Intersection Theory and the Poincaré–Hopf Theorem

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1 Introduction

The Poincaré–Hopf theorem is a beautiful theorem about vector fields on smooth oriented manifolds which relates the zeros of the vector field to the Euler characteristic of the manifold. Specifically, it states that the global sum of the indices of a vector field with finitely many zeros equals the Euler characteristic of the manifold. We will define all these terms, but one takeaway is that the types of zeros allowed on a smooth vector field are determined by a topological invariant. For example, it is not possible for a smooth vector field on the sphere to have no zeros; there is always somewhere on earth where the air is completely still. Our purpose is to present this theorem and its preliminaries as clearly as possible to someone with a background in general topology and linear algebra. We will cover manifolds, transversality, orientation, and intersection theory to some extent, but for the sake of clarity and brevity we will leave some famous theorems unproved. At the end, we will briefly connect the Poincaré–Hopf theorem to the more well-known definition of the Euler Characteristic from algebraic topology.

The primary reference for this paper was *Differential Topology* by Guillemin and Pollack [2]. Significant portions of the presentation and proofs are taken from that book, reorganized and rewritten in my own words. The proof of the Poincaré–Hopf theorem is my own, for it was left as a series of exercises by Guillemin and Pollack.

2 Manifolds and Tangent Spaces

First, we must define some basic terms.

Definition 2.1. A function $f : U \to \mathbf{R}^m$, where U is open in $\mathbf{R}^n$, is called **smooth** if it has continuous partial derivatives of all orders. A function f defined on an arbitrary subset X of $\mathbf{R}^n$ is called smooth if for all $x \in X$ there is a neighborhood U of x and a smooth map $F : U \to \mathbf{R}^m$ such that F equal f on $U \cap X$.

The term "local" will be used frequently, and it is usually referring to open sets around a point. That is, if a space X locally has a property at x, then there is an open neighborhood of x which has that property. So, the above definition can be written more simply as "f is smooth if it can be locally extended to a smooth map on open sets."

Now, just as in general topology we have a homeomorphism and in linear algebra we have an isomorphism, here we define a concept with which we can view two sets as essentially equivalent.

Definition 2.2. A smooth map $X \to Y$, where $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ is called a **diffeomorphism** if f is bijective and $f^{-1} : Y \to X$ is also smooth.

Notice that a diffeomorphism is a homeomorphism where the continuity condition is replaced with smoothness. Now we are ready to define smooth manifolds.

Definition 2.3. A set $X \subset \mathbf{R}^n$ is a k-dimensional **smooth manifold** if it is locally diffeomorphic to $\mathbf{R}^k$. That is, for all $x \in X$, there exists a diffeomorphism (called a parametrization) $\phi : U \to V$, where U is open in $\mathbf{R}^k$ and V is an open neighborhood of x in X. For convenience, we may assume $\phi(0) = x$. The inverse $\phi^{-1} : V \to U$ is called a coordinate system on V.

Every manifold that we are working with is smooth, so we will often refer to them simply as "manifolds." Numerous examples of manifolds should readily come to mind, including the space $\mathbf{R}^k$ itself, along with the familiar $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$. Using only the definition, it is not very easy to prove that a given space is a manifold because you need to find a parametrization around every point on the manifold. Luckily, there are often easier ways of showing something is a manifold.

Theorem 2.4. *If X and Y are manifolds, so is $X \times Y$, and* $\dim X \times Y = \dim X + \dim Y$*.*

Proof. Let $(x, y) \in X \times Y$. Then there exist parametrizations $\phi : U \to X$ and $\psi : W \to Y$ where U and W are open sets in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, such that $\phi(0) = x$ and $\psi(0) = y$. Then the map $\phi \times \psi : U \times W \to X \times Y$ given by $(\phi \times \psi)(u, w) = (\phi(u), \psi(w))$ is a parametrization from an open subset of $\mathbf{R}^{m+n}$ to $X \times Y$ around (x, y), and the result follows. ■

If X and Z are both manifolds in $\mathbf{R}^n$ and $Z \subset X$, then Z is called a **submanifold** of X. In order to do any sort of useful analysis on manifolds, we need to introduce a concept of derivatives of maps from one manifold to another. First, recall the usual definition of a derivative. If $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is smooth, $x \in U$ and $h \in \mathbf{R}^n$, then we have

$$Df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

The derivative is a linear map represented by the Jacobian matrix

$$Df_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

The derivative has a number of useful properties. The first is the Chain Rule, which says that if $f : U \to V$ and $g : V \to \mathbf{R}^\ell$ are smooth, then for all $x \in U$,

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x.$$

Also, if L is a linear map, then $DL_x = L$ for all x in the domain of L. Finally, the derivative of a function is its best linear approximation. Let X be a smooth manifold and $\phi : U \to X$ a local parametrization around x, where U is open in $\mathbf{R}^k$. We can assume for convenience that $\phi(0) = x$. Then the best linear approximation to ϕ at 0 is the map

$$f(u) = \phi(0) + D\phi_0(u).$$

This leads us naturally to the definition of a tangent space.

Definition 2.5. The **tangent space** $T_x(X)$ of X at x is the image of the map $D\phi_0 : \mathbf{R}^k \to \mathbf{R}^n$. It is a vector subspace of $\mathbf{R}^n$ such that $x + T_x(X)$ is the best linear approximation to X through x.

To show that the tangent space is well-defined, we have to show that it does not depend on the choice of parametrization. So, suppose $\phi : U \to X$ and $\psi : V \to X$ are both parametrizations around $x \in X$, with $\phi(0) = \psi(0) = x$. Let $W = \phi(U) \cap \psi(V)$, which is a nonempty open subset of X. Define $U' = \phi^{-1}(W)$ and $V' = \psi^{-1}(W)$. This is done so that $\phi(U') = \psi(V')$. Define $h = \psi^{-1} \circ \phi : U' \to V'$. Then, by the Chain Rule, $D\phi_0 = D\psi_0 \circ Dh_0$. This implies that the image of $D\psi_0$ contains the image of $D\phi_0$. By symmetry, the image of $D\phi_0$ contains the image of $D\psi_0$ as well, so their images are the same. Therefore, the tangent space is well-defined.

Proposition 2.6. $\dim T_x(X) = \dim X$.

Proof. If $\phi : U \to V$ is a local parametrization about $x \in X$, then $\phi^{-1} : V \to U$ can be locally extended to a smooth map $\Phi' : W \to \mathbf{R}^k$, where W is open in $\mathbf{R}^N$. Then $\Phi' \circ \phi$ is the identity map on U. By the Chain Rule, $D\Phi'_x \circ D\phi_0$ is the identity map on $\mathbf{R}^k$, so $D\phi_0$ is an isomorphism from $\mathbf{R}^k$ to $T_x(X)$. Therefore, $\dim T_x(X) = k = \dim X$. ■

Now, we'll extend the notion of derivative to smooth functions between manifolds, which in general are not open subsets of $\mathbf{R}^n$. It has the characteristics one would expect from a derivative map. Namely, if $f : X \to Y$ is smooth, then Df_x is a linear map from $T_x(X)$ to $T_{f(x)}(Y)$, which is the best linear approximation to f at x. It also satisfies the Chain Rule. This new derivative is defined as follows:

Definition 2.7. Let $f : X \to Y$ be a smooth map of manifolds, with $f(x) = y$. Suppose $\phi : U \to X$ parametrizes X about x and $\psi : V \to Y$ parametrizes Y about y, where $U \subset \mathbf{R}^k$ and $V \subset \mathbf{R}^\ell$, and let $\phi(0) = x$, $\psi(0) = y$. Define $h : U \to V$ by $h = \psi^{-1} \circ f \circ \phi : U \to V$. See Figure 2.1. Then the **derivative** of f at x is defined to be $Df_x = D\psi_0 \circ Dh_0 \circ D\phi_0^{-1}$.

Chain Rule. *If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of manifolds, then*

$$D(g \circ f)_x = Dg_{f(x)} \circ Df_x.$$

The proof of the chain rule for manifolds is just an exercise in commutative diagrams.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h=\psi^{-1}\circ f\circ\phi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Figure 2.1

3 The Derivative and Local Behavior of Maps

The purpose of looking at derivatives is to describe the local behavior of a function near a point. We can often determine everything about the local behavior of a map just from its derivative.

Let $f : X \to Y$ be a smooth map of manifolds. Let $x \in X$ and $y = f(x) \in Y$. Then, if $Df_x : T_x(X) \to T_y(Y)$ is an isomorphism, f is a local diffeomorphism at x. That is, there exist open neighborhoods U and V of x and y such that $f|_U : U \to V$ is a diffeomorphism. This is known as the **Inverse Function Theorem**. When X and Y are open in $\mathbf{R}^n$, this is a famous theorem from multivariate calculus. The proof is rather involved and is based on the definition of the derivative and properties of limits; one can be found in *Calculus on Manifolds* by Michael Spivak [6]. We will not use the theorem very much directly; of greater interest are the related Local Immersion Theorem and Local Submersion Theorem.

First, we must introduce some terminology. If Df_x is injective, then f is called an **immersion** at x. If f is an immersion everywhere, then it is simply called an immersion. On the other hand, if Df_x is surjective, then f is called a **submersion** at x.

If $k \leq \ell$, define $\rho : \mathbf{R}^k \to \mathbf{R}^\ell$ by $\rho(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$. This is called the canonical immersion. The derivative of ρ at any point $x \in \mathbf{R}^k$ can be represented by the $\ell \times k$ matrix

$$D\rho_x = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix. Since the derivative is injective at every point in the domain, ρ is an immersion, as the name "canonical immersion" would suggest. One important aspect of the canonical immersion is that ρ is a diffeomorphism onto its image. There is a theorem, called the Local Immersion Theorem, which says that every immersion is locally equivalent to the canonical immersion. Here is the theorem in more precise terms.

