Leaky Abelian Sandpile Model With Multiple Starting Points

By Lloyd Page

Abstract

The leaky abelian sandpile model with multiple starting points is a growth model in which n grains of sand are started at either a finite number of starting points or a configuration of starting points dependent on n in $\mathbb{Z}^2$. These grains of sand spread out along the vertices of $\mathbb{Z}^2$ according to a toppling rule. A site at a vertex topples if the number of grains of sand at said vertex are above a specified threshold. In such a toppling, the site sends some sand to each of its neighbors and leaks a portion, $1 - \frac{1}{d}$, of the toppled sand. A site may topple multiple times before it falls below the threshold and stops toppling.

I explored the limit shape in the symmetric case with more than 1 source point. In this case, each topple sends an equal amount of sand to each neighbor. Supposing we have k source points where k is finite, I show that the limit shape is the union of k limit shapes, each originating at one of those source points. This means that as $d \to 1$, we have a union of circles, and as $d \to \infty$, we have a union of diamonds. I also explored a starting arrangement of a square centered at the origin with $\log n$ length sides. However, I was unable to prove any significant conclusions about the limit shape in this case.

Contents

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1. **Introduction**

The Abelian Sandpile Model (ASM), originally introduced by Per Bak, Chao Tang, and Kurt Wiensenfeld in 1987 [BTW87], is a cellular automaton defined on $\mathbb{Z}^2$. The input is a sandpile configuration $s : \mathbb{Z}^2 \to \mathbb{N}$, which represents the number of grains of sand or chips present at a vertex or site $x \in \mathbb{Z}^2$. The sandpile s evolves using the following rule: If a site $x \in \mathbb{Z}^2$ has at least 4 chips, then it "topples" and gives a chip to each of its nearest neighbors. That is it gives one chip to its northern neighbor, one to its southern neighbor, one to its eastern neighbor, and one to its western neighbor. The sandpile evolves until there are no more sites that can be toppled. The "Abelian" part of the model name is in reference to a result by Deepak Dhar in 1990 [Dha90], namely that the final stable configuration does not depend on the order in which sites topple. For a visual of a final configuration, look at Figure 1 below.

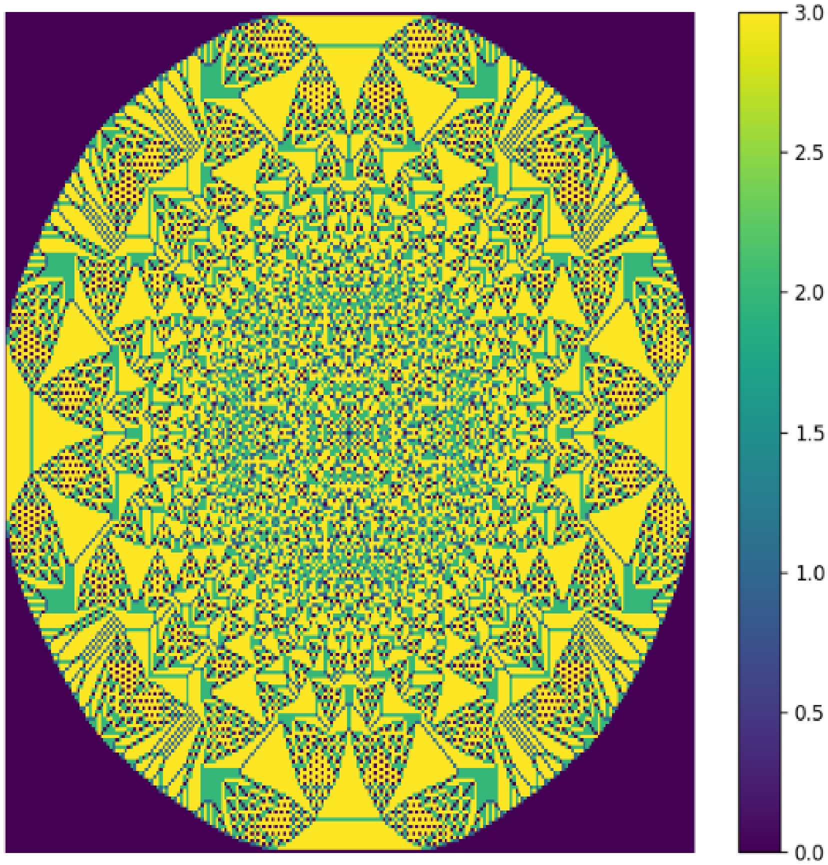

Figure 1. $n = 10^5$

The ASM has been linked to the router-router model in [HLM+08], which also provides a survey of both models. Other surveys of the mathematical literature associated with the ASM include the following papers: “Laplacian growth, sandpiles and scaling limits” by Lionel Levine and Yuval Peres [LP17], “WHAT IS a sandpile?” by Lionel Levine and James Propp [LP10], and “A sandpile model for proportionate growth” by Deepak Dhar and Tridib Sadhu [DS13]. The case of the ASM with a starting configuration of n chips at the origin has been studied extensively. In this case, simulations show the emergence of a fractal structure in the limit shape, and hint at both a convex shape and flat edges. All of these features are present in Figure 1, which is a simulation with 10^5 chips at the origin. However, there are no proofs to defend such claims at this point in time. The fractal structure and local patterns have been studied in [LPS16] and [PS20]. Thanks to the work of Hayk Aleksanyan and Henrik Shahgholian in [AS19], we know that the boundary of the limit shape is a Lipschitz graph. However, the most relevant prior result to this paper is that the boundaries for the limit shape for one source point established in [LP09] are not the union of the boundaries for the limit shape with multiple source points established in [LP10]. Thus, it is likely that the main result of this paper arises due to leakiness, which is defined later in this paper.

