Quandle Invariants of Knots and Links

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Abstract

Quandles are an algebraic structure whose axioms arise from the Reidemeister moves in knot theory. Given a knot K one can associate it with a quandle Q_K called the *fundamental quandle*. It was shown by David Joyce (1982) that if Q_K is isomorphic to $Q_{K'}$ then the knots K and K' are equivalent up to orientation. Furthermore, Fenn and Rourke (1992) showed that the fundamental quandle is a complete invariant up to mirror image for non-split links. The fundamental quandle is very powerful in that all other classical invariants of knots can be viewed through their fundamental quandles. This survey provides a quick introduction to knot and quandle theory, gives examples of how we can view weaker invariants in-terms of the fundamental quandle through *quandle-colorings*, and describes some strong invariants derived from the fundamental quandle.

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Introduction

From tying down masts on a ship to lacing shoes, our base intuition of a knot is a jumbled mess of string which defies untangling. The goal of knot theory is to distinguish knots up to our ability to maneuver them in space.

A naive approach to defining a knot may be: a simple smooth curve in $\mathbb{R}^3$ which is unable to be simplified via continuous transformation without breaking through itself. This definition is flawed as we note that it is possible to deform any simple smooth unclosed curve to the interval $I(t) = \{(t, 0, 0) | t \in [0, 1]\} \subset \mathbb{R}^3$ by taking an endpoint and feeding it along the curve to untangle it. So any two smooth, simple, and unclosed curves are equivalent, making our first definition of a knot quite useless. However, by fusing the ends of the curve together we may actually begin to distinguish them.

Definition 1. *A knot is a smooth embedding of the circle into three dimensional Euclidean space.*

Sometimes it is beneficial to view a knot as an embedding of the circle into the one-point-compactification of $\mathbb{R}^3$, the 3-sphere.

$$S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$$

This is possible by taking a standard knot in $\mathbb{R}^3$ and maneuvering it to avoid the additional point at infinity.

Definition 2. *A link is a disjoint union of knots.*

By definition a knot is always link. Since knots and links are (locally) the same, unless stated otherwise, will we use the terminology somewhat interchangeably. We proceed to state how we are allowed to maneuver a knot mathematically, in a way which aligns with our physical intuition.

Definition 3. *Given two embeddings f, g from a manifold $\mathscr{X}$ into another manifold $\mathscr{M}$, a continuous function $H : \mathscr{X} \times [0,1] \longrightarrow \mathscr{M} \times [0,1]$ is called an ambient isotopy if $(x, 0) \mapsto f(x)$ and $(x, 1) \mapsto g(x)$ where $H(x, t)$ is an embedding for all $t \in [0, 1]$, and we say f and g are ambient isotopic.*

We say that two links are equivalent if there exists an ambient isotopy between them.

It would be difficult to study knots if we always had to work with these embeddings. The main way we get around this is through *knot diagrams*: Given a knot K, a knot diagram of K is a projection of it's image in $\mathbb{R}^3$ to a suitable plane: the projection must be bounded and have a finite number of points where the projection is not one-to-one, these points are called *singularities.*

Singularities correspond to a crossings in the knot diagram. The preimage of singularities when viewed in S^1 are called *singular points.* There are exactly two singular points for each singularity – one for the overstrand and understrand respectively. If $S = U \sqcup O \subset S^1$ is the set of singular points partitioned into under/overstrand sets, then an *arc* in a knot diagram is the image of a connected component of $S^1 - U$. At each crossing in the diagram we note which arc crosses over the other two, for example:

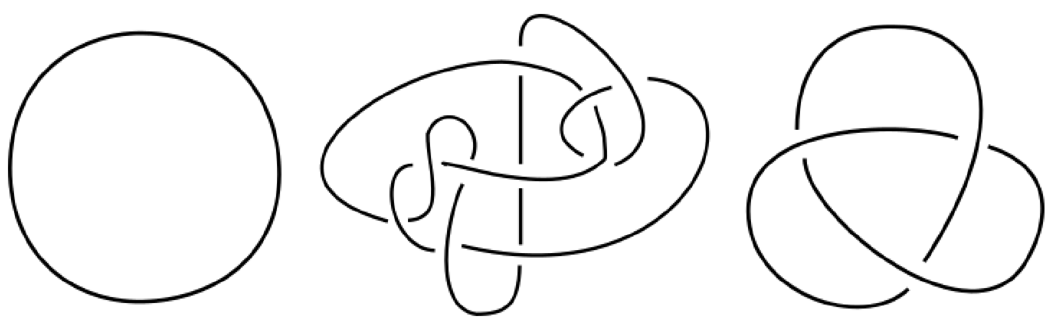

Figure 1: Three knots

Viewing the above as a link, singularities of a link diagram remain the same. However, arcs in a link diagram are the image of connected components avoiding understrand singular points when restricted to a single S^1.

Two of the three knots in Figure 1 are actually the same knot. The simplest knot, on the far left, is called the *unknot* and is obtained by the inclusion map $i : S^1 \to \mathbb{R}^3$. While the rightmost knot is the trefoil knot.

How can we show that two knots are the same? We may try to exhibit an ambient isotopy between either of the knots. But, beyond modeling the knot physically, it is difficult to show two knots are actually the same. Luckily, a theorem by Reidemeister [1] gives us an equivalent condition on knot diagrams to the existence of an ambient isotopy taking one knot to the other.

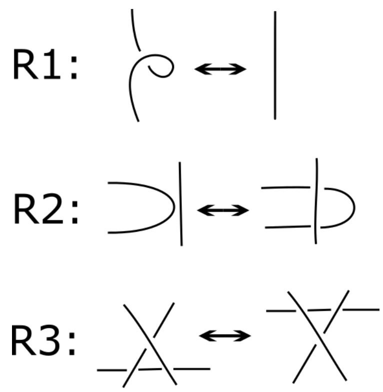

Figure 2: The Reidemeister Moves

Theorem 1. *For two links L and L', there is an ambient isotopy between them if and only if their diagrams are related by a finite sequence of moves in Figure 2 along with planar isotopies.*

The Reidemeister moves serve as a codification of the ways we maneuver a knot in three dimensional space. The first move adds a twist, the second move crosses one strand over/under another, and the third move passes a strand over/under a pre-established crossing. Using Reidemeister's theorem we are able to show that the complicated knot diagram in Figure 1 is actually the unknot, we do this in Figure 4.