Local Immersion Theorem. *Let X and Y be k- and ℓ-dimensional manifolds, respectively. If $f : X \to Y$ is an immersion at x and $y = f(x)$, then there exist parametrizations $\phi : U \to X$ and $\psi : V \to Y$ around x and y such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\rho"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Proof. Start by letting $\phi : U \to X$ and $\psi : V \to Y$ be parametrizations around x and y. Without loss of generality, let $\phi(0) = x$ and $\psi(0) = y$. Then define $g = \psi^{-1} \circ f \circ \phi$, resulting in the commutative diagram

By assumption, Df_x is injective, which implies $Dg_0 = D\psi_y^{-1} \circ Df_x \circ D\phi_0$ is injective as well. Therefore, if we use the appropriate basis for $\mathbf{R}^\ell$, we can assume Dg_0 is represented by the matrix $\begin{bmatrix} I_k \\ 0 \end{bmatrix}$. Now define $G : U \times \mathbf{R}^{\ell-k}$ by

$$G(x, z) = g(z) + (0, z).$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow{g} & V \end{array}$$

The result is that DG_0 can be represented by I_ℓ, so DG_0 is an isomorphism. Therefore, G is a local diffeomorphism at 0 by the Inverse Function Theorem, so the composition $\psi \circ G$ must also be a local diffeomorphism at 0. And $G \circ \rho = g$ by construction, which means $\psi \circ g = \psi \circ G \circ \rho$. Therefore, if we choose the appropriate open sets $U' \subset \mathbf{R}^k$ and $V' \subset \mathbf{R}^\ell$, we get a commutative diagram as desired. ■

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi\circ G} \\ U' & \xrightarrow{\rho} & V' \end{array}$$

Our sole purpose for immersions is to give us a criterion for when the image of a smooth map $f : X \to Y$ is a submanifold of Y. Unfortunately, being an immersion is not quite enough, we need a few more conditions. A map $f : X \to Y$ is called *proper* if the preimage of any compact subset of Y is a compact subset of X. Now, we define an **embedding** as a proper, injective, immersion. It turns out this is just the condition we need in order for the image of X to be a submanifold of Y.

Theorem 3.1. *An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.*

Proof. Let $f : X \to Y$ be an embedding. Let $x \in X$ and let W be a neighborhood of x such that, with the right choice of coordinates as provided by the Local Immersion Theorem, f appears to be the canonical immersion on W. Then $f|_W$ is a diffeomorphism onto its image $f(W)$. Thus, $f(W)$ can be parametrized by $f \circ \phi$, where $\phi : U \to W$ is a parametrization of W. To show $F(X)$ is a manifold, we only need to show that $f(W)$ is open in $f(X)$, for then every point in $f(X)$ will have a parametrizable neighborhood.

Suppose for a contradiction that $f(W)$ is not open in $f(X)$. Then $f(X) - f(W)$ is not closed, so it does not contain all its limit points. Thus there is a sequence y_i in $f(X) - f(W)$ that converges to a point y in $f(W)$. The set $\{y, y_i\}$ is compact, so its preimage must be compact because f is proper. Since f is injective, y must have exactly one preimage point, which we can call x, which is in W. And each y_i must have exactly one preimage point, which we can call x_i. The set $\{x, x_i\}$ is closed and bounded, so by the Bolzano-Weierstrass Theorem, x_i has a convergent subsequence x_{i_n}. Let z be the limit of x_{i_n}. Since f is continuous, $f(x_{i_n})$ converges to $f(z)$, and it also converges to $f(x)$. Since f is injective, this implies $z = x$. Therefore, x_{i_n} converges to $z = x$, which is in the open set W. So, there exists some x_i that is in W, so some y_i is in $f(W)$, a contradiction. In conclusion, $f(W)$ is open in $f(X)$, so $f(X)$ is a submanifold of Y.

We also claimed that $f : X \to f(X)$ is a diffeomorphism. This immediately follows because f is a local diffeomorphism at every point, and it is also bijective onto its image. ■

We will now switch our focus to submersions, which are maps whose derivatives are surjective. Just as immersions let us know when the image of a map is a manifold, submersions will let us know when the preimage of a point is a manifold. If $k \geq \ell$, define $\rho : \mathbf{R}^k \to \mathbf{R}^\ell$ by $\rho(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$. This is called the canonical submersion. The derivative of ρ at any point $x \in \mathbf{R}^k$ can be represented by the $\ell \times k$ matrix

$$D\rho_x = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \end{bmatrix},$$

where I_ℓ is the $\ell \times \ell$ identity matrix. Since the derivative is surjective at every point in the domain, ρ is a submersion, as the name "canonical submersion" would suggest. Just like with immersions, it is a fact that every submersion is locally equivalent to the canonical submersion. This is the Local Submersion Theorem:

Local Submersion Theorem. *Let X and Y be k- and ℓ- dimensional manifolds, respectively. If $f : X \to Y$ is a submersion at x and $y = f(x)$, then there exist parametrizations $\phi : U \to X$ and $\psi : V \to Y$ around x and y such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\rho"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Proof. The proof is effectively the same as the proof of the Local Immersion Theorem. This time, if $g = \psi^{-1} \circ f \circ \phi$ as before, we define $G : U \to \mathbf{R}^k$ by $G(x) = (g(x), x_{\ell+1}, \ldots, x_k)$, where $x = (x_1, \ldots, x_k)$. By the same logic as before, DG_0 is an isomorphism, so G is a local diffeomorphism at 0, so G^{-1} can be taken to be a diffeomorphism from an open neighborhood U' around 0 into U. This gives the commutative diagram ■

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi \circ G^{-1}"] \arrow[r, "\rho"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Before exploring what makes the Local Submersion Theorem useful, we will introduce some new terminology. For a smooth map of manifolds $f : X \to Y$, a point $x \in X$ is called a *regular point* if f is a submersion at x. It is called a *critical point* if f is not a submersion at x. If $y \in Y$ is the image of a critical point, it is called a *critical value.* If $y \in Y$ is not the image of a critical point, it is called a *regular value.*

To illustrate these definitions, consider the polynomial map $f(x) = x^4 - 2x^2 : \mathbf{R} \to \mathbf{R}$.

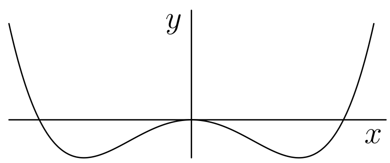

Figure 3.1

For all $x \in \mathbf{R}$, the derivative of f is represented by the 1×1 matrix $Df_x = [4x(x^2 - 1)]$. So, the derivative is surjective if and only if it is non-zero. Thus, the critical points of f are the locations where the derivative is zero: $\{-1, 0, 1\}$. The regular points are $\mathbf{R} - \{-1, 0, 1\}$. The set of critical values is the image of the set of critical points, which is $\{0, -1\}$. Finally, the regular points of f are $\mathbf{R} - \{0, -1\}$. Notice that, by definition, if $y \in Y$ is not in the image of f, then y is a regular value of f.

Preimage Theorem. *If y is a regular value of $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of X with* $\dim f^{-1}(y) = \dim X - \dim Y$.

Proof. Let $x \in f^{-1}(y)$. Then, by definition, $f(x) = y$ and Df_x is surjective. By the Local Submersion Theorem, we can choose local coordinates around x and y such that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$ and y corresponds to $(0, \ldots, 0)$. Then $f^{-1}(y)$ is locally the set of points $(0, \ldots, 0, x_{\ell+1}, \ldots, x_k)$. Since every point in $f^{-1}(y)$ has an open parametrizable neighborhood, $f^{-1}(y)$ is a submanifold of X. And it has the $k - \ell$ coordinate functions $(x_{\ell+1}, \ldots, x_k)$, so it is a $(k - \ell)$-dimensional manifold. ■

With Theorem 3.4, we now have a very simple method for showing that certain objects are smooth manifolds. For example, if we define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(x_1, \ldots, x_{k+1}) = x_1^2 + \cdots + x_{k+1}^2$, then we have

$$Df_x = \begin{bmatrix} 2x_1 & \ldots & 2x_{k+1} \end{bmatrix},$$

which is surjective whenever $|x| > 0$. Thus, 1 is a regular value of f, which implies $S^k = f^{-1}(1)$ is a smooth manifold.

If Z is a submanifold of X, then we define the codimension ("complementary dimension") of Z to be

$$\text{codim}(Z) = \dim X - \dim Z.$$

With this terminology, we can rephrase the statement $\dim f^{-1}(y) = \dim X - \dim Y$ as $\text{codim}(f^{-1}(y)) = \dim Y$.

If $g_1 \dots g_\ell$ are smooth functions from X to $\mathbf{R}$, then we say they are *independent* at x if the set $\{D(g_i)_x \mid 1 \leq i \leq \ell\}$ is a set of linearly independent row vectors. By Theorem 3.4, if $g : X \to \mathbf{R}^\ell$ is smooth and 0 is a regular value of g, then $g^{-1}(0)$ is a manifold. Equivalently, $g^{-1}(0)$ is a manifold if Dg_x has linearly independent rows for all $x \in g^{-1}(0)$. If we write $g = (g_1, \dots, g_\ell)$, then it follows that the set of Z common zeros of $g_1, \dots, g_\ell$ is a smooth manifold if $g_1 \dots g_\ell$ are independent at all $x \in Z$. Also, $\text{codim}(Z) = \ell$. This brings us to the final theorem in this section, which is a direct consequence of the Local Immersion Theorem using our new terminology.

Theorem 3.2. *If Z is a submanifold of X with codim$(Z) = \ell$, then Z can locally be written as the common zero set of ℓ independent smooth functions.*

Proof. Let $i : Z \to X$ be the inclusion map, which is an immersion. Then by the Local Immersion Theorem, for all $z \in Z$, there exists local coordinates on an open neighborhood W of z in X such that

$$i(z_1, \dots, z_{k-\ell}) = (z_1, \dots, z_{k-\ell}, 0, \dots, 0).$$

Thus Z is the zero set of the ℓ independent smooth coordinate functions $z_{k-\ell+1}, \dots, z_k$. ■

4 Transversality

Now we can build off the preliminary notions of smooth maps, submersions, and immersions to define what will be a focal point of the remainder of this paper. We will be concerned with the intersection of manifolds, but the intersection of two manifolds can, in general, be very pathological, so we will usually impose a useful condition on their intersection, which we call transversality.

Definition 4.1. Let X and Z be submanifolds of a manifold Y. We say X and Z are **transverse**, or intersect transversely, if

$$T_y(X) + T_y(Z) = T_y(Y)$$

for all $y \in X \cap Z$. This relation is denoted $X \pitchfork Y$.