The main result of this paper is that in the case of k source points where k is finite, the Leaky Abelian Sandpile Model's (Leaky-ASM) limit shape is the union of k limit shapes each generated from one of those source points. The Leaky-ASM is a one-parameter deformation of the ASM, in which dissipation is present. Sandpile Models with dissipation were first introduced in [MKK90], and the limit shape generated with 1 source point at the origin for the Leaky-ASM was first computed by Ian Alevy and Sevak Mkrtchyan in [AM21]. The equality of the limit shape and the union of limit shapes will be shown by double inclusion. However, to establish one of the inclusions, I will be providing an explicit connection between the visited sites generated by the leaky-ASM with multiple source points and death probabilities for the killed random walk with multiple starting points. I will then obtain the limit shape by relating the asymptotic death probabilities of the associated killed random walks and get the necessary inclusion through convergence of the two shapes.

The Leaky-ASM differs from the standard ASM in three notable ways. Firstly, sandpiles are now allowed to take on non-negative real numbers instead of being merely restricted to the natural numbers. So, $s : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}$. Secondly, each topple may send a different number of chips in each direction. Let $c_\downarrow, c_\uparrow, c_\rightarrow, c_\leftarrow \in \mathbb{R}_{\geq 0}$. Then, when a toppling happens at a given site x, this model sends $c_\downarrow$ south, $c_\uparrow$ north, $c_\rightarrow$ east, and $c_\leftarrow$ west from x to its nearest neighbors. Thirdly, each time there is a toppling, the site that toppled “leaks” chips. Let $d \in \mathbb{R} > 1$ and $c = c_{\downarrow} + c_{\uparrow} + c_{\rightarrow} + c_{\leftarrow}$. A site now topples if it has greater than cd chips and loses those chips, but only distributes c chips. Thus $cd - c = c(d-1)$ chips leak per toppling. Chips that leak disappear from the sandpile, and $d - 1$ is known as the leakiness parameter. An explicit formulation of this model follows. Let $S_n(x)$ be the evolution of the sandpile after n-steps. Suppose that we pick the site (a, b) as the $n + 1$st step of the evolution of the model and (a, b) has at least cd chips. Then the pile at (a, b) topples, and the new sandpile is given by:

$$\begin{aligned}
S_{n+1}(a,b) &= S_n(a,b) - cd \\
S_{n+1}(a+1,b) &= S_n(a+1,b) + c_{\rightarrow} \\
S_{n+1}(a-1,b) &= S_n(a-1,b) + c_{\leftarrow} \\
S_{n+1}(a,b+1) &= S_n(a,b+1) + c_{\uparrow} \\
S_{n+1}(a,b-1) &= S_n(a,b-1) + c_{\downarrow}
\end{aligned}$$

All other sites are unchanged after the $n + 1$st step of evolution of the model. The case in which $c_{\downarrow} = c_{\uparrow} = c_{\rightarrow} = c_{\leftarrow} = 1$ and $d > 1$ is known as the uniform leaky-ASM. Note that the set of points visited by the leaky-ASM is the limit shape of the leaky-ASM, as the height of a site can be 0 only if it is unvisited or it is in the interior of the shape of the model. [AM21] tells us that as $d \to 1$, the limit shape goes to a circle, and as $d \to \infty$, the limit shape goes to a diamond. This can be easily seen by looking at Figures 2 and 3.

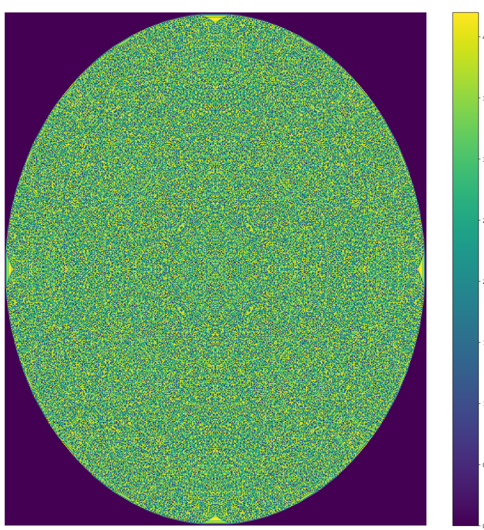

Figure 2.
$n = 10^{50}, d = 1.05$

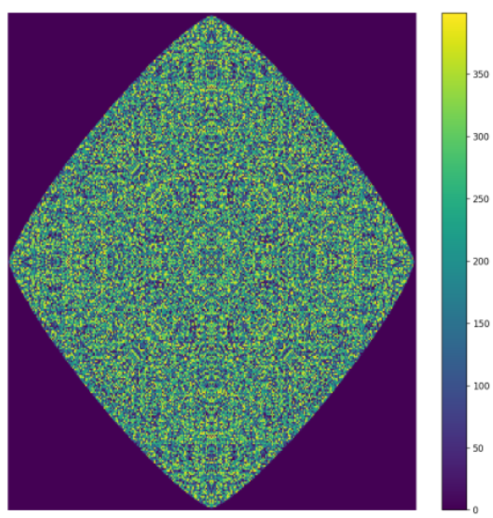

Figure 3.
$n = 10^{400}, d = 100$

A more explicit formulation of the main theorem of the paper follows:

Theorem 1.1. *Let* $V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)}$ *be the set of points visited by the Leaky-ASM when there are* k *source points located at* $(a_1,b_1),\ldots,(a_k,b_k)$ *and* n *chips distributed uniformly among those* k *source points with a leakiness parameter of* d*-1. Also, let* $d>1$ *and take* $n\to\infty$*. Then*:

$$V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)} = \bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}$$

Visually, this can be seen by comparing Figure 4 with 5 or comparing Figure 6 with 7:

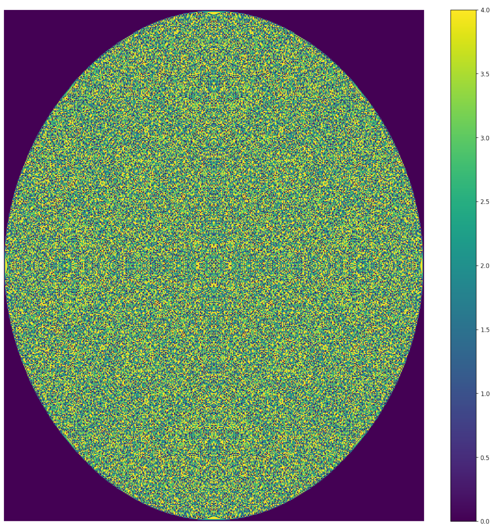

Figure 4.
$n = 10^{100}$
$d = 1.25$
Source point:$(0,0)$

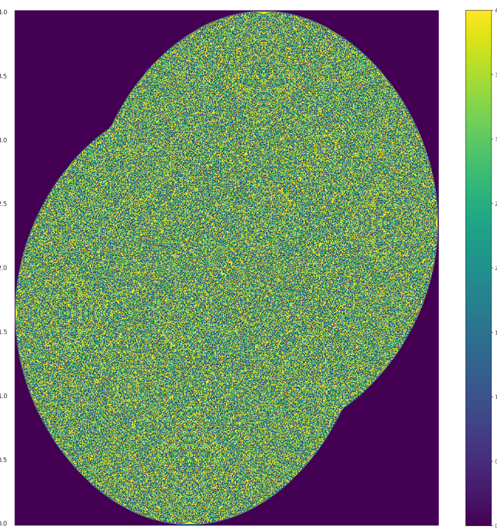

Figure 5.
$n = 2 * 10^{200}$
$d = 1.25$
Source points:$(0,0),(100,100)$

Remark 1.2. *The proof of one of the inclusions is significantly easier than the other due to the Abelian nature of the leaky ASM. That is, the proof of*

$$V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)} \supseteq \bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}$$

is much simpler than the other inclusion

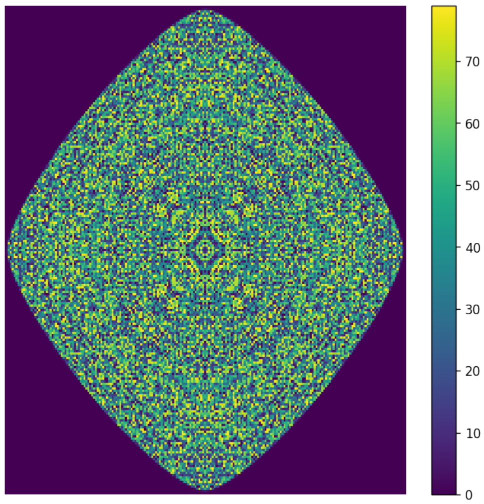

Figure 6.
$n = 10^{200}$
$d = 20$
Source points:$(0, 0)$

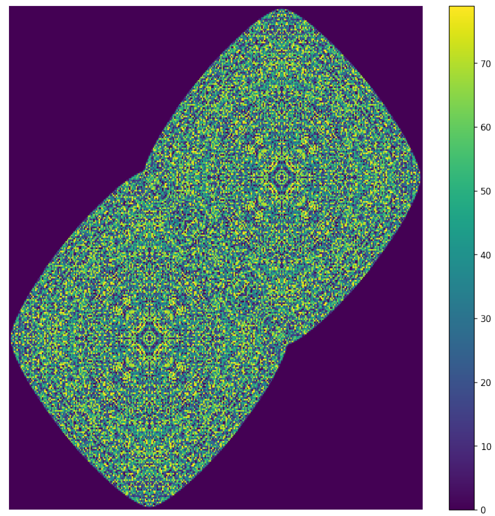

Figure 7.
$n = 2 * 10^{200}$
$d = 20$
Source points:$(0, 0), (100, 100)$

Proof. Evolve the model from the source points by toppling each site with no consideration for the interactions of the sandpiles generated by the source points. This is the same as

$$\bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}$$

and since Leaky-ASM is Abelian, and the sites with extra height due to interactions can only expand $V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)}$, we have the desired inclusion, namely:

$$V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)} \supseteq \bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}$$

□

1.1. *Outline.* This paper is organized as follows: I introduce the Killed Random Walk (KRW) with $k > 1$ starting points in section 2, and relate the death probabilities of the KRW with k starting points to coefficients in a Laurent expansion. In section 3, I relate the Leaky-ASM with k source points and the KRW with k starting points by connecting the values of the odometer function of the Leaky-ASM with k source points to the death probabilities of the KRW with k starting points. In section 4, I bound the death probabilities of the KRW with k starting points by the death probabilities associated with the union of the limit shapes, and then use this relation to prove the main theorem of the paper. In section 5, I introduce the an extension with $\log n * \log n$ square starting configuration for the leaky-ASM and show the simulation results of that start.

2. The Killed Random Walk with $k > 1$ Starting Points

In this section, I introduce the Killed Random Walk with $k > 1$ starting points and relate the death probabilities of the KRW to coefficients in a Laurent expansion. Let $(X_j)_{j=1}^n$ be a set of independent identically distributed random variables with the following distribution:

$$P\{X_j = (-1,0)\} = \frac{c_\leftarrow}{cd}$$
$$P\{X_j = (1,0)\} = \frac{c_\rightarrow}{cd}$$
$$P\{X_j = (0,-1)\} = \frac{c_\downarrow}{cd}$$
$$P\{X_j = (0,1)\} = \frac{c_\uparrow}{cd}$$
$$P\{X_j = (0,0)\} = 1 - \frac{c}{cd} = 1 - \frac{1}{d}$$

where $c, d, c_\downarrow, c_\uparrow, c_\rightarrow, c_\leftarrow$ are defined as in section 1. I will adopt the following notation for transition probabilities:

$$p_{x \to y} = P(X_j = y - x)$$

I will also be adopting the interpretation that the walk is killed when the walker does not move, which is when $X_j = (0,0)$. Let