An oriented knot is a knot along with a specified direction, this is typically signified by arrows along the knot diagram. There is an analog of Theorem 1 for oriented knots which can be found in [2]. Given an oriented knot we may reverse the orientation to obtain its *mirror image*. For some oriented knots it is possible to distinguish between mirror images – in this case we call the unoriented version of the knot *chiral* – the trefoil knot is the simplest example. One byproduct of adding an orientation is that there are now two types of crossings which we denote by left/right handedness. We do this because it is easy to determine what type of crossing you have by pointing your index-finger along the direction of the overstrand (palm down) and noting which hand has the thumb pointing along the outgoing understrand.

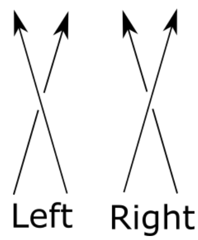

Figure 3: Left and Right Handedness

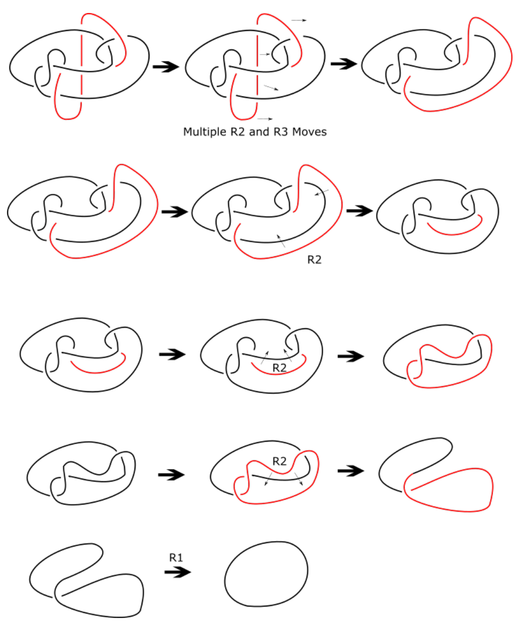

Figure 4: Unraveling a Complicated Unknot

Elementary Invariants

We've seen that the Reidemeister moves give a method to show equivalence of knots. But how do we know that the three knots in Figure 1 aren't actually all the same knot? This is a big problem since it's impossible to show that two knots are different using purely Reidemeister moves – it may be possible to simplify a knot by introducing some complexity.

A knot (resp. link) invariant is a function on the space of all knots which remains the same under ambient isotopy. For a function defined on a knot diagram to be an invariant, by Theorem 1, it is equivalent that the function be invariant under the Reidemeister moves.

Geometric Invariants

An obvious way to begin distinguishing knots is to take a geometric quantity of a knot diagram and take the minimum over all possible diagrams of the knot. Some examples are:

- Crossing Number - The minimal number of crossings of any diagram.
- Uncrossing Number - The minimal number of crossing changes (overstrand becomes the understrand) needed to obtain the unknot or unlink.
- Genus - The minimal number of holes in a surface whose boundary is a knot K.
- Length - The minimum length of a knot or link if we give the strands a uniform thickness.

Fox n-coloring

Our next two invariants are classics. Here we present the interpretation by Ralph Fox [3]. Given a knot diagram, we color each of the arcs one of three colors such that at each crossing either all of the arcs are colored the same or are unique. A trivial coloring is one which uses a single color. A knot is *tricolorable* if there exists a non-trivial coloring of its knot diagram.

Theorem 2. *Tricolorability is a knot invariant.*

Proof. It suffices to show that tricolorability is conserved under the Reidemeister moves:

□

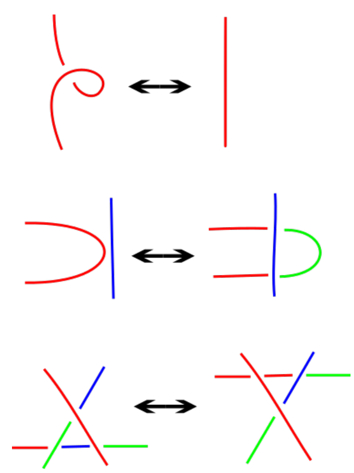

Figure 5: Visual Proof of Theorem 2

Since we know that tricolorability is an invariant we may finally be able to distinguish the knots in Figure 1. First we note that since the unknot can only be colored trivially, a tricolorable knot cannot be the unknot. Coloring each arc of the trefoil a different color shows that the trefoil $\not\simeq$ unknot.

It is natural to consider a coloring invariant which uses more than three colors. If, instead of colors, we label each arc with an element of $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ we note that the crossing conditions for tricolorability are equivalent to

$$2y - x - z \equiv 0 \pmod 3$$

where y is the label corresponding to the overstrand and x, z are the labels for the two arcs of the understrand. By considering this condition over $\mathbb{Z}_n$ we generalize tricolorings to Fox n-colorings.

Definition 4. *Given a knot diagram, a Fox n-coloring is a labeling of the arcs by elements of $\mathbb{Z}_n$ such that at each crossing the equation*

$$2y - x - z \equiv 0 \pmod{n}$$

holds. Where y is the overstrand and x, z are the understrand.

Using the same proof as for tricolorability we see that Fox n-colorability is an invariant. Furthermore, the number of Fox n-colorings for a given diagram is also invariant [4].

The Knot/Link Complement

All of our previous invariants have been built by coloring the arcs in a link diagram. Now we switch gears and consider the ambient space after we carve out the link.
Given a link $L \subset S^3$ the *link complement* (or *knot complement* resp.), is given by $S^3 \setminus L$ and is invariant up to ambient isotopy. If the link is actually a knot then it's complement is a perfect invariant – meaning it is distinct for any two unique knots. This is not the case for links. Furthermore a link complement is a $K(\pi, 1)$ space, $\pi_n(S^3 \setminus L)$ is trivial for $n > 1$. The next definition will require knowledge found in Appendix A.

