Multiplicative Correlations Between Multiplicative Functions

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1 Introduction and Statement of Results

If two functions f, g have independent behavior over some domain D, one would expect that

$$\int_D fg \approx \mu(D) \cdot \overline{f} \cdot \overline{g},$$

where $\mu(D)$ is the measure of the domain and $\overline{f}, \overline{g}$ are the averages of f and g, respectively. In contrast, if $f(x)$ is typically large when $g(x)$ is large, then we would expect that

$$\int_D fg \geq \mu(D) \cdot \overline{f} \cdot \overline{g}.$$

Hence, this integral provides information about the similarity in the behavior of f and g. We call a summation or integral of the product $f(x)g(x \star r)$ a correlation between f and g with respect to the operation "$\star$". In number theory, there are many additive correlations of multiplicative functions of the following type

$$\sum_{n \leq x} f(n)f(n+r), \text{ where } r \text{ is an integer.}$$

The purpose of this paper is to explore the existence of multiplicative correlations of multiplicative functions with a similar form

$$\sum_{n \leq x} f(kn)f(ln) \text{ for integers } k,\ l.$$

Or more generally, for multiplicative functions $f_1, ..., f_m$ and integers $k_1, ...k_m$,

$$\sum_{n \leq x} \left(\prod_{j=1}^{m} f_j(k_j n) \right).$$

Our approach will be to first derive Dirichlet series whose coefficients are expressions of the form $\prod_{j=1}^{m} f_j(k_j n)$. Then we will apply Perron's formula to the resulting Dirichlet series, which allows one to estimate the summation of the first N coefficients of a Dirichlet series.

We begin by introducing some notation which will be used throughout this paper.

Definition 1.1. For $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_j^{\alpha_j}$, define

$$\delta_p(n) = \begin{cases} \alpha_i & \text{if } p = p_i, \\ 0 & \text{else.} \end{cases}$$

Observe that we then have $n = \prod_p p^{\delta_p(n)}$. We chose this notation because our theorems involve multiple natural numbers and their prime factorizations, thus assigning a Greek letter to each would likely make it difficult to keep track of what is going on. We will begin by proving the following theorem

Theorem 2.1 Suppose that f is a multiplicative function and $k \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} \frac{f(kn)}{n^s} = \left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}}} \right) \cdot \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Applied to the divisor function $d(n)$, for example, this yields the following corollary, which can also be found in Titchmarsh [Ti, p. 9].

Corollary 2.2 Let $k \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} \frac{d(kn)}{n^s} = \zeta^2(s) \prod_p (1 + \delta_p(k)(1 - p^{-s})).$$

We also prove the following generalization of Theorem 2.1.

Theorem 2.3 Suppose $f_1, ..., f_m$ are multiplicative functions and $k_1, ..., k_m \in \mathbb{N}$. For $K = \prod_{j=1}^m k_j$,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^m f_j(k_j n)}{n^s} = \left(\prod_{p|K} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^m f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} \right) \cdot \sum_{n=1}^{\infty} \frac{\prod_{j=1}^m f_j(n)}{n^s}.$$

We note that the product $\prod_{p|K}$ here may be replaced by the product $\prod_p$ over all primes, because for any prime that does not divide K, the argument of the product evaluates to 1. Although Theorem 2.1 follows directly from Theorem 2.3, we choose to first present the proof of Theorem 2.1 to demonstrate the proof method in the simplest case.

Applying Theorem 2.3 to a product of two divisor functions, we get the following extension of Corollary 2.2.

Corollary 2.4 If $k, l \in \mathbb{N}$, then for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \frac{(1 - \delta_p(k)\delta_p(l))p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))}{(1 + p^{-s})} \right).$$

Corollary 2.5 If $k, l \in \mathbb{N}$ and are coprime, then for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)} \prod_{p|k} \left(1 + \delta_p(k) \left(\frac{p^s - 1}{p^s + 1} \right) \right) \prod_{p|l} \left(1 + \delta_p(l) \left(\frac{p^s - 1}{p^s + 1} \right) \right).$$

Corollary 2.6 If $k_1, ..., k_m \in \mathbb{N}$, then for $K = \prod_{j=1}^m k_j$ and $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^m \mu(k_j n)}{n^s} = \left(\prod_{p|K} \frac{p}{p + (-1)^m} \right) \left(\prod_{j=1}^m \mu(k_j) \right) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)^m}{n^s}.$$

For odd m, this becomes

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)}{n^s} = \frac{1}{\zeta(s)} \left(\prod_{p|K} \frac{p^s}{p^s - 1} \right) \left(\prod_{j=1}^{m} \mu(k_j) \right).$$

whereas for even m,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \left(\prod_{p|K} \frac{p^s}{p^s + 1} \right) \left(\prod_{j=1}^{m} \mu(k_j) \right). \tag{1.1}$$

Note that if even a single k_j is not square free, then the left-hand expressions are all equal to zero, as $\mu(k_j n) = 0$ for every n. This is reflected on the right-hand side in the expressions $\prod_{j=1}^{m} \mu(k_j)$, which are non zero if and only if each k_j is square free.

Since $\mu(n)^2 = |\mu(n)|$, we easily obtain the following corollary from expression (1.1).

Corollary 2.7 If $k_1, ..., k_m \in \mathbb{N}$ are square free, then for $K = \prod_{j=1}^{m} k_j$ and $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} |\mu(k_j n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \cdot \prod_{p|K} \frac{p^s}{p^s + 1}.$$

There are many arithmetical formulas of the above type that could be derived from Theorem 2.3, but we content ourselves with these few cases.

Our next results apply Perron's formula to some of the Dirichlet series above.