Since $T_y(X)$ and $T_y(Z)$ are both vector subspaces of $T_y(Y)$, their sum is the set

$$T_y(X) + T_y(Z) = \{x + z \mid x \in T_y(X), z \in T_y(Z)\}.$$

Notice that if X and Z do not intersect at all, then they intersect transversely. A sufficient condition for transversality is one of the manifolds X or Z to have the same dimension as Y. So, transversality is only interesting if Y has greater dimension than both X and Z.

Transversality is actually a very intuitive property most of the time. In theory, one has to calculate both $T_y(X)$ and $T_y(Z)$ at all points in the intersection to see if they span $T_y(Y)$, but for the examples we are working with, it's enough to look at a picture. Here are a couple simple examples:

Example 4.2. Let $Y = \mathbf{R}^2$, X be the x-axis, and Z be the graph of the function x^3. These only intersect at $(0, 0)$, and their tangent spaces at that point are the x-axis. Therefore, X and Z are not transverse in this case.

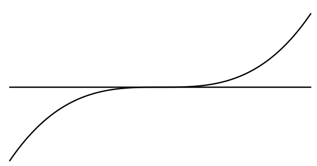

Figure 4.1

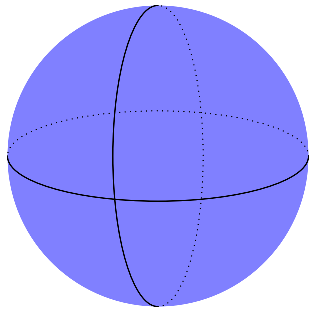

Figure 4.2

Example 4.3. Let Y be the sphere S^2, and let X and Z be great circles as pictured in Figure 4.2. Then $X \pitchfork Z$.

The above definition of transverse is the geometrically intuitive one, but there is another closely related definition whereby a *function* can be transverse to a submanifold Z of Y. Here it is:

Definition 4.4. Let $f : X \to Y$ be smooth and let Z be a submanifold of Y. We say f is transversal to Z, denoted $f \pitchfork Z$, if

$$\text{Image}\,(Df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

for all $x \in f^{-1}(Z)$.

This definition is equivalent to the first if X is a submanifold of Y and $f : X \to Y$ is the inclusion map. However, this version allows us to extend the Preimage Theorem to preimages of a set Z instead of preimages of a single point y.

Theorem 4.5. *If the smooth map $f : X \to Y$ is transversal to a submanifold Z of Y, then the preimage $f^{-1}(Z)$ is a submanifold of X, and codim$(f^{-1}(Z)) =$ codim(Z).*

Proof. Let $x \in f^{-1}(Z)$ and $y = f(x)$. By Theorem 3.2, there is a neighborhood V of y in Y and a set of independent functions $g_1, \ldots, g_\ell$ from V to $\mathbf{R}$ such that $V \cap Z$ is the common zero set $g = (g_1, \ldots, g_\ell)$, where ℓ is the codimension of Z in Y. Then, on a neighborhood U of x, $U \cap f^{-1}(Z)$ is the zero set of $g \circ f : U \to \mathbf{R}^\ell$. Using Theorem 3.4, we know that $(g \circ f)^{-1}(0)$ is a manifold if 0 is a regular value of g, which means that for all $x \in (g \circ f)^{-1}(0) = f^{-1}(Z)$, the derivative $D(g \circ f)_x = Dg_y \circ Df_x$ is surjective.

We already know Dg_y is surjective by the definition of independent functions, and the kernel of Dg_y certainly contains $T_y(Z)$. By the rank-nullity theorem, the kernel has codimension ℓ in $T_y(Y)$, so the kernel is $T_y(Z)$. Thus, $D(g \circ f)_x = Dg_y \circ Df_x$ is surjective if and only if Dg_y maps the image of Df_x onto $\mathbf{R}^\ell$, which occurs if and only if the image of Df_x and the kernel of Dg_y together span $T_y(Y)$. In other words, $f^{-1}(Z)$ is a manifold if

$$\text{Image}\,(Df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

for all $x \in f^{-1}(Z)$, as desired. The statement about the codimension of $f^{-1}(Z)$ simply follows from the local equation $(g \circ f)^{-1}(0) = f^{-1}(Z)$ and Theorem 3.4. ∎

Now, if X and Z are submanifolds of Y that intersect transversely, then we can apply the above theorem to the inclusion map $i : X \to Y$ to show that the intersection $X \cap Z$ is a submanifold of Y with codim$(X \cap Z) =$ codim$(X) +$ codim(Z). It also should be noted that transversality is a sufficient condition for $X \cap Z$ to be a manifold, but it is not a necessary one (see example 4.2).

If $f : X \to Y$ has an empty preimage $f^{-1}(Z)$, then $f \pitchfork Z$ trivially. But Theorem 4.5 is still satisfied because the empty set is a manifold of every dimension. The statement "every point x in $\emptyset$ has a neighborhood that is diffeomorphic to $\mathbf{R}^k$" is vacuously true for all k.

5 Manifolds with Boundary, Sard's Theorem, and Homotopy

This section will be a brief overview of some famous results that are used in future proofs, but are burdensome to prove themselves. A thorough treatment can be found in Guillemin and Pollack [2].

Some topological spaces, like the closed disk $D = \{x \in \mathbf{R}^2 \mid |x| \leq 1\}$, are almost manifolds but not quite, because there is no parametrization for points on the boundary of the disk. For this reason, we define manifolds with boundary.

Definition 5.1. Let H^k be the k-dimensional "upper half space" given by $H^k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \mid x_k \geq 0\}$. A set $X \subset \mathbf{R}^N$ is a k-dimensional **manifold with boundary** if it is locally diffeomorphic to H^k.

The boundary of H^k is the subset $\partial H^k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \mid x_k = 0\}$. The boundary of a manifold with boundary X, denoted ∂X, is the set of points in X which are in the image of ∂H^k under some parametrization. In fact, if $x \in X$ is in the image of ∂H^k under one local coordinate system, then the same must be true for any other local coordinate system. This implies that ∂X is a manifold of dimension $k-1$.

Since open sets in $\mathbf{R}^k$ are diffeomorphic to open sets in the interior of H^k, every smooth manifold is a manifold with boundary, technically speaking. For the manifolds we have looked at before, the boundary has been empty.

If X is a manifold with boundary and x is in the interior of X (that is, $x \in X - \partial X$), then x can be parametrized by a function $\phi : U \to X$, where U is open in $\mathbf{R}^k$, so the tangent space is defined in the same way as before. If $x \in \partial X$, then x can be parametrized by a function $\phi : U \to X$, where U is open in H^k. By the definition of smooth functions, ϕ extends to a smooth function $\phi' : U' \to X$, where U' is open in $\mathbf{R}^k$. The tangent space at x is defined to be the image of $D\phi'_0 : \mathbf{R}^k \to \mathbf{R}^n$. This tangent space does not depend on the choice of extension ϕ'.

For our purposes, we will mostly be concerned with one specific manifold with boundary: the closed interval $I = [0, 1] \subset \mathbf{R}$. This is because I is used in the next section in defining a homotopy space $X \times I$, where X is a boundaryless manifold.

Proposition 5.2. If X is a k-dimensional boundaryless manifold, then $X \times I$ is a $(k+1)$-dimensional manifold with boundary, and $\partial(X \times I) = X_0 \cup X_1$, where $X_0 = X \times \{0\}$ and $X_1 = X \times \{1\}$.

Proof. Let $(x, y) \in X \times I$. Then there are parametrizations $\phi : U \to X$ and $\psi : V \to I$ around x and y, respectively, where U is open in $\mathbf{R}^k$ and V is open in H^1. Then $\phi \times \psi$, defined by $(\phi \times \psi)(u, v) = (\phi(u), \psi(v))$, is a parametrization around (x, y), and the domain $U \times V$ is open in H^{k+1}. And if (x, y) is the image of a point in the boundary of H^{k+1}, then y must be the image of a point in the boundary of H^{k+1}, so $y = 0$ or $y = 1$. ■

If f is a smooth function defined on a manifold-with-boundary X, then ∂f is defined to be the restriction of f to the boundary of X. We want a condition on f whereby $f^{-1}(Z)$ is a manifold with boundary $\partial\{f^{-1}(Z)\} = f^{-1}(Z) \cap \partial X$, but $f \pitchfork Z$ is not a sufficient condition; we also need $\partial f \pitchfork Z$.

Theorem 5.3. *Let f be a smooth map of a manifold X with boundary onto a boundaryless manifold Y, and suppose that both $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal with respect to a boundaryless submanifold Z in Y. Then the preimage $f^{-1}(Z)$ is a manifold with boundary, and*

$$\partial\{f^{-1}(Z)\} = f^{-1}(Z) \cap \partial X$$

The other topic to briefly summarize is Sard's Theorem. It is a famous theorem from analysis that can be extended to smooth manifolds. The full statement of Sard's Theorem requires measure theory, which we will not define here. We will only need a weaker version:

Sard's Theorem. *The regular values of any smooth map $f : X \to Y$ are dense in Y, meaning every open subset of Y contains a regular value of f.*

The full version of this theorem states that the critical values of f have measure zero, but it is not worth defining "measure zero" here. The theorem was proven in 1942 by Arthur Sard in his article "The measure of the critical values of differentiable maps" [5]. Recall that smooth maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are called homotopic, denoted $f_0 \sim f_1$, if there exists a smooth map $H : X \times I \to Y$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. This is the ordinary definition of homotopy from general topology except with "continuous" replaced by "smooth." The function H is called a homotopy.

Just like in general topology, the homotopy relation $\sim$ is an equivalence relation on the set of functions from X to Y. Often, instead of writing a homotopy as a function of two variables $H(x, t)$, we will write it as a collection of functions of one variable: $f_t(x) := H(x, t)$. These notions are equivalent, but the notation $f_t(x)$ is preferable if t is fixed.

A consequence of Sard's Theorem is the Transversality Theorem, which says that for any smooth map $f : X \to Y$ and submanifold Z in Y, we can deform f by an arbitrarily small amount to make it transversal to Z.

Finally, we will state but not prove the following two theorems.