Definition 5. *We call the fundamental group of a knot complement, $\pi_1(S^3 \setminus K)$, the knot group and denote it by $\pi_1(K)$.*

Theorem 3. *For a knot K, $\pi_1(K) \cong \mathbb{Z}$ if and only if K is the unknot.*

We will only prove ($\Leftarrow$). To calculate the knot group of the unknot we first choose a basepoint. Every loop either links with the unknot or doesn't. If it doesn't then we can shrink down the loop until it is trivial. If the loop links with the unknot then (up to ambient isotopy) it must wrap around the unknot a minimum of $k \in \mathbb{Z}$ times (sign corresponds to orientation). Any loop with the same linking number is ambient isotopic. So the fundamental group is the free group generated by a single element.
A proof of ($\Rightarrow$) can be found in [5].

The knot group is generated by loops going around each arc. The *Wirtinger presentation* is the most common way to describe the knot group utilizing one of its diagrams. It has as its generators loops which go once around each arc of the diagram, and relations corresponding to each crossing.
The type of relation you get depends on the handedness of the crossing

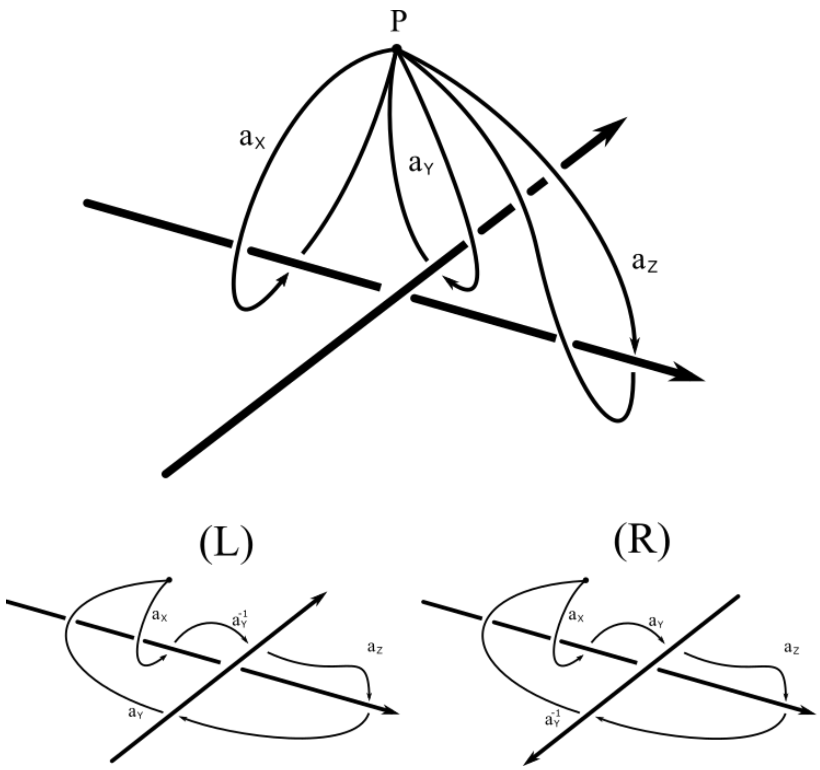

Figure 6: Wirtinger Presentation

$$a_x a_y^{-1} a_z a_y = 1 \overset{\text{or}}{\iff} a_z a_y a_x = a_y \tag{L}$$

$$a_x a_y a_z a_y^{-1} = 1 \overset{\text{or}}{\iff} a_x a_y a_z = a_y \tag{R}$$

The Wirtinger presentation of the knot group is then the free group on the generators modulo the smallest normal subgroup containing the set of relators of the form $a_x a_y^{-1} a_z a_y$ or $a_x a_y a_z a_y^{-1}$.

Let D_{2n} denote the dihedral group (the group of isometries of a regular n-gon). D_{2n} has a presentation: $D_{2n} = \{\alpha, s : \alpha^n = 1 = s^2, s\alpha s = \alpha^{-1}\}$. The rotations are the set $\{\alpha^k\}$ and is a cyclic subgroup of D_{2n} isomorphic to $\mathbb{Z}_n$. Reflections can all be written as $s_k := s\alpha^k$.

Theorem 4. *The set of Fox n-colorings of a knot K are in bijection with homomorphisms from the knot group's Wirtinger presentation to D_{2n}, which send the generators to reflections.*

Let $A = \{a_i\}$ be the arc set of a knot diagram of K. A Fox n-coloring is a map $C : A \to \mathbb{Z}_n$ which satisfies the condition $2y - x - z \equiv 0 \pmod{n}$ at each crossing. It is easy to verify that the mapping $a_k \mapsto s_{C(a_k)} \in D_{2n}$, determines a nontrivial homomorphism from the knot group to D_{2n}. Conversely, any nontrivial homomorphism arises in this way.

All the power of Fox n-colorings for knots follows from information in the knot group. Differentiating knots through presentations of their knot groups is a very difficult problem. As seen by Theorem 4 it is often easier to study maps emanating from the knot group rather than the knot group itself. This is a core idea for the invariants to come. However, to move forward we must gain a tool better at differentiating knots than the fundamental group.

Quandle Invariants

In order to improve the coloring invariants from last chapter we must realize a generalized version of our coloring set. With Fox n-colorings we took our colors to be elements of $\mathbb{Z}_n$ and then chose labels under the condition in Definition 4. After rearranging the equation we see that the label of an understrand is determined by the other two arcs. We will now consider what happens when we take a general set and impose an algebraic structure motivated by the Reidemeister moves.