Theorem 3.1 If $k_1, ..., k_m \in \mathbb{N}$ are square free, then for $K = \prod_{j=1}^{m} k_j$,

$$\sum_{n<x} \left(\prod_{j=1}^{m} |\mu(k_j n)| \right) = \left(\frac{6}{\pi^2} \prod_{p|K} \frac{p}{p+1} \right) x + O\left(C_K x^{7/12} \log(x) \right). \tag{1.2}$$

Where C_K is a constant dependent on K.

It has been proven [Br] that for a finite set T of primes and an infinite set P of primes, the proportion of numbers which are square free, divisible by all of the primes in T, and none of the primes in P is

$$\frac{6}{\pi^2} \prod_{p\in T} \frac{1}{p+1} \prod_{p\in P} \frac{p}{p+1}.$$

Although we cannot speak to the case where P is infinite, Theorem 3.1 allows us to put an error term on this proportion in the case where P is finite

Corollary 3.2 Suppose T and P are finite disjoint sets of primes. Then, the number of square free numbers up to x which are divisible by every $p \in T$ but none of the $q \in P$ is

$$\left(\frac{6}{\pi^2} \prod_{p\in T} \frac{1}{p+1} \prod_{q\in P} \frac{q}{q+1} \right) x + O\left(2^{(|T|+1)} C_K x^{7/12} \log(x) \right).$$

This proportion is what we would expect, as of the numbers which are not divisible by p^2, the natural density of those still divisible by p is precisely

$$\frac{p-1}{p^2-1} = \frac{1}{p+1}.$$

To corroborate this result, we wrote code to compute the number of square free primes up to 1000000, and computed the proportion of these which contained various primes.

Filters	Count	Expected	Proportion	Expected Proportion
None	607926	607927	1	1
2	405286	405284	0.666669957	0.666666666
3	455946	455945	0.750002467	0.75
5	506604	506605	0.833331688	0.833333333
7	531932	531936	0.874994654	0.875
2,3	303963	303963	0.5	0.5
2,5	337736	337737	0.555554459	0.555555555
2,7	354622	354624	0.583330866	0.583333333

Table 1: Counts of square free primes up to $N = 1000000$ with various primes filtered out

One may try and identify the smallest error term possible. Assuming the Riemann Hypothesis, we could get an error bound of $O(C_K x^{1/2+\varepsilon}\log(x))$. We can also consider how similar logic may apply to the k-free numbers (where n is k-free if $\forall p$, n is not divisible by p^k). Of the numbers not divisible by p^k, the the natural density those numbers which are still divisible by is

$$P_k = \frac{p^{k-1}-1}{p^k-1}.$$

We saw that in the case of the square free numbers, this proportion carried forward to the set of square free numbers. Therefore, we conjecture that of the numbers which are k-free, the natural density of those divisible by p is the proportion P_k written above.

Theorem 3.3 If $k \in \mathbb{N}$, then

$$\sum_{n<x} d(kn) = x\log(x)H_k(1) + xH_k(1)\left(2\gamma - 1 + \sum_{p|K}\frac{\delta_p(k)\log(p)}{p+\delta_p(k)(p-1)}\right) + O(C_k x^{2/3}), \tag{1.3}$$

where

$$H_k(1) = \prod_{p|k}(1+\delta_p(k)(1-p^{-1})).$$

Theorem 3.4 If $k, l \in \mathbb{N}$, then

$$\begin{aligned}\sum_{n<x} d(kn)d(ln) &= \mathrm{Res}_{w=1}\left\{\frac{\zeta(w)^4}{\zeta(2w)}H_{kl}(w)\frac{x^w}{w}\right\} + O(C_k x^{5/6}\log(x)) \\ &= x\log(x)^3\left(\frac{H_{kl}(1)}{\pi^2}\right) + xP_2(\log(x)) + O(C_{kl}x^{5/6}\log(x)),\end{aligned}$$

where $P_n(u)$ denotes an n^{th} degree polynomial of u, and

$$H_{kl}(1) = \prod_{p|kl} \left(p^{-1} + (p-1) \frac{(1 - \delta_p(k)\delta_p(l))p^{-1} + (1 + \delta_p(k))(1 + \delta_p(l))}{(p+1)} \right).$$

Corollary 3.5 If $k, l \in \mathbb{N}$ are coprime, then

$$\begin{aligned}\sum_{n<x} d(kn)d(ln) &= \mathrm{Res}_{w=1} \left\{ \frac{\zeta(w)^4}{\zeta(2w)} H_{kl}(w) \frac{x^w}{w} \right\} + O(C_k x^{5/6} \log(x)) \\ &= x \log(x)^3 \left(\frac{H_{kl}(1)}{\pi^2} \right) + xP_2(\log(x)) + O(C_{kl} x^{5/6} \log(x)),\end{aligned}$$

where $P_2(u)$ is a second degree polynomial of u dependent on p, q, and

$$H_{kl}(1) = \prod_{p|k} \left(1 + \delta_p(k) \left(\frac{p-1}{p+1} \right) \right) \prod_{p|l} \left(1 + \delta_p(l) \left(\frac{p-1}{p+1} \right) \right).$$

Suppose p, q are distinct primes. Then,

$$H_{pq}(1) = \left(1 + \delta_p(p) \left(\frac{p-1}{p+1} \right) \right) \left(1 + \delta_q(q) \left(\frac{q-1}{q+1} \right) \right) = \frac{4pq}{(p+1)(q+1)}$$

Therefore, by Corollary 3.5,

$$\sum_{n<x} d(pn)d(qn) = x \log(x)^3 \frac{4pq}{(p+1)(q+1)} + xP_2(\log(x)) + O(C_{pq} x^{5/6} \log(x)).$$

Hence for large p, q, this sum is around 4 times the sum of $d(n)^2$, which is asymptotic to $x \log(x)^3$. Consider that for arbitrary n,

$$d(pn) = \left(\frac{\delta_p(n) + 2}{\delta_p(n) + 1} \right) d(n) \leq 2d(n).$$

Thus $d(pn) = 2d(n)$ when p, n are coprime, and $d(n) < d(pn) < 2d(n)$ when $p|n$. Then, for large primes p, q which do not divide many n, we would expect that in most cases $d(pn)d(qn) = 4d(n)^2$.