$$K_n := \prod_{i=1}^{n} 1_{X_i \neq (0,0)}$$

be the indicator function as to whether or not the walker has been killed by step n or not. So, if $K_n = 1$, then the walker is still alive after having taken the nth step. Formally, the Killed Random Walk started at $x \in \mathbb{Z}^2$ is defined as the sequence $S_{n,x}$ of random variables defined by:

$$S_{n,x} = x + \sum_{j=1}^{n} X_j K_j$$

To extend this random walk from 1 starting point to k starting points, I introduce a another random variable, X_0, which is independent of all other random variables introduced so far, and has a uniform distribution over the k starting points. To be slightly more formal, for all starting points x_i, we have:

$$P\{X_0 = x_i\} = \frac{1}{k}$$

Thus, we can now define the Killed Random Walk started at k starting points as the sequence S_n of random variables defined by:

$$S_n = X_0 + \sum_{j=1}^{n} X_j K_j$$

For this Killed Random Walk, let $P_d(x)$ represent the probability that the walker dies at x. Note that this leads to the following equality:

$$P_d(x) = P(S_{\min_i K_i = 0} = x)$$

LEMMA 2.1. *Let S_k denote the set of k starting points. Let $d > 1$ and define the Laurent polynomial*:

$$P(z, w) = \frac{cdk - k(c_\uparrow z + c_\downarrow z^{-1} + c_\rightarrow w + c_\leftarrow w^{-1})}{c(d-1) \sum_{x_s \in S_k} z^{-a_s} w^{-b_s}}$$

The the death probabilities of the KRW are the coefficients of the monomials in the Laurent expansion of $P^{-1}(z, w)$, that is

$$[P^{-1}(z, w)]_{(i,j)} = P_d(i, j)$$

where the left-hand side is the coefficient of the $z^i w^j$ monomial in the Laurent series expansion of P^{-1} in the region

$$\frac{(c_\uparrow |z| + c_\downarrow |z^{-1}| + c_\rightarrow |w| + c_\leftarrow |w^{-1}|)}{c(d-1)} < 1 \tag{2.1}$$

Proof. First, expand P^{-1} as a Laurent series to obtain:

$$\begin{aligned}
P^{-1}(z, w) &= \frac{c(d-1) \sum_{x_s \in S_k} z^{-a_s} w^{-b_s}}{cdk - k(c_\uparrow z + c_\downarrow z^{-1} + c_\rightarrow w + c_\leftarrow w^{-1})} \\
&= \frac{c(d-1) \sum_{x_s \in S_k} z^{-a_s} w^{-b_s}}{cdk} \frac{1}{1 - (c_\uparrow z + c_\downarrow z^{-1} + c_\rightarrow w + c_\leftarrow w^{-1})/cd} \\
&= \frac{d-1}{dk} \sum_{x_s \in S_k} z^{-a_s} w^{-b_s} \sum_{m=0}^{\infty} (cd)^{-m} (c_\uparrow z + c_\downarrow z^{-1} + c_\rightarrow w + c_\leftarrow w^{-1})^m
\end{aligned}$$

which converges in the region defined in Equation 2.1. Note that a walker that dies at $(i, j) \in \mathbb{Z}^2$ must arrive at (i, j) and then die at the next step. Let $\Gamma_m(i, j)$ be the set of paths from the origin $(0, 0)$ to (i, j) that takes m steps. Suppose that for a path $\gamma \in \Gamma_m(i, j)$ it takes $n_{c_\uparrow}$ steps upward, $n_{c_\downarrow}$ steps downward, $n_{c_\rightarrow}$ steps rightward, and $n_{c_\leftarrow}$ steps leftward. Let the weight of a path, denoted $w(\gamma)$ be equivalent to the product of weights along the path, so $w(\gamma) = c_\uparrow^{n_{c_\uparrow}} c_\downarrow^{n_{c_\downarrow}} c_\leftarrow^{n_{c_\leftarrow}} c_\rightarrow^{n_{c_\rightarrow}}$. It is possible to move our starting point from the origin to (a_s, b_s), the $\mathbb{Z}^2$ coordinates of a starting point x_s, by going from the origin to $(i-a_s, j-b_s)$. Note that

$$(c_\uparrow z + c_\downarrow z^{-1} + c_\rightarrow w + c_\leftarrow w^{-1})^m = \sum_{(i,j)\in\mathbb{Z}^2} \sum_{\gamma_m \in \Gamma_m(i,j)} w(\gamma_m) z^i w^j$$

so we have

$$\begin{aligned} P^{-1}(z,w) &= \frac{d-1}{dk} \sum_{x_s \in S_k} z^{-a_s} w^{-b_s} \sum_{m=0}^{\infty} (cd)^{-m} \sum_{(i,j)\in\mathbb{Z}^2} \sum_{\gamma_m \in \Gamma_m(i,j)} w(\gamma_m) z^i w^j \\ &= \sum_{(i,j)\in\mathbb{Z}^2} \Bigg(\sum_{x_s \in S_k} \frac{1}{k} \sum_{m=0}^{\infty} \frac{d-1}{d} (cd)^{-m} \sum_{\gamma_m \in \Gamma_m(i-a_s, j-b_s)} w(\gamma_m) \Bigg) z^i w^j \end{aligned}$$

Note that

$$\frac{d-1}{d} (cd)^{-m} \sum_{\gamma_m \in \Gamma_m(i-a_s, j-b_s)} w(\gamma_m)$$

is the probability that the walkers dies at $(i-a_s, j-b_s)$ on their $(m+1)$st step starting from $(0,0)$. Summing over all m, dividing by k, and summing over all starting points x_s, the coefficient of $z^i w^j$ has been shown to be the probability that the walker dies at (i,j) as desired. $\square$

3. Connection Between the Killed Random Walk and Leaky Sandpiles

In this section, I justify the necessary tangent of section 2 by connecting the KRW back to the region visited by the Leaky-ASM. I will focus on the case where n chips are uniformly distributed among k source points. This initial configuration is equivalent to the following sum of mass points:

$$\sum_{j=1}^{k} \frac{n}{k} \delta_{x_j}(x)$$

I will study the evolution of such a Leaky-ASM using the odometer function, which was originally introduced in [Dha06], and is defined as follows:

$$u(x) := \text{total mass emitted from x}$$

Note that $u(x)$ includes both "leaked" chips and chips sent to its nearest neighbors. Let T be the operator that does the following:

$$\begin{aligned} Tu(x) &= \Bigg(\sum_{y \sim x} \frac{c_{y\to x}}{cd} u(y) \Bigg) - u(x) \\ &= \text{total mass received by x} - \text{total mass emitted by x} \end{aligned}$$

where $y \sim x$ denotes that y is a neighbor of x, and $c_{y \to x}$ denotes the number of chips sent from y to x. Thus, $\frac{c_{y \to x}}{cd}$ is the portion of chips emitted from y that make it to x.