Kei and Quandles

A Kei is a right-distributive groupoid which has the Reidemeister moves encoded in its structure. This is seen by first labeling each arc of a knot diagram by an element of a set X. We then say that if x is an understrand at a crossing, then overstrand y acts on x by right multiplication:

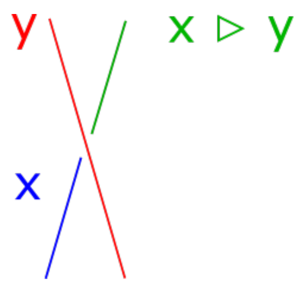

Figure 7: Kei Crossing Relation

Definition 6. *A **Kei** is a set X paired with a binary operation $\triangleright$ such that:*

(Idempotent) For all $x \in X$, $x \triangleright x = x$.

(Involutory) For all $x, y \in X$, $(x \triangleright y) \triangleright y = x$.

(Self-Distributive) For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

The Kei axioms follow by assuming the crossing relation holds and then forcing the labeling to be invariant under the Reidemeister moves. The first and second kei axioms correspond the first and second Reidemeister moves respectively. The third axiom can be see from the following figure, by evaluating the label of the orange strand in two different ways:

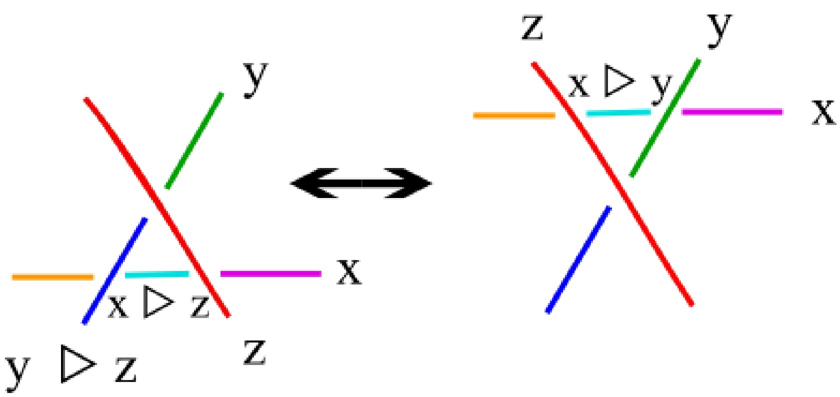

Figure 8: Kei Axiom 3

Each $y \in X$ defines a map $\beta_y : X \to X$ where $x \mapsto x \triangleright y$. The involutory condition says that this action is its own inverse.

From now on we will be primarily be dealing with oriented knots. Similar to Kei, a Quandle is an algebraic structure, which encodes the oriented version of the Reidemeister moves. We've seen that by adding an orientation we also introduce two different types of crossings called right/left handedness. So in our quandle formulation we initially start with two operations $\triangleright$ and $\triangleright^{-1}$ which will be used at left and right handed crossings respectively.

Definition 7. *A **quandle** is a set Q equipped with two binary operations $\triangleright$ and $\triangleright^{-1}$ which satisfies the following for all $x, y, z \in Q$:*

1. *(Idempotent): $x \triangleleft x = x$.*
2. *(Right-Action Invertible): $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$*
3. *(Right Self-Distributive): For all $x, y, z \in Q$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.*

See that every kei is a quandle (typically called involutory quandles) where $\triangleright = \triangleright^{-1}$. Additionally, the second axiom for quandles is equivalent to the right-action map $\beta_y : Q \to Q$ being invertible for all $y \in Q$. So we can actually just forget $\triangleright^{-1}$ and consider a quandle to be a pair $(Q, \triangleright)$ which satisfies the above.
A function $\Phi : (Q_1, \triangleright_1) \to (Q_2, \triangleright_2)$ is a *quandle homomorphism* if

$$\Phi(x \triangleright_1 y) = \Phi(a) \triangleright_2 \Phi(b)$$

for all $a, b \in Q_1$. The set of homomorphisms from Q_1 to Q_2 is denoted by $\mathrm{Hom}(Q_1, Q_2)$ and is equipped with a group structure via the composition operation.

The self-distributive property of quandles implies that β_y is a quandle homomorphism for every $y \in Q$, and so is a quandle automorphism. We call each β_y the *point-symmetry about* y and the subgroup of $Aut(Q)$ generated by the point symmetries of Q is called the *inner automorphism group of* Q and is denoted by $Inn(Q)$.

Example 1. *Given any set X, the operation $\triangleright$ such that $x \triangleright y = x$ for all $x, y \in X$ is called the* ***trivial quandle*** *on X. We let T_n denote the trivial quandle on n elements.*

Example 2. *Given any group you can construct a quandle by taking $Q = G$ and letting $\triangleright$ be the conjugation operation $x \triangleright y = y^{-1}xy$.*

Idempotent: $x \triangleright x = x^{-1}xx = x$.

Right-Action-Invertible: $x = (x \triangleright y) \triangleright y^{-1}$

Right Self-Distributive:

$$\begin{aligned} (x \triangleright y) \triangleright z &= z^{-1}(y^{-1}xy)z \\ &= z^{-1}y^{-1}(zz^{-1})x(zz^{-1})yz \\ &= (z^{-1}yz)^{-1}(z^{-1}xz)(z^{-1}yz) \\ &= (x \triangleright z) \triangleright (y \triangleright z) \end{aligned}$$

Example 3. *For any vector space V and invertible linear transformation $M : V \to V$, we can define a quandle structure on V called the* ***Alexander quandle****:*

$$\vec{u} \triangleright \vec{v} := M(\vec{u} - \vec{v}) + \vec{v}$$

Example 4. *Let $Q = \mathbb{Z}_n$ and $a \triangleright b = 2b - a$. This is a quandle structure on Z_n and is called the* ***Dihedral quandle****, denoted by $\mathcal{D}_{2n}$. This is because it is isomorphic to the quandle obtained from the dihedral group D_{2n} through the process in Example 2.*

The Fundamental Quandle

The fundamental quandle is a similar construction to the knot group and allows us to construct better invariants! To begin using quandle invariants we must describe how to obtain a quandle from any set. This will later be applied to the arc set of a knot diagram along with additional conditions derived from crossings.