Extension Theorem. *Suppose that Z is a closed submanifold of Y, both boundaryless, and C is a closed subset of X. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then there exists a smooth map $g : X \to Y$ homotopic to f, such that $g \pitchfork Z$, $\partial g \pitchfork Z$, and on a neighborhood of C we have $g = f$.*

Tubular Neighborhood Theorem. *Let Z be a submanifold of Y, where $Y \subset \mathbf{R}^m$. The **normal bundle** to Z in Y is defined as the set*

$$N(Z;Y) = \{(z,v) \mid z \in Z, v \in T_z(Y) \text{ and } v \perp T_z(Z)\}.$$

It is a smooth manifold with the same dimension as Y. Then there exists a diffeomorphism from an open neighborhood of Z in $N(Z;Y)$ onto an open neighborhood of Z in Y.

6 Orientation

Let V be a finite-dimensional real vector space. Let V be a finite-dimensional real vector space. If β_1 and β_2 are bases of V, then there exists a unique linear isomorphism $A : V \to V$ such that $A\beta_1 = \beta_2$. We say β_1 and β_2 are **equivalently oriented** if $\det A > 0$. An orientation of V is an assignment of $+1$ to one equivalence class and -1 to the other. Thus every basis is either positively or negatively oriented.

This definition of orientation is perfectly natural. Recall from linear algebra that the determinant of an $n \times n$ matrix can be interpreted as an n-volume scaling factor. For instance, the determinant of a 2×2 matrix is equal to the area of the parallelogram spanned by the images of the standard basis vectors as shown in Figure 7.1. If the determinant is negative, then the image has been flipped. We will give $\mathbf{R}^k$ the standard orientation, which is the assignment of positive orientation to the standard ordered basis. On $\mathbf{R}^2$, the standard orientation is counterclockwise: a basis is positively oriented if v_2 is counterclockwise from v_1 by less than 180 degrees. On $\mathbf{R}^3$, the standard orientation is given by the right-hand rule.

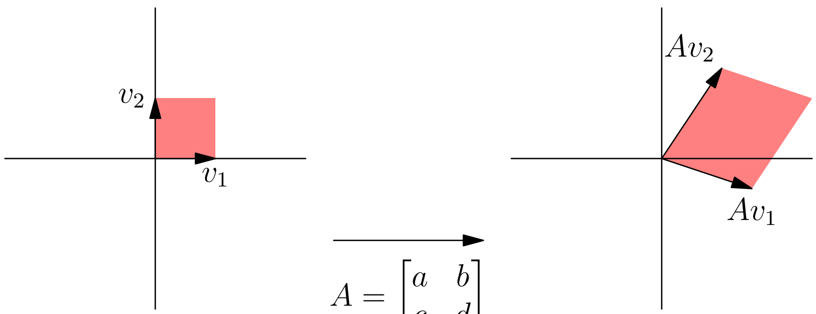

Figure 7.1

If V is a zero-dimensional real vector space, then an orientation of V is just a choice of either $+1$ or -1. A linear isomorphism A between oriented vector spaces V and W can either preserve or reverse orientation. Notably, if A is a linear isomorphism of V with itself, then A is orientation-preserving exactly when its determinant is positive.

As usual, we now wish to extend this definition from $\mathbf{R}^k$ to smooth manifolds using local parametrizations. If X is a k-dimensional manifold with boundary, then we can orient X by assigning an orientation for each tangent space $T_x(X)$. The orientation on X must be smooth, meaning that for all $x \in X$, there must exist a parametrization $\phi : U \to X$, where $\phi(0) = x$ and U is open in H^k, such that $D\phi_u$ is orientation-preserving for all $u \in U$. In other words, there are no sudden jumps in an orientation on X. Consequently, not every manifold with boundary is orientable, the most famous example being the Möbius loop.

Because of the smoothness restriction on orientation, assigning an orientation to a single tangent space $T_x(X)$ will induce an orientation on the whole manifold as long as X is connected. This implies that an orientable, connected manifold with boundary has exactly two orientations. Take, for example, the sphere S^2. Since $\mathbf{R}^2$ has two orientations, which we can call "counterclockwise" and "clockwise," there exist two distinct orienations on S^2.

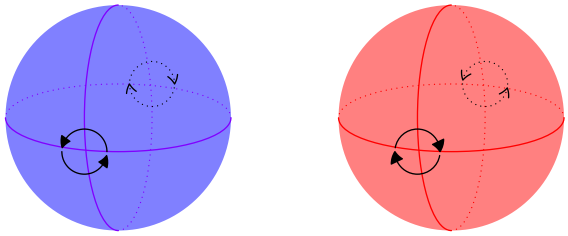

Figure 7.2

If X is oriented, then $-X$ denotes the same manifold with the opposite orientation. It will often be desirable to have orientations on different manifolds "agree" in some way. For instance, when does the orientation of ∂X agree with the orientation of X? The rest of this section will be devoted to four instances of this idea: the boundary orientation, the product orientation, the direct sum orientation, and, most importantly, the preimage orientation.

If X is an oriented manifold with boundary, then for $x \in \partial X$, there are three types of tangent vectors. There are vectors tangent to the boundary, which form a $(k-1)$-dimensional subspace $T_x(\partial X)$ of $T_x(X)$, there are vectors pointing outward from the manifold, and there are vectors pointing inward to the manifold. Let n_x be the unit vector in $T_x(X)$ which is normal to $T_x(\partial X)$ and points outward from the manifold. Then we assign the *boundary orientation* to ∂X by saying that a basis $\{v_1, \ldots, v_{k-1}\}$ of $T_x(\partial X)$ is positively oriented if and only if $\{n_x, v_1, \ldots, v_{k-1}\}$ is a positively oriented basis for $T_x(X)$. For example, referring back to Figure 7.2, notice that the sphere S^2 is the boundary of the closed unit ball $D^3 \subset \mathbf{R}^3$. If we give D^3 the standard orientation as a submanifold of $\mathbf{R}^3$, then the boundary orientation on S^2 is the one drawn in blue. As shown in Figure 7.3 below, the basis $\{n_x, v_1, v_2\}$ is positively oriented in $\mathbf{R}^3$, so $\{v_1, v_2\}$ is positively oriented on $T_x(S^2)$; the counterclockwise orientation. This is summed up in the following definition.

Definition 6.1. We orient $T_x(\partial X)$ by declaring the sign of any ordered basis $\beta = \{v_1, \ldots, v_{k-1})$ to be the sign of the ordered basis $\{n_x, \beta\} = \{n_x, v_1, \ldots, v_{k-1}\}$ for $T_x(X)$, where n_x is the outward unit normal at x.

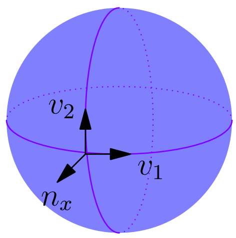

Figure 7.3

The following is a very useful consequence of the definition of boundary orientation:

Proposition 6.2. The sum of the orientation numbers at the boundary points of any compact oriented one-dimensional manifold with boundary is zero.

Proof. Note that the boundary of a one-dimensional manifold is an oriented zero-dimensional manifold. Here, "orientation number" simply refers to the number $+1$ or -1 which is a assigned to every point in the boundary. Every compact, connected one-dimensional manifold with boundary is diffeomorphic either to the unit interval or the circle. We will leave this fact unproved. The boundary of the circle is empty, so the sum of the orientation numbers at the boundary points is zero. Similarly, the boundary of the unit interval consists of two points, one with orientation $+1$ and the other with orientation -1. Again, the sum is zero. Since every compact oriented one-dimensional manifold can be written as the disjoint union of circles and intervals, the result follows. ■

Next is the product orientation, which is very simple. If V and W are finite-dimensional real vector spaces with bases $\beta_1 = \{v_1, \ldots, v_k\}$ and $\beta_2 = \{w_1, \ldots, w_\ell\}$, respectively, then let $(\beta_1 \times 0, 0 \times \beta_2)$ be the ordered basis

$$\{(v_1,0),\ldots,(v_k,0),(0,w_1),\ldots,(0,w_\ell)\}$$

for $V \times W$. Using this notation, we can define product orientation.

Definition 6.3. If X and Y are oriented and one of them is boundaryless, then the **product orientation** on $T_x(X) \times T_y(Y)$ is given by declaring

$$\operatorname{sign}(\beta_1 \times 0, 0 \times \beta_2) = \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2)$$

for any ordered bases β_1 and β_2 of $T_x(X)$ and $T_y(Y)$, respectively.

Combining these definitions, we can now orient the homotopy space $I \times X$, where I is the unit interval $[0,1]$ with standard orientation and X is an oriented manifold. We can draw the homotopy space in three dimensions, pretending for simplicity that X is a circle.

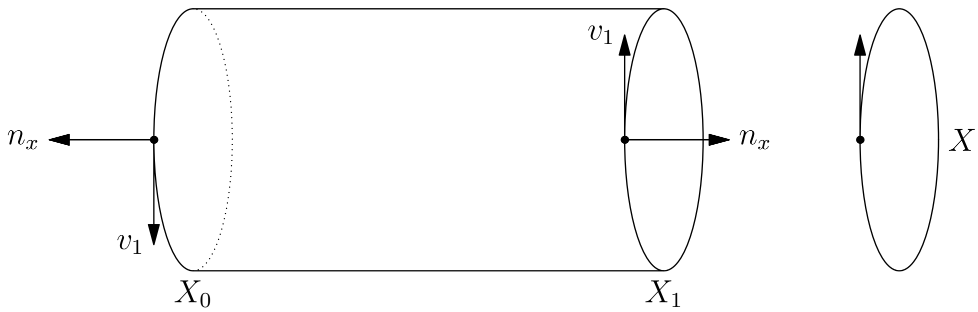

Figure 7.4

Recall from Proposition 5.2 that $\partial(I \times X) = X_1 \cup X_0$. As pictured in Figure 7.4, it appears that, as an oriented manifold,

$$\partial(I \times X) = X_1 \cup (-X_0) = X_1 - X_0,$$

and indeed this will always be the case. For if β is a positively oriented basis for $T_x(X)$ and $(1,x) \in X_1$, then $0 \times \beta$ is a positively oriented basis for $T_{(1,x)}(X_1)$. Likewise, $0 \times \beta$ is a negatively oriented basis for $T_{(0,x)}(X_0)$.

We can also define orientation for a direct sum of vector spaces. If $V = V_1 \oplus V_2$ and β_1 and β_2 are bases for V_1 and V_2, respectively, then $\beta = (\beta_1, \beta_2)$ is a basis for V. If any two of these spaces are oriented, we can give the direct product orientation on the third by requiring $\operatorname{sign}(\beta) = \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2)$. This is summarized in the following definition.