REMARK 3.1. *Note that T can be rewritten as a function of a weighted Laplacian, as shown below*:

$$T = \frac{1}{d}\Delta - \frac{d-1}{d}I$$

where Δ is the weighted Laplacian as defined below:

$$\Delta u(x) = \sum_{y \sim x} e_{y \to x} u(y) - u(x)$$

where $e_{y \to x}$ is the weight of the edge between y and x with $\sum\limits_{x : y \sim x} e_{y \to x} = 1$. That is $e_{y \to x} = \frac{c_{y \to x}}{c}$.

Note that under this rewrite, when we consider the ASM, we get that $T = \Delta$ where Δ is the standard Laplacian since all edge weights are $\frac{1}{4}$ in the ASM.

Let $f(x)$ denote the final configuration, and run the Leaky-ASM starting with k evenly distributed source points $x_1, ..., x_k$ with $\frac{n}{k}$ chips each. Then the odometer function satisfies the following equation:

$$Tu(x) = f(x) - \sum_{j=1}^{k} \frac{n}{k} \delta_{x_j}(x) \tag{3.1}$$

Note that $\frac{c_{y \to x}}{cd} = p_{y \to x}$, so there is a probabilistic interpretation for the operator T, which is given below:

LEMMA 3.2. *Applying the operator T to the death probabilities P_d leads to the following result*:

$$TP_d(x) = -\frac{d-1}{kd}\Big(\sum_{j=1}^{k} \delta_{x_j}(x)\Big) \tag{3.2}$$

Proof. Let S_m be the position of the walker after m steps. Let $P_d^m(x)$ be the probability that the walker dies after m steps at site x. As the walker can only either move or die,

$$P_d(x) = \sum_{m=0}^{\infty} P_d^m(x)$$

Suppose that x is not one of the k source points. Then the walker will need to take at least 2 steps to die at x, as the walker will need to first get to x, which will take at least one step, and then die there, which will take another step. Thus,

$$
\begin{aligned}
P_d(x) &= \sum_{m=2}^{\infty} P(S_{m-1}=x, K_{m-1}=1)\frac{d-1}{d} \\
&= \sum_{m=2}^{\infty}\sum_{y\sim x} P(S_{m-2}=y, K_{m-2}=1)P(S_{m-1}=x|S_{m-2}=y, K_{m-2}=1)\frac{d-1}{d} \\
&= \sum_{m=2}^{\infty}\sum_{y\sim x} P(S_{m-2}=y, K_{m-2}=1)p_{y\to x}\frac{d-1}{d} \\
&= \sum_{m=2}^{\infty}\sum_{y\sim x} P_d^{m-2}(y)p_{y\to x} \\
&= \sum_{y\sim x} p_{y\to x}\sum_{m=2}^{\infty} P_d^{m-2}(y) \\
&= \sum_{y\sim x} p_{y\to x}P_d(y) \\
&= TP_d(x) + P_d(x)
\end{aligned}
$$

Thus, $TP_d(x)$=0 for all x that is not a source point. To cover the remaining cases, suppose that x is a source point. The sole difference in this case is that the walker may just spawn and die at the site x with probability $\frac{d-1}{kd}$. Thus, we have the following result

$$
\begin{aligned}
P_d(x) &= \frac{d-1}{kd} + \sum_{m=2}^{\infty} P(S_{m-1}=x, K_{m-1}=1)\frac{d-1}{d} \\
&= \frac{d-1}{kd} + TP_d(x) + P_d(x)
\end{aligned}
$$

so $TP_d(x) = -\frac{d-1}{kd}$ for all source points. Stacking these points by using indicator functions to check whether or not x is a source point, we have the equation described in Equation 3.2. ☐

As the odometer and death probabilities satisfy similar equations, we can relate the 2, as shown below:

LEMMA 3.3. *For any $x \in \mathbb{Z}^2$ we have:*

$$\text{If } P_d(x) \geq \frac{cd}{n} \text{ then } u(x) \geq cd \tag{3.3}$$

$$\text{If } P_d(x) < \frac{c(d-1)}{n} \text{ then } u(x) = 0 \tag{3.4}$$

Proof. Combining Equation 3.2 and Equation 3.1 and applying the linearity of T results in

$$T(\frac{d-1}{dn}u(x) - P_d(x)) = \frac{d-1}{dn}f(x)$$

As $f(x)$ is a stabilized configuration, $0 \leq f(x) < cd$ since no sites are allowed to topple. This implies

$$0 \leq T(\frac{d-1}{dn}u(x) - P_d(x)) < \frac{c(d-1)}{n} \tag{3.5}$$

The proof of Lemma 3.2 shows that T is invertible and has the following inverse:

$$(T^{-1}f)(x) = -\frac{d}{d-1}\sum_{y\in\mathbb{Z}^2} P_d(x-y)f(x) \tag{3.6}$$

Note that the constant function 1 is an eigenvector of T and has a eigenvalue of $-\frac{d-1}{d}$. As such, 1 is an eigenvector of T^{-1} with a eigenvalue of $-\frac{d}{d-1}$. Applying T^{-1} to Equation 3.5, and using these eigenvalue results along with the fact that T^{-1} has only negative coefficients, the following emerges:

$$0 \geq \frac{d-1}{dn}u(x) - P_d(x) > -\frac{cd}{n}$$

This reduces to:

$$\frac{dn}{d-1}P_d(x) \geq u(x) > \frac{dn}{d-1}(P_d(x) - \frac{cd}{n})$$

So, if $P_d(x) \geq \frac{cd}{n}$, $u(x) > 0$, which results in $u(x) \geq cd$, as the minimum amount of emission per topple is cd, and if $u(x) > 0$, the site x has toppled. Similarly, if $\frac{dn}{d-1}P_d(x) < cd$ or $P_d(x) < \frac{c(d-1)}{n}$ then $u(x) < cd$ implies $u(x) = 0$, as a site cannot topple and emit less than cd. ☐

PROPOSITION 3.4. *For any $x \in \mathbb{Z}^2$ the following holds*:

$$\text{If } P_d(x) \geq \frac{cd}{n} \text{ then } x \in V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)} \tag{3.7}$$

$$\text{If } P_d(x) < \frac{c(d-1)}{n} \text{ then } x \notin V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)} \tag{3.8}$$

Proof. This follows immediately from Lemma 3.3 ☐

4. Limit Shape of the Leaky-ASM with k source points

In this section, I relate the death probabilities associated with the union of k Leaky-ASMs with 1 source point and the death probabilities associated with the Leaky-ASM with k source points. I will then use this relation to successfully prove the main theorem of this paper. Let $d > 1$, and $c_\uparrow = c_\leftarrow = c_\downarrow = c_\rightarrow = 1$. Thanks to Lemma 2.1, we can compute the coefficients in the

Laurent expansion of $P^{-1}(z,w)$ to find our death probabilities. A useful thing to note is the following:

PROPOSITION 4.1. *Let $Q^{-1}(z,w)$ be the Laurent expansion for the Laurent polynomial $\frac{cd-(c_{\uparrow}z+c_{\downarrow}z^{-1}+c_{\rightarrow}w+c_{\leftarrow}w^{-1})}{c(d-1)}$. Let S_k denote the set of source points. Then,*

$$P^{-1}(z,w)=\frac{1}{k}\Big(\sum_{x_s\in S_k} z^{-a_s}w^{-b_s}\Big)Q^{-1}(z,w) \tag{4.1}$$

Proof.

$$\begin{aligned}P^{-1}(z,w)&=\Big(\sum_{x_s\in S_k} z^{-a_s}w^{-b_s}\Big)\frac{c(d-1)}{cdk-k(c_{\uparrow}z+c_{\downarrow}z^{-1}+c_{\rightarrow}w+c_{\leftarrow}w^{-1})}\\&=\frac{1}{k}\Big(\sum_{x_s\in S_k} z^{-a_s}w^{-b_s}\Big)Q^{-1}(z,w).\end{aligned}$$

□

Note that $Q(z,w)$ has been shown to be the Laurent polynomial associated with 1 source point at the origin in [AM21]. We also know that the monomials in the sum are adjustments to the Laurent expansions to move the source point from the origin to a source point located at $x_s \in S_k$ since this procedure is the same procedure that was used in the proof of Lemma 2.1. As such, the Laurent expansion is the sum of the Laurent expansions associated with 1 source point adjusted for location of origin and then divided by k. Note that for the union, instead of the sum, $\max_{x_s\in S_k} z^{-a_s}w^{-b_s}Q^{-1}(z,w)$ is the appropriate Laurent expansion to find corresponding death probabilities. This is due to the fact that for a point x to be in the limit shape of the union, it only needs to clear the death probability threshold from one source point. Thus, for a given $x=(i,j)\in\mathbb{Z}^2$, the following emerges:

PROPOSITION 4.2. *Let $f(x)$ be the coefficients of $P^{-1}(z,w)$ for z^iw^j. Also let $f_s(x)$ be the coefficients of $z^{-a_s}w^{-b_s}Q^{-1}(z,w)$ for z^iw^j. Then the following inequality holds:*

$$k*f(x)\geq \max_s f_s(x)\geq f(x) \tag{4.2}$$

Proof. The inequality follows immediately from the fact that $f(x)=\frac{1}{k}\sum_{s=1}^{k} f_s(x)$, which is an immediate consequence of Proposition 4.1 since adding Laurent expansions together just involves combining like terms, and all that is happening here is comparing coefficients of said like terms. □

The following lemma is a necessary step for the proof of the main theorem

Lemma 4.3. *For sufficiently large n, given a point $x \in \mathbb{Z}^2$ such that*

$$d\Big(\bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}, x\Big) \geq \sqrt{\log(\frac{n}{k})}$$

where (a_j, b_j) are the coordinates of the jth source point and $d(\cdot,\cdot)$ is the distance function in $\mathbb{Z}^2$, then $x \notin V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)}$

Proof. Thanks to the paper of Alevy and Mkrtchyan [AM21], we know that the radius of the limit shape from one source point is on the order of $\log(m) - \frac{1}{2}\log\log(m)$ where m is the initial number of chips at the source point and the associated coefficient is bounded by values dependent on d. So the distance between x and the nearest source point is greater than or equal to $\log(\frac{n}{k}) + \sqrt{\log(\frac{n}{k})} - \frac{1}{2}\log\log(\frac{n}{k})$. Consider the limit shape of the union where each source point gets n chips instead of $\frac{n}{k}$ chips. The radius of this shape is similarly, on the order of $\log(n) - \frac{1}{2}\log\log(n)$ with the same bounds on the coefficient. Subtracting this second radius from the first, we find:

$$\begin{aligned}
&\log(\frac{n}{k}) + \sqrt{\log(\frac{n}{k})} - \frac{1}{2}\log\log(\frac{n}{k}) - (\log(n) - \frac{1}{2}\log\log(n)) \\
&= \sqrt{\log(\frac{n}{k})} - \log(k) + \frac{1}{2}\log\log(k) > 0
\end{aligned}$$

where the inequality follows from the fact that n is large. So, x is outside the radius of the limit shape for n chips started at any source point x_s. Thus, thanks to Alevy and Mkrtchyan's paper [AM21], we also get that $f_s(x) < \frac{c(d-1)}{n}$ for sufficiently large n where $f_s(x)$ is the death probability for the KRW started at x_s. Obviously, this inequality holds for all source points x_s. Thus, by Proposition 4.2, we have that $f(x) \leq \frac{c(d-1)}{n}$, and so by Propositon 3.4, $x \notin V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)}$. □