2 Dirichlet Series of Multiplicative Correlations

Proof of Theorem 2.1: Because each $n \in \mathbb{N}$ can be expressed as a unique product of primes,

$$\left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}}}\right) \cdot \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}}}\right) \cdot \prod_{p} \sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}}.$$

For any p which does not divide k, $\delta_p(k) = 0$, thus

$$\begin{aligned}\left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}}}\right) \cdot \prod_{p} \sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}} &= \prod_{p|k} \sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}} \cdot \prod_{p\nmid k} \sum_{r=0}^{\infty} \frac{f(p^r)}{p^{rs}} \\ &= \prod_{p} \sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})}{p^{rs}} \\ &= \sum_{n=1}^{\infty} \left(\frac{\displaystyle\prod_p f(p^{\delta_p(k)+\delta_p(n)})}{n^s} \right).\end{aligned}$$

As the n^{th} term in the right hand sum is the product of the $r = \delta_p(n)$ element of the left hand sums for each prime. Because f is a multiplicative function,

$$\sum_{n=1}^{\infty} \frac{\displaystyle\prod_p f(p^{\delta_p(k)+\delta_p(n)})}{n^s} = \sum_{n=1}^{\infty} \frac{f\left(\displaystyle\prod_p p^{\delta_p(k)+\delta_p(n)}\right)}{n^s} = \sum_{n=1}^{\infty} \frac{f(kn)}{n^s}.$$

$\cdot /\!/ \cdot$

Proof of Corollary 2.2: By Theorem 2.1, we have that

$$\sum_{n=1}^{\infty} \frac{d(kn)}{n^s} = \left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{d(p^{\delta_p(k)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{d(p^r)}{p^{rs}}}\right) \cdot \sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\delta_p(k)+r+1}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{r+1}{p^{rs}}}\right) \cdot \zeta^2(s).$$

We can compute that for an arbitrary constant c,

$$\sum_{r=0}^{\infty} \frac{c+r}{p^{rs}} = \sum_{r=0}^{\infty} \frac{c}{p^{rs}} + \sum_{r=0}^{\infty} \frac{r}{p^{rs}} = \frac{c}{1-p^{-s}} + \frac{p^{-s}}{(1-p^{-s})^2} = \frac{c(1-p^{-s})+p^{-s}}{(1-p^{-s})^2}.$$

Therefore, for an arbitrary prime p,

$$\frac{\displaystyle\sum_{r=0}^{\infty} \frac{\delta_p(k)+r+1}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{r+1}{p^{rs}}} = \frac{\dfrac{(\delta_p(k)+1)(1-p^{-s})+p^{-s}}{(1-p^{-s})^2}}{\dfrac{(1-p^{-s})+p^{-s}}{(1-p^{-s})^2}} = (1+\delta_p(k)(1-p^{-s})).$$

Plugging this back into the above expression, we get

$$\sum_{n=1}^{\infty} \frac{d(kn)}{n^s} = \left(\prod_{p|k} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\delta_p(k)+r+1}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{r+1}{p^{rs}}} \right) \cdot \zeta^2(s) = \zeta^2(s) \prod_{p|k} (1+\delta_p(k)(1-p^{-s})).$$

Observe that for $p \nmid k$, we will have $(1+\delta_p(k)(1-p^{-s})) = 1$, thus this expression can also be written as the product over all primes.

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As a remark, Titchmarsh [Ti] has an alternate proof of this result where he first considers the multiplicative function $\sigma_a(n)$, defined by

$$\sigma_a(n) = \sum_{d|n} d^a.$$

If $a > 0$, then for a power of a prime p^k,

$$\sigma_a(p^k) = \sum_{m=0}^{k} (p^m)^a = \frac{1-p^{(k+1)a}}{1-p^a}.$$

Therefore, for an arbitrary non negative integer m,

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{\sigma_a(p^{m+r})}{p^{rs}} &= \sum_{r=0}^{\infty} \frac{(1-p^{(r+m+1)a})}{(1-p^a)p^{rs}} \\
&= \frac{1}{(1-p^a)} \left(\sum_{r=0}^{\infty} \frac{1}{p^{rs}} - \sum_{r=0}^{\infty} \frac{p^{(r+m+1)a}}{p^{rs}} \right) \\
&= \frac{1}{(1-p^a)} \left(\frac{1}{(1-p^{-s})} - \frac{p^{(m+1)a}}{(1-p^{a-s})} \right) \\
&= \frac{1-p^{a-s}-p^{(m+1)a}+p^{(m+1)a-s}}{(1-p^a)(1-p^{-s})(1-p^{a-s})}.
\end{aligned}$$

Because $\sigma_0 = d$, the result then follows by applying Theorem 2.1 and taking the limit $a \to 0$.