Definition 6.4. Direct sum orientation on $V = V_1 \oplus V_2$ is defined by

$$\operatorname{sign}(\beta) = \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2),$$

where $\beta = (\beta_1, \beta_2)$.

Finally, we must define preimage orientation. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, and Z are oriented manifolds and Y and Z are boundaryless. Then the manifold with boundary $S = f^{-1}(Z)$ gets a preimage orientation as follows.

Let $x \in S$, with $f(x) = z$. In order to orient S, we just have to specify an orientation on $T_x(S)$. The derivative map $Df_x : T_x(X) \to T_z(Y)$ is the best linear approximation to f at x, and $T_x(S)$ is the preimage of $T_z(Z)$ under Df_x. Let $N_x(S;X)$ be the orthogonal complement to $T_x(S)$ in $T_x(X)$. Then, by definition of orthogonal complement,

$$N_x(S;X) \oplus T_x(S) = T_x(X),$$

We already know the orientation of $T_x(X)$, so we just have to choose an orientation on $N_x(S;X)$. From there, the orientation on $T_x(S)$ will be determined by the direct product orientation. By the transversality condition, we have

$$Df_x T_x(X) + T_z(Z) = T_z(Y).$$

Since $T_x(S)$ is the preimage of $T_z(Z)$, we get a direct sum

$$Df_x N_x(S;X) \oplus T_z(Z) = T_z(Y).$$

We already know the orientations of $T_z(Z)$ and $T_y(Y)$, so $Df_x N_x(S;X)$ gets a direct product orientation. Since Df_x maps $N_x(S;X)$ isomorphically onto its image, the orientation on $Df_x N_x(S;X)$ defines an orientation on $N_x(S;X)$. This is summarized in the following definition.

Definition 6.5. Let $S = f^{-1}(Z)$ and $N_x(S;X)$ be the orthogonal complement to $T_x(S)$ in $T_x(X)$. Then the **preimage orientation** on $T_x(S)$ is given by the equations

$$Df_x N_x(S;X) \oplus T_z(Z) = T_z(Y),$$

$$N_x(S;X) \oplus T_x(S) = T_x(X).$$

This definition is a little bit convoluted and abstract, so let us look at an example. Let X be the closed unit disk embedded in $\mathbf{R}^2$ with the standard orientation. Let $Y = \mathbf{R}^3$ with standard orientation, and let Z be the y-z plane in $\mathbf{R}^3$ with standard orientation. That is, $\{(0,1,0),(0,0,1)\}$ is a positively oriented basis for $T_z(Z)$, for all z. Let $f : X \to Y$ be the inclusion map which sets X in the x-y plane. Then $S = f^{-1}(Z)$ is a diameter of the circle. Figure 7.5 shows how S is oriented at x according to the preimage orientation. Arrows drawn on tangent spaces indicate the positive orientation on those spaces.

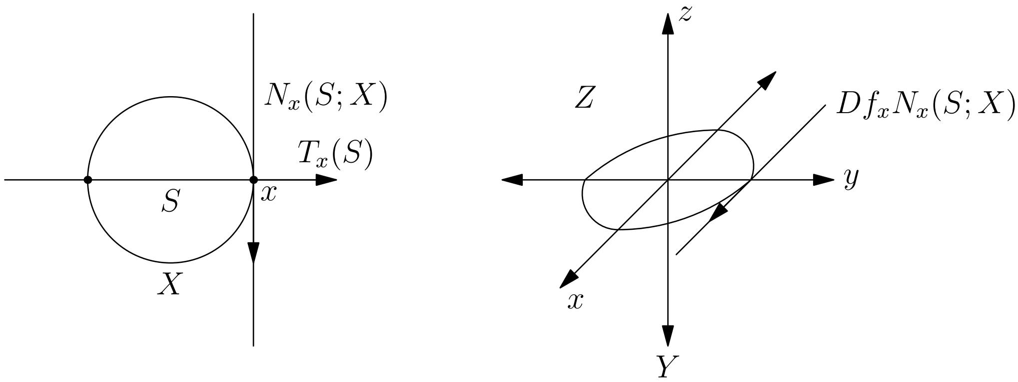

Figure 7.5

7 Oriented Intersection Number

Now we have the technology to talk about oriented intersection theory. If X, Y, and Z are boundaryless oriented manifolds, X is compact, Z is a closed submanifold of Y, and $\dim X + \dim Z = \dim Y$, then we say $f : X \to Y$ and Z are appropriate for intersection theory. The reason for these conditions will be made clear once we define the oriented intersection number of a function and a manifold:

Definition 7.1. Let $f : X \to Y$ and Z be appropriate for intersection theory. If f is transversal to Z, then the **intersection number** of f with Z $I(f,Z)$ is defined to be the sum of the orientation numbers in $f^{-1}(Z)$.

Notice that, since $\dim X + \dim Z = \dim Y$, we have $\operatorname{codim}(Z) = \dim X$. Thus, the codimension of $f^{-1}(Z)$ in X is equal to the dimension of X, which means $f^{-1}(Z)$ is a 0-dimensional manifold. Since Z is closed and X is compact, $f^{-1}(Z)$ is compact, so it is a finite number of points. Thus, we can take the sum of the orientation numbers at these points.

In the case where $\dim f^{-1}(Z) = 0$, the preimage orientation from Definition 7.6 becomes much simpler because $N(S;X) = T_x(X)$ as sets, but they might not have the same orientation. The equation

$$N_x(S;X) \oplus T_x(S) = T_x(X)$$

simply means that $T_x(S)$ has orientation $+1$ if the orientation on $N_x(S;X)$ is the same as $T_x(X)$, and it has orientation -1 otherwise. From the second equation in Definition 6.5:

$$Df_x N_x(S;X) \oplus T_z(Z) = T_z(Y),$$

we conclude that if $x \in f^{-1}(Z)$, then the contribution of x to $I(f,Z)$ is $+1$ if the direct sum orientation on $Df_x T_x(X) \oplus T_z(Z)$ is the same as the given orientation on $T_z(Y)$ and -1 otherwise.

Here comes the anticipated proposition which was the sole purpose of introducing manifolds with boundary.

Proposition 7.2. If $X = \partial W$, where W is compact, and $f : X \to Y$ extends to W, then $I(f, Z) = 0$.

Proof. This is a consequence of the Extension Theorem, which we stated but did not prove at the end of Section 5. Let $F : W \to Y$ be an extension of f, and apply the Extension theorem to F, with $C = X$. Since $F \pitchfork Z$ on X and $\partial F \pitchfork Z$ on $C \cap \partial W$, there exists a smooth map $F' : W \to Y$ homotopic to F such that $F \pitchfork Z$, $\partial F' \pitchfork Z$, and in a neighborhood of C we have $F' = F$. In other words, F' is also an extension of f which is transverse to Z.

So, we can assume without loss of generality that the extension F of f is transverse to Z, which means $F^{-1}(Z)$ is a compact oriented one-dimensional manifold with boundary, and $\partial F^{-1}(Z) = f^{-1}(Z)$. By Proposition 6.2, the sum of the orientation numbers at points in $f^{-1}(Z)$ must be zero. ■

Corollary 7.3. Homotopic maps always have the same intersection numbers.

Proof. Recall that if X is appropriate for intersection theory, then it is compact. Thus the homotopy space $I \times X$ is compact. Let f_0 and f_1 be homotopic and both transversal to Z. If $F : I \times X \to Y$ is a homotopy between them, then $I(\partial F, Z) = 0$ by Proposition 7.2. Since $\partial(I \times X) = X_1 - X_0$, it follows that

$$\partial F^{-1}(Z) = f_1^{-1}(Z) - f_0^{-1}(Z).$$

Thus $I(f_1, Z) - I(f_0, Z) = I(\partial F, Z) = 0$, so $I(f_1, Z) = I(f_0, Z)$, as desired. ■

The homotopy invariance of intersection numbers is absolutely crucial, for it allows us to define $I(f, Z)$ even when f is not transversal to Z. Simply pick a function f' which is homotopic to f and transversal to Z, and define $I(f, Z) = I(f', Z)$. Homotopy invariance guarantees that this is well-defined.

Using orientation numbers, we can define something called the *degree* of a smooth function.

Definition 7.4. Let Y be a connected manifold of the same dimension as X. The **degree** of a smooth function $f : X \to Y$ is the intersection number of f with any point y,

$$\deg(f) = I(f, \{y\}).$$

To motivate this definition a little, consider maps from the unit circle to itself. If we consider S^1 as a subset of the complex plane, we can define a family of maps

$$f_n(z) = z^n : S^1 \to S^1,$$

where $n \geq 0$. We can calculate the degree of f_n by picking a regular value y and counting the number of preimage points, where each preimage point x contributes $+1$ if Df_x preserves orientation, and -1 otherwise. Thus, we have $\deg(f_n) = n$. Since degree is an intersection number, which is a homotopy invariant, it follows that there are infinitely many homotopically distinct maps from S^1 to itself. We have not yet shown that degree is well defined, but this will follow shortly.

If X is also a submanifold of Y, then we define $I(X, Z)$ to be the intersection number of the inclusion map of X with Z. We can also extend the notion of intersection number to two functions in a natural way. If X and Z are compact, boundaryless oriented manifolds (Z is no longer a submanifold of Y), $f : X \to Y$ and $g : Z \to Y$ are smooth, and $\dim X + \dim Z = \dim Y$, we say $f \pitchfork g$ if $Df_x T_x(X) + Dg_z T_z(Z) = T_y(Y)$ whenever $f(x) = y = g(z)$. The local intersection number at (x, z) is defined to be $+1$ if the orientations on $Df_x T_x(X)$ and $Dg_z T_z(Z)$ add up to the orientation on $T_y(Y)$, in that order, and -1 otherwise. The intersection number $I(f, g)$ is the sum of the intersection numbers from all pairs (x, z) satisfying $f(x) = g(z)$.

The purpose of defining the intersection number of two functions is so that we can perturb Z without changing the intersection number. If Z is a submanifold of Y and $g : Z \to Y$ is the inclusion map, then by construction $I(f, g) = I(f, Z)$. We can homotopically vary g without changing the intersection number.