We are finally in position to finish the proof of the main theorem

Proof of Theorem 1.1. Remark 1.2 covers inclusion in one direction, so all that remains is to show inclusion in the other direction. Note that as $n \to \infty$, $\frac{\log(\frac{n}{k}) + \sqrt{\log(\frac{n}{k})} - \frac{1}{2}\log\log(\frac{n}{k})}{\log(\frac{n}{k}) - \frac{1}{2}\log\log(\frac{n}{k})} = 1$. Thus, by Lemma 4.3, the set of points not in $\bigcup_{j=1}^{k} V_{d,\frac{n}{k},(a_j,b_j)}$ converges to the set of points not in $V_{d,n,(a_1,b_1),\ldots,(a_k,b_k)}$ as $n \to \infty$. As the sets of excluded points converge, the set of included points must also converge, and so, inclusion in the other direction arises and we are done. □

5. Extension: log-box starts

In this section I will motivate and showcase the simulations for the starting configuration of having all points in the square with side-lengths of $\log(n)$

centered at the origin as source points. The earlier work of this paper defines the limit shape for finite source points, so an obvious extension is to start considering sets of source points that are not necessarily finite. However, to enable this situation to not purely be the starting configuration for some leakiness parameter, the starting configuration needs to grow at a rate such that $\lim_{n\to\infty} \frac{\text{number of source points}}{n} = 0$. Since the limit shape generated by one source point has a radius on the order of $\log n$ and 8-fold symmetry, a configuration that may be interesting to explore is a square with side-lengths of $log(n)$. Such a configuration will satisfy the 2 conditions listed, have 8-fold symmetry along the same axes as the limit shape generated by one source point centered at the origin, and also have a starting configuration that has radius on the same order of the limit shape generated by one source point. It also creates issues with the argument using the killed random walk, as the death probabilities all go to 0 for every point $x \in \mathbb{Z}^2$. Since there cannot exist a Laurent polynomial whose inverse is 0, the critical Lemma 2.1 fails. Thus, the proof above does not hold for such a configuration and may generate a different limit shape. Heuristically, as

$$\lim_{n\to\infty} \frac{\log(\frac{n}{\log^2(n)}) - \frac{1}{2}\log\log(\frac{n}{\log^2(n)})}{\log(n) - \frac{1}{2}\log\log(n)} = 0$$

the limit shape should be the union of the limit shapes generated from singular source points in the square. However, when we look at the simulations shown by Figures 9 to 15, this claim seems significantly more dubious.

Visually speaking, it is interesting to note that Figures 1 and 8 look very similar. However, given the lack of understanding of the limit shape of the ASM, more commentary on this particular result is not possible at this time by me. What is significantly more unexpected is what happened to the simulation results as leakiness increased. Figures 9 and 10 appear as expected based on the heuristic result. That is, they look approximately like a union of a bunch of circles. However, thanks to the work of Alevy and Mkrtchyan in [AM21], we know that as leakiness increases, the limit shape from one source point approaches a diamond. While the corners of the diamonds probably do explain the straight edges we see in Figures 11 to 15, the diamonds do not cleanly explain the curves that we see on those same figures. When looking beyond just the limit shapes, the simulations also showcase some interesting internal phenomenon. Namely, the central square that appears in Figures 12 to 15, as there is nothing in the finite case simulations to suggest that such a structure should arise. It is of note that the relative values of the box depend on both n and d as can be seen by comparing figures 12, 13, and 14. The set of straight lines with increasing length as they approach the edge present in Figures 11 to 15 are also an unexpected feature in the visualization, but similarly, I do not have an explanation for their existence at this time.