Proof of Theorem 2.3: For a prime p which does not divide any of the k_j,

$$\frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{m} f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} = \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{m} f_j(p^{0+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} = \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} = 1.$$

A prime p does not divide any of the k_j if and only if it does not divide $K = \prod_{j=1}^{m} k_j$, thus

$$\left(\prod_{p|K} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{m} f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} \right) \cdot \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} f_j(n)}{n^s} = \left(\prod_{p} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{m} f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}} \right) \cdot \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} f_j(n)}{n^s}.$$

The proof then follows by taking the same steps as in Theorem 2.1:

$$\begin{aligned}
\left(\prod_p \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^m f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}}\right) \cdot \sum_{n=1}^{\infty} \frac{\prod_{j=1}^m f_j(n)}{n^s} & \\
&= \left(\prod_p \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^m f_j(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f_j(p^r)^m}{p^{rs}}}\right) \cdot \prod_p \sum_{r=0}^{\infty} \frac{\prod_{j=1}^m f_j(p^r)}{p^{rs}} \\
&= \prod_p \sum_{r=0}^{\infty} \frac{\prod_{j=1}^m f_j(p^{\delta_p(k_j)+r})}{p^{rs}} \\
&= \sum_{n=1}^{\infty} \frac{\prod_{j=1}^m f_j(k_j n)}{n^s}.
\end{aligned}$$

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For the sake of discussing some basic examples, we will write out this formula for $m = 2$:

$$\sum_{n=1}^{\infty} \frac{f(kn)f(ln)}{n^s} = \left(\prod_{p|kl} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^{\delta_p(k)+r})f(p^{\delta_p(l)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{f(p^r)^2}{p^{rs}}}\right) \cdot \sum_{n=1}^{\infty} \frac{f(n)^2}{n^s}. \tag{2.1}$$

Proof of Corollary 2.4: By Theorem 2.3, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} &= \left(\prod_p \frac{\displaystyle\sum_{r=0}^{\infty} \frac{d(p^{\delta_p(k)+r})d(p^{\delta_p(l)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{d(p^r)^2}{p^{rs}}}\right) \cdot \sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} \\
&= \left(\prod_p \frac{\displaystyle\sum_{r=0}^{\infty} \frac{(\delta_p(k)+r+1)(\delta_p(l)+r+1)}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{(r+1)^2}{p^{rs}}}\right) \frac{\zeta(s)^4}{\zeta(2s)}.
\end{aligned}$$

We can compute that for an arbitrary constants a, b, c,

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{ar^2+br+c}{p^{rs}} &= \sum_{r=0}^{\infty} \frac{ar^2}{p^{rs}} + \sum_{r=0}^{\infty} \frac{br}{p^{rs}} + \sum_{r=0}^{\infty} \frac{c}{p^{rs}} \\
&= \frac{ap^{-s}(1+p^{-s})}{(1-p^{-s})^3} + \frac{bp^{-s}}{(1-p^{-s})^2} + \frac{c}{1-p^{-s}} \\
&= \frac{ap^{-s}(1+p^{-s}) + bp^{-s}(1-p^{-s}) + c(1-p^{-s})^2}{(1-p^{-s})^3}.
\end{aligned}$$

In the case $a = 1, b = 2, c = 1$ (which is what we get from $(r + 1)^2$),

$$\sum_{r=0}^{\infty} \frac{r^2 + 2r + 1}{p^{rs}} = \frac{p^{-s}(1 + p^{-s}) + 2p^{-s}(1 - p^{-s}) + (1 - p^{-s})^2}{(1 - p^{-s})^3} = \frac{1 + p^{-s}}{(1 - p^{-s})^3}.$$

Combining this result with our previous expression, we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} \\
&\quad = \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \frac{(2 + \delta_p(k) + \delta_p(l))p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))(1 - p^{-s})}{(1 + p^{-s})} \right) \\
&\quad = \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \frac{(2 + \delta_p(k) + \delta_p(l))p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))(1 - p^{-s})}{(1 + p^{-s})} \right) \\
&\quad = \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \frac{(1 - \delta_p(k)\delta_p(l))p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))}{(1 + p^{-s})} \right).
\end{aligned}$$

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Proof of Corollary 2.5: If k, l are coprime, then $\forall p$, $\delta_p(k)\delta_p(l) = 0$, thus by Corollary 2.4,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} &= \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \frac{(1 - \delta_p(k)\delta_p(l))p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))}{(1 + p^{-s})} \right) \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \left(\frac{p^{-s} + (1 + \delta_p(k))(1 + \delta_p(l))}{(1 + p^{-s})} \right) \right) \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(p^{-s} + (1 - p^{-s}) \left(1 + \frac{\delta_p(k) + \delta_p(l)}{(1 + p^{-s})} \right) \right) \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(1 + (1 - p^{-s}) \left(\frac{\delta_p(k) + \delta_p(l)}{(1 + p^{-s})} \right) \right) \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_p \left(1 + \left(\frac{p^s - 1}{p^s + 1} \right) (\delta_p(k) + \delta_p(l)) \right).
\end{aligned}$$

As discussed before, the argument of this product is 1 if p does not divide k or l. If k, l are coprime, then any other prime either divides k or divides l. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(kn)d(ln)}{n^s} &= \frac{\zeta(s)^4}{\zeta(2s)} \prod_{p|k} \left(1 + \left(\frac{p^s - 1}{p^s + 1} \right) (\delta_p(k) + \delta_p(l)) \right) \prod_{p|l} \left(1 + \left(\frac{p^s - 1}{p^s + 1} \right) (\delta_p(k) + \delta_p(l)) \right) \\
&= \frac{\zeta(s)^4}{\zeta(2s)} \prod_{p|k} \left(1 + \delta_p(k) \left(\frac{p^s - 1}{p^s + 1} \right) \right) \prod_{p|l} \left(1 + \delta_p(l) \left(\frac{p^s - 1}{p^s + 1} \right) \right).
\end{aligned}$$