The fact that degree is well-defined follows from the requirement that Y is connected. For if y_0 and y_1 are in Y, and i_0 and i_1 are their respective inclusion maps, then i_0 is homotopic to i_1. Thus $I(f, \{y_0\}) = I(f, \{y_1\})$.

We can now define the Differential Topology version of the Euler characteristic, which is a famous topological invariant.

Definition 7.5. If X is a compact, oriented manifold, the **Euler characteristic** of X, denoted $\chi(X)$, is defined to be the self-intersection number of the diagonal $\Delta = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$:

$$\chi(X) = I(\Delta, \Delta)$$

Part of the significance of the Euler Characteristic is made clear in the next section.

8 Vector Fields and the Poincaré–Hopf Theorem

With the preliminaries out of the way, we can finally state and prove the Poincaré–Hopf Theorem. But first, a few more definitions are in order. A **vector field** on a manifold X in $\mathbf{R}^n$ is a smooth map $\overrightarrow{v} : X \to \mathbf{R}^N$ such that $\overrightarrow{v}(x) \in T_x(X)$ for every x. The most interesting points of a vector fields are the zeros, which are points $x \in X$ such that $\overrightarrow{v}(x) = 0$. When X is compact, certain types of vector fields are not possible. For example, on the sphere S^2, there does not exist a smooth vector field without any zeros. The Poincaré–Hopf Theorem explains precisely why this is the case.

Let $\overrightarrow{v}$ be a vector field on $\mathbf{R}^k$ with an isolated zero at the origin. Pick $\epsilon > 0$ small enough that the sphere S_ϵ of radius ϵ centered at 0 does not contain any other zeros of $\overrightarrow{v}$. Then we define the **index** of $\overrightarrow{v}$ at 0, denoted $\operatorname{ind}_0(\overrightarrow{v})$, to be the degree of the directional map $x \to \overrightarrow{v}(x)/|\overrightarrow{v}(x)|$ from S_ϵ to S^{k-1}.

To calculate $\operatorname{ind}_0(\overrightarrow{v})$ in the two-dimensional case, simply count the number of times $\overrightarrow{v}$ rotates counterclockwise while we walk counterclockwise around the zero. If $\overrightarrow{v}$ completes a clockwise rotation, it contributes -1 to the index. For example, consider the following vector fields on $\mathbf{R}^2$:

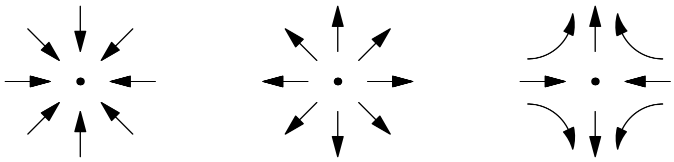

Figure 8.1

For the first two, we have $\operatorname{ind}_0(\overrightarrow{v}) = 1$ because the direction of the vector field completes one counterclockwise rotation as we walk counterclockwise around the zero. For the third, we have $\operatorname{ind}_0(\overrightarrow{v}) = -1$ because the direction of the vector field completes a clockwise rotation.

Now we want to define index on a k-dimensional manifold X. Suppose $\overrightarrow{v}$ is a smooth vector field on X with an isolated zero at x, and let $\phi : U \to X$ be a parametrization with $\phi(0) = x$. For all $u \in U$, define the corresponding *pullback* vector field on U by

$$\overrightarrow{w}(u) = D\phi_u^{-1}\overrightarrow{v}(\phi(u)).$$

Then we define

$$\operatorname{ind}_x(\overrightarrow{v}) := \operatorname{ind}_0(\overrightarrow{w}).$$

We can now state the Poincaré–Hopf Theorem.

Poincaré–Hopf Theorem. *If $\overrightarrow{v}$ is a smooth vector field on the compact oriented manifold X with only finitely many zeros, then the global sum of the indices of $\overrightarrow{v}$ equals the Euler characteristic of X.*

This theorem has some immediate implications. We can now find the Euler characteristic of a manifold just by constructing a smooth vector field on it. For example, the sphere S^2 admits a vector field with one source and one sink, so its Euler characteristic is 2. It then follows that there is no smooth vector field on S^2 with no zeros. On the other hand, the circle S^1 and the torus $S^1 \times S^1$ have vector fields with no fixed points, so they must have Euler characteristic 0.

Recall that our definition of the Euler characteristic of X is the orientation number $I(\Delta, \Delta)$ in $X \times X$. Thus, the Poincaré–Hopf theorem indicates that there is a relationship between the indices of $\overrightarrow{v}$ and the intersection numbers in $X \times X$. This is what our proof revolves around.

If X is a smooth k-dimensional manifold, then the **tangent bundle** of X, denoted $T(X)$, is a smooth $2k$-dimensional manifold given by

$$T(X) = \{(x, v) \mid x \in X, v \in T_x(X)\}.$$

Since X and $T_x(X)$ are both subsets of $\mathbf{R}^n$, $T(X)$ is a subset of $\mathbf{R}^{2n}$. It contains a copy of X given by $X_0 = \{(x, 0) \subset T(X) \mid x \in X\}$. Let $\pi : T(X) \to X$ be the projection $\pi(x, v) = x$.

We will now prove the Poincaré–Hopf Theorem. Due to its length, we will divide the proof into three propositions. These propositions contain some terms that have not been defined yet, but they will be defined in their corresponding sections:

Proposition 8.1. If x is a nondegenerate zero of $\overrightarrow{v}$, then $\operatorname{ind}_x(\overrightarrow{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_v$.

Proposition 8.2. If x is an isolated zero of $\overrightarrow{v}$ and U is a neighborhood of x containing no other zeros of $\overrightarrow{v}$, then $\overrightarrow{v}$ is homotopically equivalent to a vector field $\overrightarrow{v}_1$ which agrees with $\overrightarrow{v}$ outside some compact subset of U and has only nondegenerate fixed points inside U.

Proposition 8.3. If $X_0 \subset T(X)$ is the zero section and $\Delta \in X \times X$ is the diagonal, then $I(X_0, X_0) = I(\Delta, \Delta)$.

8.1 Proof of Proposition 8.1

Let X be a compact, oriented manifold. For a smooth vector field $\overrightarrow{v}$ on X there is a smooth map $\sigma : X \to T(X)$ such that $\sigma(x) = (x, \overrightarrow{v}(x))$. Let X_v be the image of σ. We want to show that σ is an embedding, so that X_v submanifold of $T(X)$ diffeomorphic to X. That is, we have to show that σ is a proper, injective immersion. We get proper for free because X is compact, and the equation $\pi \circ \sigma = Id_X$ implies that σ is injective. So, the only interesting part is to show that σ is an immersion.

Fix $(x, v) \in T(X)$. Let $\phi : U \to X$ be a parametrization of X at x, and let $\theta : U \times \mathbf{R}^k \to T(X)$ be a parametrization of $T(X)$ at (x, v) given by $\theta(a, b) = (\phi(a), D\phi_a(b))$. Then define $h = \theta^{-1} \circ \sigma \circ \phi : U \to U \times \mathbf{R}^k$. We have

$$h(a) = \theta^{-1}(\sigma(\phi(a))) = \theta^{-1}(\phi(a), b) = (a, D\phi_a^{-1}(b)),$$

for some $b \in T_{\phi(a)}(X)$. It follows that Dh_0 is injective, so

$$D\sigma_x = D\theta_0 \circ Dh_0 \circ D\phi_0^{-1}$$

is also injective. Therefore, σ is an immersion, hence an embedding.

The tangent space of X_v at a point $(x, \overrightarrow{v}(x))$ is the best linear approximation to X_v at that point: $\{(a, D\overrightarrow{v}_x(a)) \mid a \in T_x(X)\}$.

A zero x of $\overrightarrow{v}(x)$ is called **nondegenerate** if $D\overrightarrow{v}_x : T_x(X) \to T_x(X)$ is an isomorphism. Zeroes of $\overrightarrow{v}$ correspond to points in the intersection of X_v with X_0. The transversality condition at such points is

$$T_{(x,0)}(X_0) + T_{(x,0)}(X_v) = T_{(x,0)}(T(X)).$$

But because $T_{(x,0)}(T(X)) = T_x(X) \times T_x(X)$, we can rewrite this condition as

$$T_x(X) \times \{0\} + \{(a, D\overrightarrow{v}_x(a)) \mid a \in T_x(X)\} = T_x(X) \times T_x(X).$$

This condition holds if and only if the kernel of $D\overrightarrow{v}_x$ is trivial. Thus, x is a nondegenerate zero of $\overrightarrow{v}$ if and only if $X_v \pitchfork X_0$ at $(x, 0)$.

As an example, consider a vector field on the manifold $\mathbf{R}$. The tangent space of $\mathbf{R}$ at any point x is just all of $\mathbf{R}$, so a smooth vector field on $\mathbf{R}$ is just a smooth map $\overrightarrow{v} : \mathbf{R} \to \mathbf{R}$. We can graph this map in the $\mathbf{R}^2$ plane. Consider the following two vector fields:

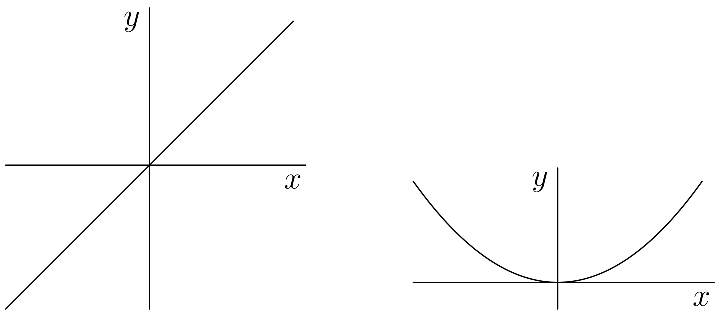

Figure 8.2

The zero of the first vector field is nondegenerate because $D\overrightarrow{v}_0$ can be represented by the 1×1 matrix [1], which is invertible. The zero of the second vector field is degenerate because $D\overrightarrow{v}_0$ can be represented by [0], which is not invertible. Notice the correspondence of nondegeneracy with transversality, as mentioned above.

So, a nondegenerate zero on a 1-dimensional manifold can either be a source or a sink, but it cannot be a source on one side and a sink on the other. However, notice that we can deform the vector field in Figure 4 just a little bit to split the degenerate zero into two nondegenerate zeroes. This will be important in section 8.2.