Given a set X the set of *quandle words* $W_Q(X)$ is defined recursively by

1. $x \in X \implies x \in W_Q(X)$
2. $x, y \in W_Q(X) \implies x \triangleright y \in W_Q(X)$

A quandle word is then a finite string of elements and $\triangleright$'s paired with parenthesis such that it makes sense as a quandle product.

Definition 8. *The* ***free quandle*** *on X is $W_Q(X)/\sim$ where $\sim$ is defined by:*

$$\begin{aligned}(x \triangleright x) &\sim x\\ (x \triangleright y) \triangleright y &\sim x\\ (x \triangleright y) \triangleright z &\sim (x \triangleright z) \triangleright (y \triangleright z)\end{aligned}$$

Note: we follow the same conventions for quandle presentations as we do for group presentations: if $\{g_i\}_{i\in I}$ is a set of generators and $\{r_j\}_{j\in J}$ a set of relations, then

$$\langle \{g_i\} | \{r_j\} \rangle$$

is the free quandle on $\{g_i\}_{i\in I}$, modulo the relations $\{r_j\}_{j\in J}$.

Now, given an oriented link L, if A is the arc-set of a given diagram of L we may interpret each crossing as giving us one of the following *relations*

$$x \triangleright y = z$$

$$x \triangleright^{-1} y = z$$

where z is the outgoing understrand. Which relation we use depends on the type of crossing, as seen below.

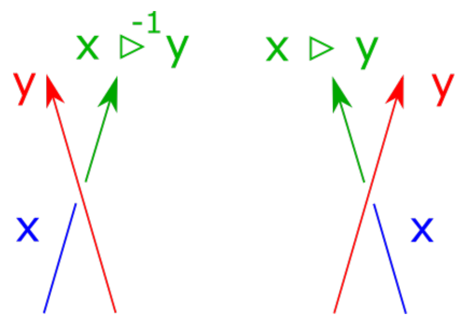

Figure 9: The Quandle Crossing Relation

Definition 9. *Given a diagram of a link L, the* ***fundamental quandle*** Q_L *is the free quandle on the arc-set A modulo the equivalence relations generated by the crossing relation.*

Theorem 5. *The fundamental quandle is a link invariant.*

Proof. We will show how the quandle axioms are motivated by the Reidemeister moves in such a way that the fundamental quandle is locally invariant.

R1: Going from one strand, labeled x, to a twist we know that two of the arcs must be labeled x. The other strand is $x \triangleright x$, so in order for it to be invariant we must have $x \triangleright x = x$ which follows from the first quandle axiom.

R2: Comparing the left and right sides of the R2 move, we require $y \triangleright z = x$. See that given any $z, x \in Q_L$ there should be a unique y such that $y \triangleright z = x$. This means that the map $\beta_x(z) : Q_L \to Q_L$ defined is injective. Since the strand on the left could have been any label in Q_L, this map β_x is also surjective.

R3: The left and middle strand labels match already. All that's left is to require $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$, which is the right distributive condition required in quandle axiom 3. $\square$

Example 5. *Here we will calculate the fundamental quandle of the oriented trefoil knot T. We start with three generators a, b, c, each corresponding to one of the arcs in Figure 1. The following choice is used only to exploit the three-fold symmetry of T. First choose an orientation, then label the strands as we traverse the knot so as to label them in reverse alphabetical order. We obtain the following crossing relations:*

$$\begin{aligned} a \triangleright b &= c \\ b \triangleright c &= a \\ c \triangleright a &= b \end{aligned}$$

Thus Q_T is partially given by the following operation table.

$\triangleright$	a	b	c
a	a	c	$\square$
b	$\square$	b	a
c	b	$\square$	c

We get that Q_T is infinite and given by:

$$Q_T = \langle a, b, c | a \triangleright b = c, b \triangleright c = a, c \triangleright a = b \rangle$$

We now give a geometric description of the fundamental quandle of a knot (called knot quandle) adapted from [**6**]. Let K be an oriented knot in $\mathbb{R}^3$, and let $N(K)$ be a small tubular neighborhood about K, let $E(K) = \overline{(\mathbb{R}^3 \setminus N(K))}$. We let Γ_K be the set of homotopy classes of paths in the space $E(K)$ with a fixed initial point, p, and endpoint on $\partial N(K)$. Let $m_y \subset E(K)$ be an oriented meridian of the tubular neighborhood hooking an arc, y, of the knot. Define $x \triangleright y = [\overline{x} \circ \overline{y^{-1}} \circ \overline{m_y} \circ \overline{y}]$, where $\overline{x}$ is a representative path of $x \in \Gamma_K$ and we view each arc a as an element of Γ_K where $\overline{a}$ is a path from p to a point on the boundary of the torus $\partial N(K)$ about the arc a and the path must travel only 'over' the knot. The quandle axioms are easily checked, To see how Γ_K is equivalent to Q_K from Definition 9, see Theorem 3.1 in [**6**].

The knot group acts naturally on the knot quandle. Fix a point p outside of the tubular neighborhood used as a basepoint for both the quandle and group. For a loop $\gamma \in \pi_1(K)$ and element δ of the quandle, $\gamma(\delta) = \delta \circ \gamma \in \Gamma_K$. Furthermore, under this interpretation there is a natural map from the knot quandle to the knot group. For each element x of the knot quandle (a path from p to $\partial E(K)$) we may associate the loop $x^{-1} \circ m \circ x$, where m is the meridian passing through the endpoint of x. This shows that the knot group can be constructed from the knot quandle by taking meridians as generators and replacing all relations of the form $x \triangleright y = z$ with $yxy^{-1} = z$.

Fenn and Rourke [**7**], proved that the fundamental quandle is a complete invariant up to mirror image for non-split links. For two non-equivalent knots K, K' their fundamental quandles will not be isomorphic. Though, if K' is simply the reverse orientation of K then their fundamental quandles may or may not be isomorphic. The trefoil knot is chiral, meaning that you can distinguish between the trefoil's two orientations; to see the limitation of fundamental quandle above simply reverse the orientation in Example 5.