$\cdot/\!\!/\cdot$

Proof of Corollary 2.6: For each $p|K$, at least one k_j satisfies $\delta_p(k_j) \geq 1$. Then, if $r \geq 1$, for this k_j, $p^{\delta_p(k_j)+r}$ is not square free. Therefore,

$$\mu(p^{\delta_p(k_j)+r}) = 0$$

Then, by Theorem 2.3, we can compute that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)}{n^s} &= \left(\prod_{p|K} \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{m} \mu(p^{\delta_p(k_j)+r})}{p^{rs}}}{\displaystyle\sum_{r=0}^{\infty} \frac{\mu(p^r)^m}{p^{rs}}} \right) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)^m}{n^s} \\
&= \left(\prod_{p|K} \frac{\prod_{j=1}^{m} \mu(p^{\delta_p(k_j)})}{1 + \frac{(-1)^m}{p^s}} \right) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)^m}{n^s} \\
&= \left(\prod_{p|K} \frac{p^s}{p^s + (-1)^m} \right) \left(\prod_{j=1}^{m} \mu(k_j) \right) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)^m}{n^s}.
\end{aligned}$$

As on the first line, only the $r = 0$ term of the sum in the numerator is nonzero for the reasoning mentioned above. It is stated in Titchmarsh [Ti] that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \qquad\qquad \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.$$

Therefore, for odd m, $\mu(n)^m = \mu(n)$, hence

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)}{n^s} = \frac{1}{\zeta(s)} \left(\prod_{p|K} \frac{p^s}{p^s - 1} \right) \left(\prod_{j=1}^{m} \mu(k_j) \right).$$

Likewise, for even m, $\mu(n)^m = |\mu(n)|$, hence

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \left(\prod_{p|K} \frac{p^s}{p^s + 1} \right) \left(\prod_{j=1}^{m} \mu(k_j) \right).$$

$\cdot/\!/\cdot$

Proof of Corollary 2.7: Consider the $2m$ integers given by $k_1, k_1, ..., k_m, k_m$. By Corollary 2.6,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \left(\prod_{p|K^2} \frac{p^s}{p^s + 1} \right) \left(\prod_{j=1}^{m} \mu(k_j)^2 \right).$$

$p|K^2$ if and only if $p|K$, thus we can write this as

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \left(\prod_{p|K} \frac{p^s}{p^s + 1} \right) \left(\prod_{j=1}^{m} \mu(k_j)^2 \right).$$

If each k_j is square free, then

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} |\mu(k_j n)|}{n^s} = \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \mu(k_j n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \cdot \prod_{p|K} \frac{p^s}{p^s + 1}.$$

$\cdot/\!/\cdot$

3 Application of Perron's Formula

Suppose that

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is a Dirichlet series such that $a(n) = O(\psi(n))$ for some $\psi(n)$ which is non-decreasing, and

$$\sum_{n=1}^{\infty} \frac{|a(n)|}{n^\sigma} = O\left(\frac{1}{(\sigma-1)^\alpha}\right).$$

Perron's formula [Ti] states that if $c > 0$ and $c + \sigma > 1$, x is not an integer, and N is the closest integer to N, then

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s+w)\frac{x^w}{w}\,dw + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) \tag{3.1}$$
$$+ O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right) + O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right).$$

We are interested in applying this formula for $s = 0$, hence we will be using $c > 1$ and computing the integral by pulling the contour to the left. The expressions we are interested in are in terms of $\zeta(s)$, hence we will be calculating the residue of $F(s)$ at $s = 1$. In order to compute this residue, we will use the Laurent series for $\zeta(s)$ at $s = 1$ [Ber], which is given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k. \tag{3.2}$$

Where

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{x\to\infty}\left\{\sum_{n<x}\frac{\log(n)^k}{n} - \frac{\log(x)^{k+1}}{k+1}\right\} \tag{3.3}$$

We will also need the Taylor series for $1/\zeta(2s)$ at $s = 1$. Consider that for $\sigma > 0.5$,

$$\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}$$

This is a uniformly convergent sum, thus the k^{th} order derivative of this expression is

$$(-2)^k \sum_{n=1}^{\infty} \frac{\mu(n)\log(n)^k}{n^{2s}}.$$

Each of which is convergent for $\sigma > 1$. Therefore,

$$\frac{1}{\zeta(2s)} = \sum_{k=0}^{\infty} c_k (s-1)^k.$$

Where $c_0 = 1/\zeta(2)$, and generally,

$$c_k = \frac{(-2)^k}{k!}\sum_{n=1}^{\infty}\frac{\mu(n)\log(n)^k}{n^2}.$$

In addition, we will also be using the Taylor series

$$x^s = \sum_{k=0}^{\infty} \frac{x \log(x)^k}{k!}(s-1)^k \qquad\qquad \frac{1}{s} = \sum_{k=0}^{\infty}(-1)^k (s-1)^k.$$

For $\sigma > 1$,

$$\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(s) \ll \frac{1}{\sigma - 1}. \tag{3.4}$$

Proof of Theorem 3.1: Denote

$$F(s) = \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} |\mu(k_j n)|}{n^s}.$$

As proven in Corollary 2.7, for $\sigma > 1$,

$$F(s) = \frac{\zeta(s)}{\zeta(2s)} \cdot \prod_{p|K} \frac{p^s}{p^s + 1} \equiv \frac{\zeta(s)}{\zeta(2s)} \cdot H_K(s).$$

Note that $H_K(s)$ is analytic everywhere except where $p^s = -1$ for some $p|K$, which occurs when

$$s = i \log(p)(\pi + 2\pi k) \quad k \in \mathbb{Z}.$$

Hence the poles of $H_k(s)$ all occur along the $\mathrm{Re}\{s\} = 0$ line. We want to compute for $c > 1$