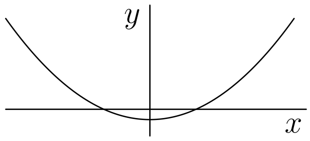

Figure 8.3

Now, suppose x is a nondegenerate zero of $\overrightarrow{v}$. Then, since $X_v \pitchfork X_0$ at $(x, 0)$, we have

$$T_x(X) \times \{0\} \oplus \{(a, D\overrightarrow{v}_x(a)) \mid a \in T_x(X)\} = T_x(X) \oplus T_x(X).$$

We want to show that $\text{ind}_x(\overrightarrow{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_v$. Let $\beta = \{v_1, \ldots, v_k\}$ be a positively oriented basis for $T_x(X)$. Then

$$\{(v_1, 0), \ldots, (v_k, 0)\}$$

and

$$\{(v_1, D\overrightarrow{v}_x(v_1)), \ldots, (v_k, D\overrightarrow{v}_x(v_k))\}$$

are positive oriented bases for $T_{(x,0)}(X_0)$ and $T_{(x,0)}(X_v)$, respectively. Therefore the intersection number at $(x, 0)$ is $+1$ if and only if

$$\{(v_1, 0), \ldots, (v_k, 0), (v_1, D\overrightarrow{v}_x(v_1)), \ldots, (v_k, D\overrightarrow{v}_x(v_k))\}$$

is a positively oriented basis for $T_x(X) \times T_x(X)$. But the sign of this basis is equal to the sign of

$$\{(v_1, 0), \ldots, (v_k, 0), (0, D\overrightarrow{v}_x(v_1)), \ldots, (0, D\overrightarrow{v}_x(v_k))\} = \text{sign}(\beta) \cdot \text{sign}(\{D\overrightarrow{v}_x(v_1)), \ldots, D\overrightarrow{v}_x(v_k))\})$$

Therefore, the intersection number at $(x, 0)$ is $+1$ if the determinant of $D\overrightarrow{v}_x$ is positive, and -1 otherwise.

To complete the proof, we need to show that $\text{ind}_x(\overrightarrow{v})$ is $+1$ if the determinant of $D\overrightarrow{v}_x$ is positive and -1 otherwise. Let $\phi : U \to X$ be a parametrization around x with $\phi(0) = x$. Let $\overrightarrow{w}$ be the pullback vector field on U: $\overrightarrow{w} = D\phi_u^{-1}\overrightarrow{v}(\phi(u))$. By the chain rule, we have

$$D\overrightarrow{w}_0 = D\phi_0^{-1} \circ D\overrightarrow{v}_x \circ D\phi_0.$$

Thus the sign of the determinant of $D\overrightarrow{w}_0$ is equal to the sign of the determinant of $D\overrightarrow{v}_x$. By definition, we have

$$\text{ind}_x(\overrightarrow{v}) = \text{ind}_0(\overrightarrow{w}) = \deg(f),$$

where $f(u) = \frac{\overrightarrow{w}(u)}{|\overrightarrow{w}(u)|}$. Since $\overrightarrow{w}(0) = 0$, we can write $\overrightarrow{w}(u) = D\overrightarrow{w}_0(u) + \epsilon(u)$, where $\epsilon(u)/|u| \to 0$ as $u \to 0$. Then, if we define

$$f_t(u) = \frac{D\overrightarrow{w}_0(u) + t\epsilon(u)}{|D\overrightarrow{w}_0(u) + t\epsilon(u)|},$$

we see that $f_1(u) = f(u)$ and $f_0(u) = \frac{D\overrightarrow{w}_0(u)}{|D\overrightarrow{w}_0(u)|}$. This simplifies the problem because the degree is invariant under homotopy, and the degree of $f_0(u)$ is easier to calculate. However, in order for this to be a smooth homotopy, the denominator $|D\overrightarrow{w}_0(u) + t\epsilon(u)|$ must never be zero. But remember that x is a nontrivial zero, so $D\overrightarrow{w}_0(u)$ is an isomorphism. This means the image of the unit ball under $D\overrightarrow{w}_0(u)$ contains a closed ball of radius $c > 0$, and linearity implies $|D\overrightarrow{w}_0(u)| > c|u|$ for all $u \in \mathbf{R}^k$. By choosing a ball S_ϵ small enough (where S_ϵ is the domain of $f(u)$), we can say that $\frac{|\epsilon(u)|}{|u|} < \frac{c}{2}$ on S_ϵ. Therefore, by the triangle inequality, we have

$$|D\overrightarrow{w}_0(u) + t\epsilon(u)| \geq ||D\overrightarrow{w}_0(u)| - |t\epsilon(u)|| \geq \frac{c}{2}|u| > 0.$$

Thus f_t is a smooth homotopy, and we have reduced the problem to showing that the degree of $f_0(u) = \frac{D\overrightarrow{w}_0(u)}{|D\overrightarrow{w}_0(u)|}$ is $+1$ if and only if $D\overrightarrow{w}_0(u)$ is orientation-preserving. To complete the proof from here, we will need a lemma from linear algebra.

Lemma 8.4. *If A is an orientation-preserving linear isomorphism on $\mathbf{R}^k$, then there exists a homotopy A_t such that $A_0 = A$, $A_1 = I_k$, and A_t is an isomorphism for all t. Similarly, if A is orientation-reversing, then there exists a homotopy A_t such that $A_0 = A$, A_1 is the reflection $A_1(x_1, \ldots, x_k) = (-x_1, \ldots, x_k)$, and A_t is an isomorphism for all t.*

Proof. Suppose A is a linear isomorphism. Recall that the eigenvalues of A are the roots of the characteristic polynomial of A. So, by the Fundamental Theorem of Algebra, A has some real or complex eigenvalue. If the eigenvalue is a real number λ, then $\mathbf{R}^k$ has a one-dimensional invariant subspace V given by the span of an eigenvector with eigenvalue λ. If the eigenvalue is a complex number $a+bi$ with $b \neq 0$, then there is a complex eigenvector $\vec{w} = \vec{u} + i\vec{v}$, where $\vec{u}$ and $\vec{v}$ are nonzero real vectors. Then we have $A\vec{v} = a\vec{v} + b\vec{u}$ and $A\vec{u} = -b\vec{v} + a\vec{u}$. Therefore, $V = \operatorname{Span}\{\vec{u}, \vec{v}\}$ is an invariant subspace. In any case, there is a one- or two-dimensional invariant subspace V. We can write $\mathbf{R}^k = V \oplus W$. Let β_1 be an ordered basis for V and β_2 be an ordered basis for W. Then the matrix representation of A with respect to the combined ordered basis $\{\beta_1, \beta_2\}$ is a block matrix of the form

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where B is a $\dim V \times \dim V$ square matrix. Define

$$A_t = \begin{bmatrix} B & (1-t)C \\ 0 & D \end{bmatrix}.$$

Then every A_t is an isomorphism and A_1 has both V and W as invariant subspaces. We have $\det(A_1) = \det(A_0)$, so A_1 is orientation-preserving if and only if A_0 is. Moreover, $\det(A_1) = \det(B)\det(D)$.

We will proceed by induction, starting with the base cases of $k = 1$ and $k = 2$. If $k = 1$, then $A = [a]$ is a 1×1 matrix. If $a > 0$, then $A_t = (1-t)A + tI$ is a homotopy of the desired type. If $a < 0$, then $A_t = (1-t)A - tI$ works. When $k = 2$, we only need to consider the complex eigenvalue case, because if there is a real eigenvalue we can find a 1-dimensional invariant subspace and apply the inductive step. So, if A is a 2×2 matrix with a non-real complex eigenvalue z, then $\overline{z}$ is the other eigenvalue, so $\det(A) = z\overline{z} = |z|^2 > 0$. Thus, A is orientation-preserving. And for all $t < 0$, the matrix $(1-t)A$ is an isomorphism with eigenvalues $(1-t)z$ and $(1-t)\overline{z}$. Thus $A_t = tI + (1-t)A$ is always an isomorphism, as desired.

For the inductive step, suppose the lemma holds for $k-1$ and $k-2$. Let A be an orientation-preserving isomorphism on $\mathbf{R}^k$. Using the previous construction, A is homotopy equivalent to

$$\begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}.$$

If B is orientation-preserving, then so must D be. By the inductive assumption, it follows that A is homotopy equivalent to the identity matrix. If B is not orientation-preserving, then so must D be, so A is homotopy equivalent to

$$\begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

which is homotopic to the identity under rotation in the first two coordinates. The proof for when A is orientation-reversing is essentially the same. ■

Now we can finally conclude the proof that $\operatorname{ind}_x(\vec{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_v$. If $D\vec{w}_0(u)$ is orientation-preserving, then

$$\deg\left(\frac{D\vec{w}_0(u)}{|D\vec{w}_0(u)|}\right) = \deg\left(\frac{u}{|u|}\right) = 1.$$

If $D\vec{w}_0(u)$ is orientation-reversing, then the above map is homotopic to $\frac{u}{|u|}$ composed with reflection on the first variable, so the degree is -1. Therefore, $\operatorname{ind}_x(\vec{v})$ is the orientation number of the point $(x, 0)$ in $X_0 \cap X_v$.

8.2 Proof of Proposition 8.2

For the next part of the proof, we have to deal with the case when x is a degenerate zero of $\vec{v}$. As hinted earlier, the solution will be to homotopically deform $\vec{v}$ around x to "split" the zero into nondegenerate zeros. By homotopy invariance, this will not change either the index at x nor the intersection number $I(X_0, X_v)$.

We'll start with the simple case of $\mathbf{R}^k$. Suppose $\vec{w}$ is a smooth vector field on $\mathbf{R}^k$ with an isolated zero at the origin, and let U be an open neighborhood of the origin containing no other zeros of the vector field. The goal is to construct a homotopy $\vec{w}_t$ of $\vec{w}$ such that $\vec{w}_1$ has only nondegenerate zeros in U, and each $\vec{w}_t$ equals $\vec{w}$ outside some compact subset of U.

