f(z)-z|}$$

defines a smooth map $F : \partial B \to S^{k-1}$. We call the degree of this map the local Lefschetz number of f at x, denoted $L_x(f)$.

Proposition 8.14. *At Lefschetz fixed points, the two definitions of $L_x(f)$ agree.*

Proposition 8.15. *Let $f : X \to X$ be any smooth map on a compact manifold, with only finitely many fixed points. Then the global Lefschetz number (which is a homotopy invariant) equals the sum of the local Lefschetz numbers*

$$L(f) = \sum_{f(x)=x} L_x(f).$$

9 Vector Fields and the Poincaré–Hopf Theorem

Definition 9.1. *A vector field on a manifold X in $\mathbf{R}^N$ is a smooth map $\overrightarrow{v} : X \to \mathbf{R}^N$ such that $\overrightarrow{v}(x) \in T_x(X)$ for every x.*

If X is a smooth k-dimensional manifold, then the **tangent bundle** of X, denoted $T(X)$, is a smooth $2k$-dimensional manifold given by

$$T(X) = \{(x, v) \mid x \in X, v \in T_x(X)\}.$$

Since X and $T_x(X)$ are both subsets of $\mathbf{R}^N$, $T(X)$ is a subset of $\mathbf{R}^{2N}$. It contains a copy of X given by $X_0 = \{(x, 0) \subset T(X) \mid x \in X\}$.

Let X and Y be manifolds. A smooth map $f : X \to Y$ induces a smooth "derivative" map $Df : T(X) \to T(Y)$ given by $Df(x, v) = (f(x), Df_x(v))$. Now, if $g : S \to M$ and $f : M \to N$ are smooth maps of manifolds, then by the chain rule for derivatives on manifolds, we have

$$Df \circ Dg(x, v) = Df(g(x), Dg_x(v)) = (f \circ g(x), Df_{g(x)} \circ Dg_x(v)) = (f \circ g(x), D(f \circ g)_x(v)) = D(f \circ g)(x, v).$$

Let $\pi : T(X) \to X$ be the projection $\pi(x, v) = x$.

10 Exercise 6

Let X be a compact, oriented manifold. For a smooth vector field $\overrightarrow{v}$ on X there is a smooth map $\sigma : X \to T(X)$ such that $\sigma(x) = (x, \overrightarrow{v}(x))k$. We want to show that σ is an embedding, so that its image is a submanifold of $T(X)$ diffeomorphic to X. That is, we have to show that σ is a proper, injective immersion. We get proper for free because X is compact, and the equation $\pi \circ \sigma = Id_X$ implies that σ is injective. So, the only interesting part is to show that σ is an immersion.

Fix $(x, v) \in T(X)$. Let $\phi : U \to X$ be a parametrization of X at x, and let $\theta : U \times \mathbf{R}^k \to T(X)$ be a parametrization of $T(X)$ at (x, v) given by $\theta(a, b) = (\phi(a), D\phi_a(b))$. Then define $h = \theta^{-1} \circ \sigma \circ \phi : U \to U \times \mathbf{R}^k$. We have

$$h(a) = \theta^{-1}(\sigma(\phi(a))) = \theta^{-1}(\phi(a), b) = (a, D\phi_a^{-1}(b)),$$

for some $b \in T_{\phi(a)}(X)$. It follows that Dh_0 is injective, so

$$D\sigma_x = D\theta_0 \circ Dh_0 \circ D\phi_0^{-1}$$

is also injective. Therefore, σ is an immersion, hence an embedding.

Let X_σ be the image of X under σ. The tangent space of X_σ at a point $(x, \overrightarrow{v}(x))$ is the best linear approximation to X_σ at that point: $\{(a, D\overrightarrow{v}_x(a)) \mid a \in T_x(X)\}$.

A zero x of $\overrightarrow{v}(x)$ is called nondegenerate if $D\overrightarrow{v}_x : T_x(X) \to T_x(X)$ is an isomorphism. Zeroes of $\overrightarrow{v}$ correspond to points in the intersection of X_σ with X_0. The transversality condition at such points is

$$T_{(x,0)}(X_0) + T_{(x,0)}(X_\sigma) = T_{(x,0)}(T(X)).$$

But because $T_{(x,0)}(T(X)) = T_x(X) \oplus T_x(X)$, we can rewrite this condition as

$$T_x(X) \times \{0\} + \{(a, D\overrightarrow{v}_x(a)) \mid a \in T_x(X)\} = T_x(X) \oplus T_x(X).$$

This condition holds if and only if the kernel of $D\overrightarrow{v}_x$ is trivial. Thus, x is a nondegenerate zero of $\overrightarrow{v}$ if and only if $X_\sigma \pitchfork X_0$ at $(x, 0)$.

As an example, consider a vector field on the manifold $\mathbf{R}$. The tangent space of $\mathbf{R}$ at any point x is just all of $\mathbf{R}$, so a smooth vector field on $\mathbf{R}$ is just a smooth map $\overrightarrow{v} : \mathbf{R} \to \mathbf{R}$. We can graph this map in the $\mathbf{R}^2$ plane. Consider the following two vector fields:

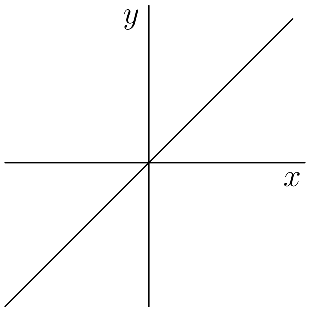

Figure 3

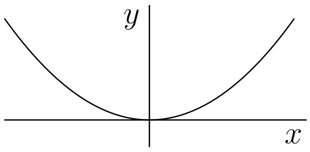

Figure 4

The zero of the first vector field is nondegenerate because $D\overrightarrow{v}_0$ can be represented by the 1×1 matrix [1], which is invertible. The zero of the second vector field is degenerate because $D\overrightarrow{v}_0$ can be represented by [0], which is not invertible. Notice the correspondence of nondegeneracy with transversality, as mentioned above.

So, a nondegenerate zero on a 1-dimensional manifold can either be a source or a sink, but it cannot be a source on one side and a sink on the other.

However, notice that we can perturb the vector field in Figure 4 just a little bit to split the degenerate zero into two nondegenerate zeroes:

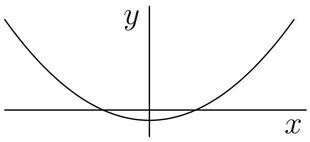

Figure 4

Now, suppose x is a nondegenerate zero of $\overrightarrow{v}$.