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(w) \frac{x^w}{w}\, dw = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{\zeta(w)}{\zeta(2w)} \cdot \prod_{p|K} \frac{p^w}{p^w+1}\right) \frac{x^w}{w}\, dw.$$

$\zeta(s)$ has a pole of order 1 at $s = 1$, hence to compute the residue of the integrand at $s = 1$, we only need the first term from the Taylor series of the remaining expressions.

$$\mathrm{Res}_{w=1} \left\{\frac{\zeta(w)}{\zeta(2w)} \cdot \frac{x^w}{w} \cdot \prod_{p|K} \frac{p^w}{p^w+1}\right\} = \frac{x H_K(1)}{\zeta(2)} = \left(\frac{6}{\pi^2} \prod_{p|K} \frac{p}{p+1}\right) x.$$

By residue theorem, for $b > 1/2$,

$$\begin{aligned}
&\mathrm{Res}_{w=1} \left\{\frac{\zeta(w)}{\zeta(2w)} \cdot \frac{x^w}{w} \cdot H_K(w)\right\} \\
&\quad = \frac{1}{2\pi i} \left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{c-iT}\right) \left(\frac{\zeta(w)}{\zeta(2w)} \cdot \frac{x^w}{w} \cdot H_K(w)\right) dw.
\end{aligned}$$

Take $b = (1/2) + 1/\log(x)$. If $\mathrm{Im}\{w\} = \pm T$ and $b \leq \mathrm{Re}\{w\} \leq c$, then

$$\left|\prod_{p|K} \frac{p^w}{p^w+1}\right| = \left|\prod_{p|K} 1 - \frac{1}{p^w+1}\right| \leq \prod_{p|K} \left(1 + \frac{1}{p^\sigma - 1}\right) \leq \prod_{p|K} \left(1 + \frac{1}{p^{1/2} - 1}\right) \equiv C_K.$$

By equation (3.4), for $\sigma \geq b$,

$$\left|\frac{1}{\zeta(2w)}\right| \leq \zeta(2b) = \zeta\left(1 + \frac{2}{\log(x)}\right) \ll \log(x).$$

For $\sigma > 1/2$, $\zeta(\sigma+it) \ll t^{1/6}$ ([Ti], pg 115). Therefore,

$$\left| \int_{c+iT}^{b+iT} \frac{\zeta(w)}{\zeta(2w)} \frac{x^w}{w} H_K(w)\, dw \right| \ll \frac{C_K x^c \log(x)}{T^{5/6}}.$$

And the bottom edge has the same bound. If we consider the left edge, then

$$\begin{aligned} \left| \int_{b-iT}^{b+iT} \frac{\zeta(w)}{\zeta(2w)} \frac{x^w}{w} H_K(w)\, dw \right| &= \left| \int_{-T}^{T} \frac{\zeta(b+it)}{\zeta(2b+i2t)} \frac{x^{b+it}}{b+it} H_k(w)\, dt \right| \\ &\ll C_K x^{1/2} x^{\log(x)} \log(x) \int_{-T}^{T} \left| \frac{t^{1/6}}{b+it} \right| dt \\ &\ll C_K x^{1/2} \log(x) T^{1/6}. \end{aligned}$$

Therefore,

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(w) \frac{x^w}{w}\, dw = \left(\frac{6}{\pi^2} \prod_{p|K} \frac{p}{p+1} \right) x + O\left(\frac{C_K x^c \log(x)}{T^{5/6}} \right) + O\left(C_K x^{1/2} \log(x) T^{1/6} \right).$$

Now, we want to apply Perron's formula. $|\mu(n)| \leq 1$, hence we can take $\psi(n) = 1$. In addition,

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} |\mu(k_j n)|}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma) \ll \frac{1}{(\sigma-1)}.$$

Hence we can take $\alpha = 1$. Taking x to be half of an odd integer, the error terms from Perron's formula are

$$\begin{aligned} O\left(\frac{x^c}{T(\sigma+c-1)^\alpha} \right) &= O\left(\frac{x^c}{T(c-1)} \right), \\ O\left(\frac{\psi(2x) x^{1-\sigma} \log(x)}{T} \right) &= O\left(\frac{x \log(x)}{T} \right), \\ O\left(\frac{\psi(N) x^{1-\sigma}}{T|x-N|} \right) &= O\left(\frac{x}{T} \right). \end{aligned}$$

Take $c = 1 + 1/\log(x)$. Then,

$$O\left(\frac{x^c}{T(c-1)} \right) = O\left(\frac{x \log(x)}{T} \right), \qquad O\left(\frac{C_K x^c \log(x)}{T^{5/6}} \right) = O\left(\frac{C_K x \log(x)}{T^{5/6}} \right).$$

In order to minimize error, we want

$$\frac{x \log(x)}{T^{5/6}} = x^{1/2} \log(x) T^{1/6} \implies T = x^{1/2}.$$

Then, by Perron's formula,

$$\sum_{n<x} \prod_{j=1}^{m} |\mu(k_j n)| = \left(\frac{6}{\pi^2} \prod_{p|K} \frac{p}{p+1} \right) x + O\left(C_K x^{7/12} \log(x) \right).$$

$\cdot/\!/\!\cdot$

Proof of Corollary 3.2: Suppose n is a square free, but one of its prime factors is p. Then, $|\mu(n)| = 1$, but $|\mu(pn)| = 0$. If n is not square free, then both are 0. In general, if $p_1, ..., p_m$ are distinct primes, then $\prod_{j=1}^{m} |\mu(p_j n)| = 0$ if n contains any of the p_j as a prime factor. Therefore, by Theorem 3.1, we can interpret that

$$\sum_{n<x} \left(\prod_{j=1}^{m} |\mu(p_j n)| \right) = \left(\frac{6}{\pi^2} \prod_{j=1}^{m} \frac{p_j}{p_j + 1} \right) x + O\left(C_K x^{7/12} \log(x) \right).$$

is the count of all $n < x$ which are square free and are not divisible by $p_1, ..., p_m$.