Let $\rho : \mathbf{R}^k \to [0,1]$ be a smooth function that is 1 on a neighborhood V of the origin and zero outside a compact subset $K \subset U$. Such a function does exist, but we will not construct it here. We will show that there exists $a \in \mathbf{R}^k$ with $|a|$ arbitrarily small such that

$$\vec{w}_t(u) = \vec{w}(u) + t\rho(u)a$$

works. We need $|a|$ to be small enough that $\vec{w}_t$ has no fixed points outside V. This is possible because, since $\vec{w}$ has no zeros on the compact set $K - V$,

$$|\vec{w}_t(u)| > c > 0$$

there. By demanding $|a| < c/2$, we get

$$|\vec{w}_t(u)| \geq |w_t(u)| - |t\rho(u)a| > \tfrac{c}{2}.$$

So $\vec{w}_t(u)$ has no zeros outside V.

By Sard's theorem, there exists a such that $|a| < c/2$ and $-a$ is a regular value of $\vec{w}$. If x is a zero of $\vec{w}_1$, then $x \in V$, so $\vec{w}_1 = \vec{w} + a$ in a neighborhood of x. Thus, $D(\vec{w}_1)_x = D\vec{w}_x$. And since $\vec{w}_1(x) = 0$, we have $\vec{w}(x) = -a$. Since $-a$ is a regular value of $\vec{w}$, $D\vec{w}_x$ is an isomorphism, so $D(\vec{w}_1)_x$ is as well, so x is a nondegenerate zero of $\vec{w}_1$, as desired.

Now we just have to show that the same holds for a vector field $\vec{v}$ on a manifold X with an isolated zero at x. Let $\phi : U \to X$ be a parametrization around x with $\phi(0) = x$, and apply the above case to the pullback vector field $\vec{w}(u) = D\phi_u^{-1}\vec{v}(\phi(u))$. Define

$$\vec{v}_t(x) = D\phi_{\phi^{-1}(x)} \circ \vec{w}_t \circ \phi^{-1}(x).$$

If u is a zero of $\vec{w}_t$, then $\phi(u)$ is certainly a zero of $\vec{v}_t$, and the Chain Rule implies that $D(\vec{v}_t)_{\phi(z)}$ is nonsingular if and only if $D(\vec{w}_t)_u$ is. Thus $\vec{v}_t$ works.

8.3 Proof of Proposition 8.3

We will show that there is a diffeomorphism of a neighborhood of X_0 in $T(X)$ with a neighborhood of the diagonal Δ extending the natural diffeomorphism $X_0 \to \Delta$ defined by $(x,0) \to (x,x)$. To do this, we will use the Tubular Neighborhood Theorem from the end of section 5. First, notice that $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$. This is because if $\phi : U \to X$ is a parametrization of X at x, then $\theta(u) = (\phi(u), \phi(u))$ is a parametrization of Δ at (x,x). Then $D\theta_0 = \begin{bmatrix} D\phi_0 \\ D\phi_0 \end{bmatrix}$, so

$$T_{(x,x)}(\Delta) = \text{Image}(D\theta_0) = \{(v,v) \mid v \in T_x(X)\}.$$

Thus, the orthogonal complement to $T_{(x,x)}(\Delta)$ in $T_{(x,x)}(X \times X)$ is the set $\{(v,-v) \mid v \in T_x(X)\}$.

Therefore, an arbitrary element of $N(\Delta; X \times X)$ can be written as $((x,x),(v,-v))$, where $v \in T_x(X)$. If we define $f : T(X) \to N(\Delta; X \times X)$ by $f(x,v) = ((x,x),(v,-v))$, then f is surjective, and it is also injective. Smoothness is clear from the formula, so f is a diffeomorphism. Therefore, $T(X)$ is diffeomorphic to $N(\Delta; X \times X)$.

By the Tubular Neighborhood Theorem, there is a diffeomorphism of an open neighborhood of Δ_0 in $N(\Delta; X \times X)$ with an open neighborhood of Δ in $X \times X$, where $\Delta_0 = \Delta \times \{0\} \in N(\Delta; X \times X)$. So, in conclusion, there is a diffeomorphism of an open neighborhood of Δ in $X \times X$ to an open neighborhood of Δ_0 in $N(\Delta; X \times X)$, and there is also a diffeomorphism from $N(\Delta; X \times X)$ to $T(X)$ which maps Δ_0 onto X_0. Therefore, there is a diffeomorphism of a neighborhood of X_0 in $T(X)$ with a neighborhood of the diagonal Δ extending the natural diffeomorphism $X_0 \to \Delta$. Proposition 8.3 folllows.

8.4 Concluding the Proof

Now, let $\overrightarrow{v}$ be a smooth vector field on a compact oriented manifold X with only finitely many zeros. Let s be the global sum of the indices of $\overrightarrow{v}$. Propositions 8.1 and 8.2 together imply that $s = I(X_0, X_v)$. Since X_v can be smoothly deformed into X_0 by the homotopy $f_t(x) = (x, (1-t)\overrightarrow{v}(x))$, this is equal to $I(X_0, X_0)$. By Proposition 8.3, $I(X_0, X_0) = I(\Delta, \Delta) = \chi(X)$. Therefore, the global sum of the indices of $\overrightarrow{v}$ equals the Euler characteristic of X, as desired.

9 The Euler Characteristic and Simplicial Complex Structures

To get a fuller grasp of the Euler characteristic, we will briefly look at an alternative definition from algebraic topology. The Euler characteristic can be defined on a set of topological spaces more broad than smooth manifolds, namely, spaces that have a simplicial complex structure. We start with the following definition.

Definition 9.1. Let $[v_0, v_1, ..., v_n]$ be an ordered set of affine linearly independent points in $\mathbf{R}^m$, with $n \leq m$. An n-**simplex** Δ^n is the smallest convex subset of $\mathbb{R}^m$ containing $v_0, ..., v_n$ [1].

For example, Δ^0 is a point, Δ^1 is a line segment, and Δ^2 is a triangle. The points v_i are called the **vertices** of the simplex, and the simplex itself is denoted by $[v_0, \ldots, v_n]$. A **face** of an n-simplex is a simplex defined by a $n-1$ of its vertices.

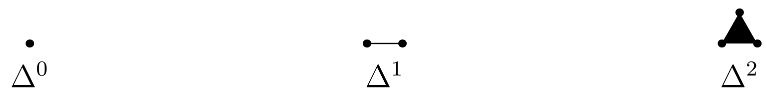

Figure 9.1

Definition 9.2. Let X be a topological space. A **simplicial complex structure** (or Δ-**complex structure**) on X is a collection of continuous functions $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that

1. *The restrictions* $\sigma_\alpha : \mathrm{Int}(\Delta^n) \to X$ *are injective, and each* $x \in X$ *belongs to exactly one* $\sigma_\alpha(\mathrm{Int}\Delta^n)$.
2. *Each restriction of* σ_α *to a face of* Δ^n *is one of the maps* $\sigma_\beta : \Delta^{n-1} \to X$. *Here we are identifying the face of* Δ^n *with* Δ^{n-1} *by the canonical linear homeomorphism between them that preserves the ordering of the vertices.*
3. *A set* $U \subset X$ *is open if and only if* $\sigma_\alpha^{-1}(U)$ *is open in* Δ^n *for all* α. [3]

For example, consider the sphere in Figure 9.2, which satisfies the definition of simplicial complex structure. There are eight maps σ_α. Notice that the restriction of σ_α to a face of one of the 2-simplices is equal to one of the maps on a 1-simplex, and the restriction of σ_α to a face of one of the 1-simplices is equal to one of the maps on a 0-simplex, as long as we identify the faces of Δ^n with Δ^{n-1} appropriately. It's also worth noting that a 0-simplex has an empty boundary, which means it is its own interior. Thus, every point on the sphere belongs to exactly one $\sigma_\alpha(\mathrm{Int}\Delta^n)$.

Figure 9.2

Definition 9.3. Let X be a topological space with a Δ-complex structure. For a fixed $n \geq 0$, let $D_n(X)$ denote the collection of functions σ_α whose domain is an n-simplex. Then the n-**chain group of** X, denoted by $C_n(X)$, is the free abelian group generated by $D_n(X)$. That is, elements of $C_n(X)$ are formal sums

$$\gamma = \sum_\alpha n_\alpha \sigma_\alpha,$$

where $n_\alpha \in \mathbb{Z}$ and $\sigma_\alpha \in D_n(X)$. By convention, we define $C_n(X) = \{0\}$ for all $n < 0$ [3].

Now, if X is a topological space with a simplicial complex structure, then we can define the Euler characteristic in the more common way:

Definition 9.4. The **Euler characteristic** of a topological space with a finite simplicial complex structure is

$$\chi(X) = \sum_n (-1)^n \cdot \mathrm{rank}(C_n(X)),$$

where the rank of $C_n(X)$ is the number of elements that generate $C_n(X)$ [4]. So, we can equivalently write

$$\chi(X) = \sum_n (-1)^n |D_n(X)|$$

Defined in this way, the Euler characteristic is a topological invariant, and does not depend on the choice of simplicial complex structure, but we will not prove that here. Looking at the sphere in Figure 9.2, we see that the Euler characteristic is $2 - 3 + 3 = 2$, as expected. Since every convex polyhedron is homeomorphic to the sphere, every convex polyhedron has Euler characteristic 2. This is Euler's famous polyhedron formula: $F - E + V = 2$.

If X is a smooth manifold with a simplicial complex structure, then the two definitions of Euler characteristic agree with each other, which is actually an intuitive result given the Poincaré–Hopf theorem. If X is 2-dimensional and has a simplicial complex structure, then there exists a smooth vector field on X that has a source in each face, a saddle on each edge, a sink at each vertex, and no other zeros. Thus, by Poincaré–Hopf, $\chi(X) = F - E + V$, where $\chi(X)$ is defined according to Definition 7.5. The same type of informal proof works for higher dimensions, but it becomes less intuitive.

References

[1] M. A. Armstrong. *Basic Topology.* Springer New York, 2013. ISBN: 9781475717938.

[2] Victor Guillemin and Alan Pollack. *Algebraic Topology.* Prentice-Hall, 1974. ISBN: 0132126052.

[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2009. ISBN: 9780521795401.

[4] James R. Munkres. *Elements of Algebraic Topology.* CRC Press, 2018. ISBN: 9780201045864.

[5] Arthur Sard. *The measure of the critical values of differentiable maps.* Vol. 48. Bulletin of the American Mathematical Society, 1942.

[6] Michael Spivak. *Calculus on Manifolds.* Perseus Books, 1968.