Quandle Colorings

The way we obtain the fundamental quandle is typically through a presentation determined by a given link diagram. Similar to the theory of groups, distinguishing between two quandle representations extremely difficult – no easier than differentiating the link diagrams themselves. Thus, quandle invariants will typically employ mappings from the fundamental quandle in order to exploit linear algebra. This is analagous to Theorem 4 where Fox n-colorings are viewed as maps from the knot group and, more broadly, to representation theory where we can study of a group G by looking at the homomorphisms into the general linear group of a vector space, $GL_n(V)$. The following three sections have been adapted from [**8, 9, 10**].

Definition 10. *Let L be an oriented link with fundamental quandle Q_L and X be a finite quandle which we will call the coloring quandle. The set of quandle homomorphisms from the fundamental quandle to X,* $\mathrm{Hom}(Q_L, X)$*, is called the coloring space. The* ***quandle coloring invariant*** *is the cardinality of the coloring space,* $|\mathrm{Hom}(Q_L, X)| := \Phi_X(L)$.

To see how each element of $\mathrm{Hom}(Q_L, X)$ can be interpreted as a coloring of the diagram of L, remember that arc labels of L are the generators of the fundamental quandle Q_L. A valid coloring is then an assignment of an element of X to each element of the arcset A of a given diagram of L which respects the quandle operation for each crossing relation [9] when the labels are viewed in X. See that this means a coloring for one diagram can naturally be made into an admissible coloring for any other diagram.
For any coloring of L by the quandle X we may associate it with a coloring map $C : Q_L \to X$ where if an arc labeled a in the fundamental quandle and

is assigned the color $x \in X$, then $a \mapsto x$. Furthermore, the map $C : Q_L \to X$ is a homomorphism. Take a crossing as in [9] where $C(a) = x$, $C(b) = z$, and $C(a \triangleright b) = y$, then since the crossing relation requires that $x \triangleright z = y$ we get that $C(a) \triangleright C(b) = C(a \triangleright b)$ for any two generators x, y of Q_L.

Theorem 6. *The Fox n-coloring invariant is related to $\Phi_X(L)$ where X is taken to be the dihedral quandle on n elements.*

Where our definition of Fox n-colorability was conveniently chosen to ignore trivial colorings obtained by using the same color throughout a diagram, the quandle coloring invariant (as a consequence of the first quandle axiom) does not differentiate. In other words, for any finite quandle X with $|X| = n$, $\mathrm{Hom}(Q_L, X)$ will have at least n elements corresponding to the constant maps. This is seen through the trefoil knot T, which has six Fox 3-colorings but $\Phi_{\mathscr{D}_{2\cdot 3}}(T) = 9$. To see the latter simply note that homomorphisms are uniquely determined by where we send the generators of Q_T. By Example 5 there are three generators of Q_T, but once we choose where to send any two of them the one remaining is locked-in. There are three choices for each generator for a total of 9 homomorphisms.

The Quandle Coloring Quiver

We now have a wide array of integer-valued link invariants we can employ. Each invariant corresponds to a finite quandle X; the number of which is strictly larger than the number of finite groups! However, we lose a lot of information by going from $\mathrm{Hom}(Q_L, X)$ to $|\mathrm{Hom}(Q_L, X)|$. The goal of this section is to create an invariant which in some sense captures the structure of $\mathrm{Hom}(Q_L, X)$. To do this we consider a directed graph (called a *quiver*) obtained from the elements of $\mathrm{Hom}(Q_L, X)$.

Definition 11. *Let X be a finite quandle and L an oriented link. The associated **quandle coloring quiver**, denoted $\mathbb{Q}_X(L)$, is the directed graph with a vertex for every element $f \in Hom(Q_L, X)$ and an edge directed from f to f' when $f' = \sigma f$ for an element $\sigma \in \mathrm{Hom}(X, X)$.*

Theorem 7. *$\mathbb{Q}_X$ is a link invariant.*

Proof. Given a link L, Q_L is fixed by Theorem 5. Thus $\mathrm{Hom}(Q_L, X)$ is completely determined by our choice of the finite quandle X. $\square$

See that $\mathbb{Q}_X$ is an enhancement of the quandle coloring invariant since we can recover $\Phi_X(L)$ from the cardinality of the vertex set of $\mathbb{Q}_X$. By considering endomorphisms on X we are able to glean information about the structure of the coloring space. This is because the structure of the quandle quiver tells us if two elements of the fundamental quandle are related by an endomorphism on X. For examples of the quandle coloring quiver in action see Examples 5, 6, and 7 in [**8**]. One can also find certain polynomial invariants derived from the quandle coloring quiver, for this see [**9, 11**].

Definition 12. *A* ***category****,* $\mathscr{C}$*, is a class of objects* $\mathbb{O}$ *along with a set of maps between the objects called morphisms. Additionally a category must satisfy the following:*

> *1. For each object* $a \in \mathscr{C}$ *there is an identity morphism* $\mathbb{I}_a$ *such that for any two morphisms* $f : a \to b$ *and* $g : c \to a$ *we have* $f \circ \mathbb{I}_a = f$ *and* $\mathbb{I}_a \circ g = g$*.*
>
> *2. For any pair of morphisms* $f : a \to b$*,* $g : b \to c$*, there exists a composition morphism* $g \circ f : a \to c$*, and the composition of morphisms is associative.*

The quandle coloring invariant is a fairly useful, but it is integer valued and not functorial: the invariant does not associate anything to a map between spaces. The quandle coloring quiver is its *categorification*; for a fixed finite quandle X it associates each link to a set of vertices, and to every endomorphism of X a directed path on these vertices.