Suppose that T and P are disjoint finite sets of primes. Let $K = \prod_{p \in T} p \prod_{q \in P} q$ and C_K be the coefficient in error term used in Theorem 3.1. Observe that for any subset of these primes, the error term coefficient is less than or equal to this C_K. For any $S \subseteq T$, let

$$N_S = \sum_{n<x} \left(\prod_{p \in S} |\mu(pn)| \prod_{j=1}^{l} |\mu(q_j n)| \right) = \left(\frac{6}{\pi^2} \prod_{p \in S} \frac{p}{p+1} \prod_{q \in P} \frac{q}{q+1} \right) x + O\left(C_K x^{7/12} \log(x) \right).$$

Then, N_S is the count of all the square free numbers up to x which are not divisible by any $q \in P$ and not divisible by any $p \in S$. We can then compute the number of square free numbers up to x which are not divisible by $q \in P$ but are divisible by all $p \in T$ as

$$\sum_{S \subseteq T} (-1)^{|S|} N_S.$$

We want to show that

$$\sum_{S \subseteq T} (-1)^{|S|} N_S = \left(\frac{6}{\pi^2} \prod_{p \in T} \frac{1}{p+1} \prod_{q \in P} \frac{q}{q+1} \right) x + O\left(2^{(|T|+1)} C_K x^{7/12} \log(x) \right).$$

Consider the case where $|T| = 1$, where $T = \{p\}$. Then, $S = \varnothing$ or $\{p\}$, thus

$$\begin{aligned}
N - N_T &= \left(\frac{6}{\pi^2} \prod_{q \in P} \frac{q}{q+1} \right) x - \left(\frac{6}{\pi^2} \left(\frac{p}{p+1} \right) \prod_{q \in P} \frac{q}{q+1} \right) x + 2O\left(C_K x^{7/12} \log(x) \right) \\
&= \left(1 - \frac{p}{p+1} \right) \left(\frac{6}{\pi^2} \prod_{q \in P} \frac{q}{q+1} \right) x + 2O\left(C_K x^{7/12} \log(x) \right) \\
&= \left(\frac{6}{\pi^2} \left(\frac{1}{p+1} \right) \prod_{q \in P} \frac{q}{q+1} \right) x + O\left(2^2 C_K x^{7/12} \log(x) \right).
\end{aligned}$$

Suppose true for $|T| = k$. Suppose $|T| = k+1$, and select an arbitrary $p_0 \in T$. If $S \subseteq T$ does not contain p_0, then

$$\begin{aligned}
N_{S \cup \{p_0\}} &= \left(\frac{6}{\pi^2} \prod_{p \in S \cup \{p_0\}} \frac{p}{p+1} \prod_{q \in P} \frac{q}{q+1} \right) x + O\left(C_K x^{7/12} \log(x) \right) \\
&= \frac{p_0}{p_0 + 1} \left(\frac{6}{\pi^2} \prod_{p \in S} \frac{p}{p+1} \prod_{q \in P} \frac{q}{q+1} \right) x + O\left(C_K x^{7/12} \log(x) \right) \\
&= \left(\frac{p_0}{p_0 + 1} \right) N_S + 2O\left(C_K x^{7/12} \log(x) \right).
\end{aligned}$$

Thus, we can compute that

$$\begin{aligned}
\sum_{S\subseteq T}(-1)^{|S|}N_S &= \sum_{\substack{S\subseteq T\\ p_0\notin S}}(-1)^{|S|}N_S+\sum_{\substack{S\subseteq T\\ p_0\in S}}(-1)^{|S|}N_S\\
&= \sum_{\substack{S\subseteq T\\ p_0\notin S}}(-1)^{|S|}\left(N_S-\left(\frac{p_0}{p_0+1}\right)N_S\right)+\sum_{\substack{S\subseteq T\\ p_0\in S}}2O\left(C_K x^{7/12}\log(x)\right)\\
&= \frac{1}{p_0+1}\left(\sum_{S\subseteq(T\setminus\{p_0\})}(-1)^{|S|}N_S\right)+2^{|T|}O\left(C_K x^{7/12}\log(x)\right)\\
&= \frac{1}{p_0+1}\left(\frac{6}{\pi^2}\prod_{p\in T\setminus\{p_0\}}\frac{1}{p+1}\prod_{q\in P}\frac{q}{q+1}\right)x+\left(2^{|T|}+2^{|T|}\right)O\left(C_K x^{7/12}\log(x)\right)\\
&= \left(\frac{6}{\pi^2}\prod_{p\in T}\frac{1}{p+1}\prod_{q\in P}\frac{q}{q+1}\right)x+O\left(2^{(|T|+1)}C_K x^{7/12}\log(x)\right).
\end{aligned}$$

Where the fourth line follows from our induction hypothesis.