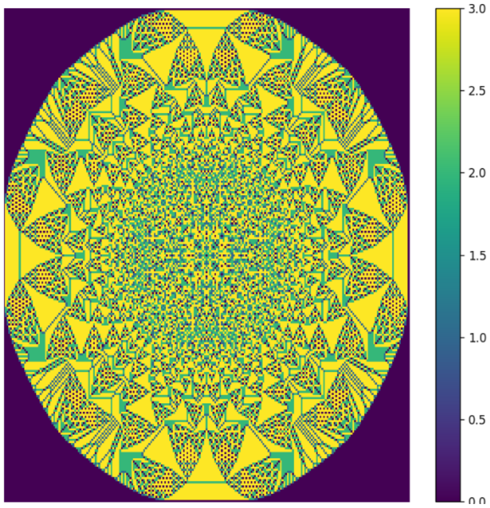

Figure 8.
$n = 10^5, d = 1$

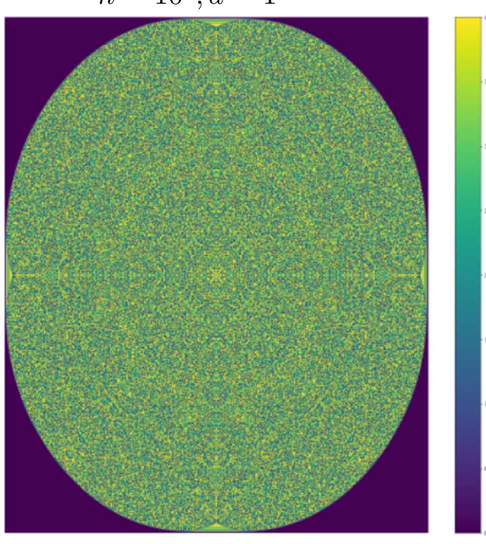

Figure 9.
$n = 10^{100}, d = 1.25$

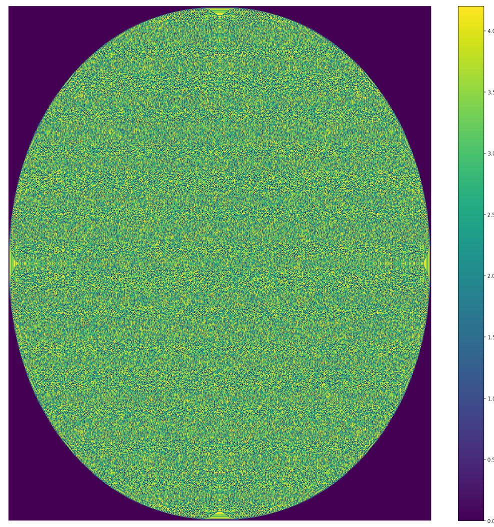

Figure 10.
$n = 10^{50}, d = 1.05$

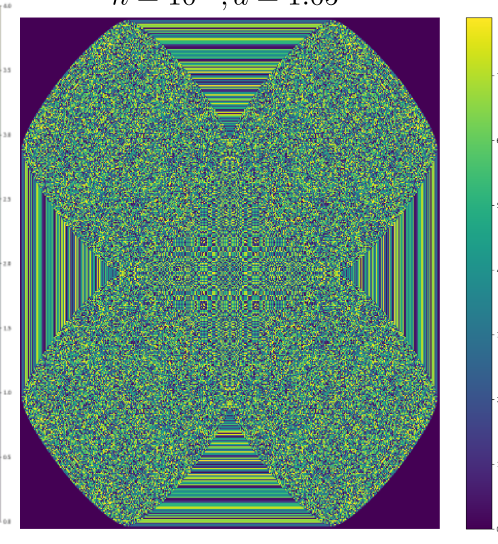

Figure 11.
$n = 10^{200}, d = 20$

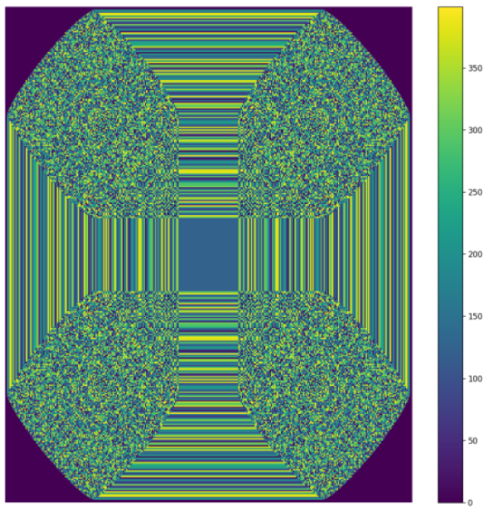

Figure 12.
$n = 10^{200}, d = 100$

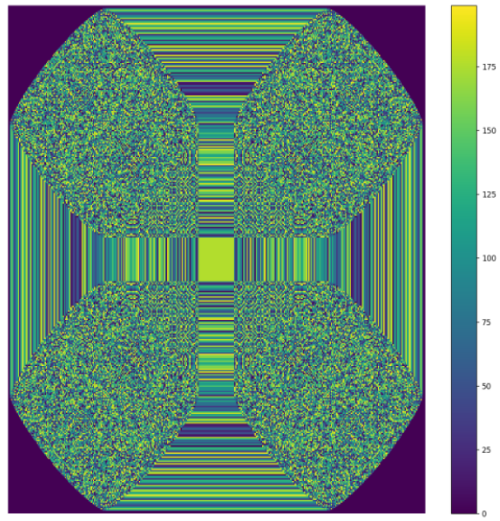

Figure 14.
$n = 10^{200}, d = 50$

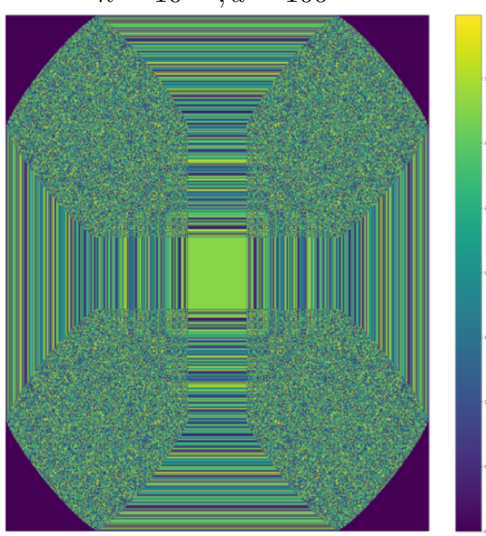

Figure 13.
$n = 10^{400}, d = 100$

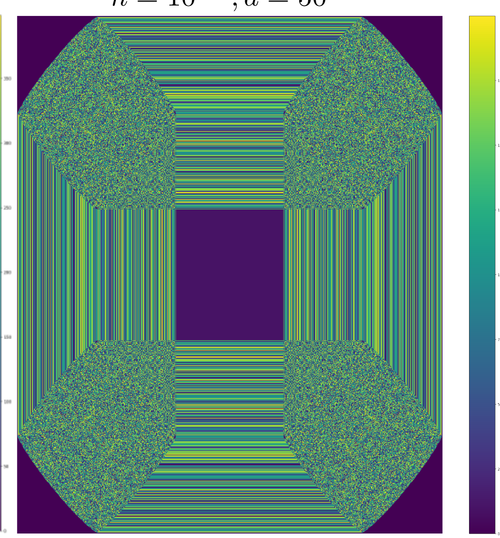

Figure 15.
$n = 10^{500}, d = 500$

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