Theorem 8. *The quandle coloring quiver is a categorization of the quandle coloring invariant, with* X*-colorings of* L *as objects and elements of* $\mathrm{Hom}(X, X)$ *as morphisms.*

Proof. The identity map $\mathbb{I} \in \mathrm{Hom}(X.X)$ satisfies the first axiom. Since composition of endomorphisms is an endomorphism, and composition is associative we are done. □

Quandle Cohomology

This section requires knowledge of homology and cohomology. For a primer see Appendix B. A **rack** is a quandle without the first (idempotent) axiom. For a finite quandle X, let $C_n^R(X)$ be the free abelian group generated by

$(x_1, \ldots, x_n)$ for $x_i \in X$. The superscript "R" stands for rack. We define the boundary map:
$\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ as the following:

$$\partial(x_1, \ldots, x_n) := \sum_{i=2}^{n} (-1)^i [(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ - (x_1 \rhd x_i, x_2 \rhd x_i, \ldots, x_{i-1} \rhd x_i, x_{i+1}, ..., x_n)]$$

for $n \geq 2$ and $\partial_n = 0$ for $n < 2$, and extend linearly. The chain complex is then:

$$\cdots \to C_n^R(X) \xrightarrow{\partial_n} C_{n-1}^R(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0^R(X) \to 0$$

In order to gain an intuition on the boundary map ∂_n we look to the case $n = 2$:

$$\partial_2(x, y) = [(y) - (y)] - [(x) - (x \rhd y)] = \underbrace{y - y - x + (x \rhd y)}$$

Suppose we are given a right-handed crossing in a link diagram with the incoming understrand colored x and the overstrand colored y, the outgoing understrand must then be colored $x \rhd y$. Thus $\partial_2(x, y)$ signifies the sum of the colors at the crossing where the sign of each element of the sum corresponds to direction ('tail' $= -1$ and 'head' $= +1$). This idea holds for higher dimensional knotted surfaces. However, since we are dealing with 2-dimensional knots we don't gain information by considering homology groups of dimension greater than 2.

C_n^R already satisfies quandle axioms 2 and 3. In order make this construction satisfy axiom 1 we let $C_n^D(X)$ be the subgroup of $C_n^R(X)$ generated by elements $(x_1, \ldots, x_n)$ where $x_i = x_{i+1}$ for some i. Elements of C_n^D are *degenerate chains* in the rack complex so a quandle complex can be obtained by modding them out:

Definition 13. *The **quandle chain complex** is then $C_*^Q(X) = \{C_n^Q(X), \partial_n^Q\}$ Where $C_n^Q(X) = C_n^R(X)/C_n^D(X)$, and ∂_n^Q is the induced homomorphism of ∂_n on the quotient.*

In order to obtain quandle cohomology we must use a contravariant functor to dualize quandle chain complex. In this case we will use $\mathrm{Hom}_{\mathbb{Z}}(-, A)$, the set of homomorphisms from the place-holder abelian group to a given abelian group A.

For abelian groups A, B, C and homomorphism $f : B \to C$, we will let $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{f}, \mathbf{A}) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ be the homomorphism mapping $\phi \to \phi \circ f$ for all $\phi \in \mathrm{Hom}(C, A)$.

Definition 14. *Given an abelian group A, let $C_Q^n(X; A) = \mathrm{Hom}(C_n^Q(X), A)$ and $\delta^n : C_Q^n(X; A) \to C_Q^{n+1}(X; A)$ is given by:*

$$(\partial^n f)(x_1, \ldots, x_{n+1}) := f \circ \partial_{n+1}(x_1, \ldots, x_{n+1})$$

*The **quandle cochain complex** is $C_Q^*(X; A) = \{C_Q^n, \delta^n\}$.*

We can then define the *k-th cohomology module* as $H^k = \ker(\delta^n)/\,\mathrm{Im}(\delta^{n-1})$, where elements of $\ker(\delta^n)$ are called *k-cocycles* and elements of $\mathrm{Im}(\delta^{n-1})$ are *k-coboundaries*. We are most interested in quandle 2-cocycles, which are maps $\phi : A[X \times X] \to A$, which can be written as linear combinations of simpler functions:

$$\chi_{i,j}(x_1, x_2) = \begin{cases} 1 & \text{for } i = x_1, j = x_2 \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to a map $\phi : X \times X \to A$ satisfying the following for any $x, y, z \in X$

$$\phi \circ \partial(x, y, z) = [\phi(x, z) - \phi(x \triangleright y, z)] - [\phi(x, y) - \phi(x \triangleright z, y \triangleright z)] = 0.$$

equivalently:

$$\phi(x, y) + \phi(x \triangleright y, z) = \phi(x, z) + \phi(x \triangleright z, y \triangleright z)$$

We can then construct a **quandle cocycle quiver** much in the same way as in Definition 11, where vertices again correspond to X-colorings of a link L, but now we assign a weight to the vertices equal to the quandle co-cycle evaluated at that crossing. The quandle cocycle quiver is a stronger invariant than the quandle coloring quiver as seen in Example 3 of [**10**].

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Appendix A

The Fundamental Group

First formulated by Henri Poicare (April 29th, 1854 - July 17th, 1912), the fundamental group is a group associated to each topological space (in this paper we used a subset of $\mathbb{R}^3$). We first define *homotopy* as it allows use to define equivalence classes of functions. In particular: paths.

Definition 15. *Let X be a topological space and $x, y \in X$. A path from x to y is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Two paths α and β with endpoints $\alpha(0) = x = \beta(0)$ and $\alpha(1) = y = \beta(1)$ are called path homotopic ($\alpha \simeq \beta$) if there exists a continuous map $H : [0,1] \times [0,1] \to X$ which satisfies:*

$$\begin{aligned} H(s,0) &= \alpha(s) \\ H(s,1) &= \beta(s) \\ H(0,t) &= x \\ H(1,t) &= y \end{aligned}$$

One may think of H as a function along the space of paths in X where the endpoints are fixed. The time interval t is then a continuous deformation of path α to the path β. Thus path homotopy gives an equivalence relation on the set of paths in X from x to y. We denote $[\gamma]$ as the *homotopy class* containing the path γ. Thus $[\alpha] = [\beta] \iff \alpha \simeq \beta$. We also get an equivalence relation on elements of the set X. For $x, y \in X$ we say that they are path connected if there exists a path in X between x and y. For nice spaces (locally path connected), the path connected congruence classes correspond to the connected components of X.

We may define a binary operation, called *path composition*, between paths where the endpoint of one equals the intial point on the other.

Definition 16. *Let $x, y, z \in X$ and α be a path from x to y and β a path from y to z. Since $\alpha(1) = \beta(0)$ we can define:*

$$(\alpha\beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Furthermore, this product is associative and can be extended to the equivalence classes of paths. If $\alpha_1(1) = \beta_1(0)$, $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$ then

$$[\alpha_1][\beta_1] = [\alpha_1\beta_1] = [\alpha_2\beta_2] = [\alpha_2][\beta_2].$$

We are now able to define the *fundamental group*:

Definition 17. *Let X be a topological space. Fix an element $x \in X$. A loop based at x is a path with initial and endpoint equal to x. The set of equivalence classes of loops, up to homotopy, in X equipped with the binary operation determined by path composition determines a group structure denoted $\pi_1(X, x)$, called the fundamental group.*

By construction, the fundamental group is invariant under homotopies of the topological space X. If X is path connected then $\pi_1(X, x)$ is isomorphic for any $x \in X$, and is denoted $\pi_1(X)$. It turns out that an ambient isotopy between knots K and K' induces an isomorphism between fundamental groups of the knot complements $\pi_1(\mathbb{R}^3 \setminus K^{(')})$. Thus two knots with distinct knot groups cannot be equivalent.

Appendix B

Homology and Cohomology

Cohomology is one of the greatest contributions to mathematics of the last century. It is derived from homology, a powerful tool used as a Rosetta Stone between questions in geometry/topology and algebra. Homology was originally used as a method for defining and categorizing holes in a manifold.

A *cell-decomposition* of a subset X of $\mathbb{R}^n$ is a division of X into *cells* (spaces homeomorphic to n-balls, B^n) of various dimensions. The boundary of these 'nice' objects are linear combinations of cells one dimension down. We may then view X as a set of vector spaces generated by the cells related to one-another by linear transformations which encode the boundary maps. Because the boundary maps are linear maps, the composition of two boundary maps must be the zero map [**11**].

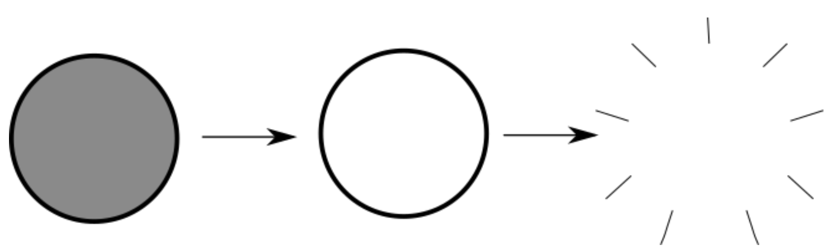

Figure B.1: Boundary Map Intuition

Homology begins with an object, in this paper we used quandles X, which we will use to define a chain complex $\{C_n(X), \partial_n\}_{n\geq 0}$. A *chain complex* is a sequence of abelian groups or modules, $C_n(X)$, connected by homomorphisms

$\partial_n : C_n \to C_{n-1}$ called *boundary maps*:

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_n(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

We require that composition of boundary maps is the constant map which sends all elements in $C_{n+1}(X)$ to the identity of $C_{n-1}(X)$:

$$\partial_n \circ \partial_{n+1} = 0_{n+1,n-1}$$

Or equivalently, $\operatorname{Im}(\partial_{n+1}) \subseteq \ker(\partial_n)$. Furthermore, $\operatorname{Im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n)$. Elements of $\operatorname{Im}(\partial_{n+1})$ are called *n-boundaries*, and elements of $\ker(\partial_n)$ are called *n-cycles*.

Following the cell-decomposition idea, a cycle roughly corresponds to a closed submanifold. While, a boundary is a cycle which is also the boundary of a submanifold. A cycle which is not the boundary of any submanifold is said to represent a hole in the space X: a manifold whose boundary would be the cycle but is instead ‘missing’. These holes correspond to elements of the homology group:

$$H_n(X) := \ker(\partial_n)/\operatorname{Im}(\partial_{n+1})$$

elements of which are called *homology classes*.

We construct cohomology from the chain-complex used for homology. Fix an abelian group A, and replace each $C_n(X)$ by it’s dual group $C^n(X, A) := \operatorname{Hom}(C_n(X); A)$ the group of homomorphisms from $C_n(X)$ to A. We obtain dual boundary map homorphisms between the dual groups as pullbacks of a contravariant functor (also called cofunctor) which we will call $\mathbf{F}$. All we need to know for now is that the cofunctor will take $\partial_n \mapsto \delta^n$, where $\delta^n : C^n(X, A) \to C^{n+1}(X, A)$. This has the effect of reversing the arrows in the (co)chain-complex below.

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+2}"] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "\operatorname{Hom}(-;A)"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "\operatorname{Hom}(-;A)"] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "\operatorname{Hom}(-;A)"] & \ldots \\
\ldots & C^{n+1}(X, A) \arrow[l, "\delta^{n+1}"'] & C^n(X, A) \arrow[l, "\delta^n"'] & C^{n-1}(X, A) \arrow[l, "\delta^{n-1}"'] & \ldots \arrow[l, "\delta^{n-2}"']
\end{tikzcd}$$

The n^{th}-cohomology group, $H^n_A(X)$, is then the *n-cocylces* $\ker(\delta^n)$ modulo the *n-coboundaries* $\operatorname{Im}(\delta^{n-1})$.

Cohomology is powerful. While homology is just a certain sequence of abelian groups or modules, we are able to turn the set of cohomology groups into a ring structure. This is done through the *cup product*, which serves as the multiplication operation. Furthermore, we may vary the abelian group A to get many different cochain-complexes which may or may not give us more information. Because cohomology is representable it is more accessible to study, particularly via computers. For a more in-depth look at cohomology see **[11, 12]**.