$\cdot/\!\!/\cdot$

Proof of Theorem 3.3: Denote

$$F(s)=\sum_{n=1}^{\infty}\frac{d(kn)}{n^s}.$$

By Corollary 2.2, for $\sigma>1$,

$$F(s)=\zeta^2(s)\prod_{p|k}(1+\delta_p(k)(1-p^{-s}))\equiv\zeta^2(s)H_k(s).$$

To compute the residue for Perron's formula, we will need the first two terms of the Taylor series for $H_k(s)$ about $s=1$. Using logarithmic differentiation, we can compute that

$$H_k'(s)=H_k(s)\sum_{p|K}\frac{\delta_p(k)\log(p)p^{-s}}{1+\delta_p(k)(1-p^{-s})}=H_k(s)\sum_{p|K}\frac{\delta_p(k)\log(p)}{p^s+\delta_p(k)(p^s-1)}.$$

Thus, we get that

$$H_k(1)=\prod_{p|k}(1+\delta_p(k)(1-p^{-1})),\qquad H_k'(1)=H_k(1)\sum_{p|K}\frac{\delta_p(k)\log(p)}{p+\delta_p(k)(p-1)}.$$

The residue of the expression is thus

$$\begin{aligned}
\mathrm{Res}_{w=1}\left\{\zeta^2(w)H_k(w)\frac{x^w}{w}\right\} &= 2\gamma H_k(1)x+H_k'(1)x+H_k(1)x\log(x)-H_k(1)x\\
&= x\log(x)H_k(1)+xH_k(1)\left(2\gamma-1+\sum_{p|K}\frac{\delta_p(k)\log(p)}{p+\delta_p(k)(p-1)}\right).
\end{aligned}$$

By residue theorem, for $b > 1/2$,

$$\mathrm{Res}_{w=1}\left\{\zeta^2(w)H_k(w)\frac{x^w}{w}\right\} = \frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{c-iT}\right)\left(\zeta^2(w)H_k(w)\frac{x^w}{w}\right)\, dw.$$

Take $b = (1/2) + 1/\log(x)$. If $\mathrm{Im}\{w\} = \pm T$ and $b \le \mathrm{Re}\{w\} \le c$, then

$$|H_k(w)| = \left|\prod_{p|k}(1 + \delta_p(k)(1 - p^{-s}))\right| \le \prod_{p|k}(1 + \delta_p(k)(1 + p^{-1/2})) \equiv C_k.$$

Then, we have that

$$\left|\int_{c+iT}^{b+iT} \zeta^2(w)H_k(w)\frac{x^w}{w}\, dw\right| \ll \frac{T^{1/3}C_k x^c}{T} = \frac{C_k x^c}{T^{2/3}}.$$

And the bottom edge has the same bound. If we consider the left edge, then

$$\begin{aligned}\left|\int_{b-iT}^{b+iT} \zeta^2(w)H_k(w)\frac{x^w}{w}\, dw\right| &= \left|\int_{-T}^{T} \zeta^2(b+it)H_k(b+it)\frac{x^{b+it}}{b+it}\, dt\right| \\ &\ll C_k x^{(1/2)+\log(x)} \int_{-T}^{T}\left|\frac{t^{1/3}}{b+it}\right| dt \\ &\ll C_k x^{1/2}T^{1/3}.\end{aligned}$$

Therefore,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(w)\frac{x^w}{w}\, dw = \mathrm{Res}_{w=1}\left\{\zeta^2(w)H_k(w)\frac{x^w}{w}\right\} + O\left(\frac{C_k x^c}{T^{2/3}}\right) + O\left(C_k x^{1/2}T^{1/3}\right).$$

Now, we want to apply Perron's formula. $|d(kn)| \le (kn)^\delta$, hence take $\psi(n) = k^\delta n^\delta$. In addition,

$$\sum_{n=1}^{\infty}\frac{d(kn)}{n^\sigma} = k^\sigma\sum_{n=1}^{\infty}\frac{d(kn)}{(kn)^\sigma} \le k^\sigma\sum_{n=1}^{\infty}\frac{d(n)}{n^\sigma} = k^\sigma\zeta^2(\sigma) \ll \frac{1}{(\sigma-1)^2}.$$

Thus $\alpha = 2$. Taking x to be half of an odd integer, the error terms from Perron's formula are

$$\begin{aligned}O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) &= O\left(\frac{x^c}{T(c-1)^2}\right) \\ O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right) &= O\left(\frac{x^{1+\delta}\log(x)}{T}\right) \\ O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right) &= O\left(\frac{x^{1+\delta}}{T}\right).\end{aligned}$$

We will take $c = 1 + 1/\log(x)$. Then,

$$O\left(\frac{x^c}{T(c-1)^2}\right) = O\left(\frac{x\log(x)^2}{T}\right) \ll O\left(\frac{x^{1+\delta}\log(x)}{T}\right) \qquad O\left(\frac{C_k x^c}{T^{2/3}}\right) = \left(\frac{C_k x}{T^{2/3}}\right).$$

In order to minimize error, we want

$$\frac{x}{T^{2/3}} = x^{1/2}T^{1/3} \implies T = x^{1/2}.$$

Then, by Perron's formula,

$$\sum_{n<x} d(kn) = x\log(x)H_k(1) + xH_k(1)\left(2\gamma - 1 + \sum_{p|K}\frac{\delta_p(k)\log(p)}{p+\delta_p(k)(p-1)}\right) + O(C_k x^{2/3}).$$

$\cdot /\!\!/ \cdot$

Proof of Theorem 3.4: Denote

$$F(s) = \sum_{n=1}^{\infty}\frac{d(kn)d(ln)}{n^s}.$$

For $\sigma > 1$,

$$F(s) = \frac{\zeta(s)^4}{\zeta(2s)}\prod_{p|kl}\left(p^{-s} + (1-p^{-s})\frac{(1-\delta_p(k)\delta_p(l))p^{-s} + (1+\delta_p(k))(1+\delta_p(l))}{(1+p^{-s})}\right) \equiv \frac{\zeta(s)^4}{\zeta(2s)}H_{kl}(s).$$

The Taylor expansion of $\zeta^4(s)$ at $s=1$ is given by

$$\begin{aligned}\zeta(s)^4 &= \frac{1}{(s-1)^4} + \frac{4\gamma}{(s-1)^3} + \frac{6\gamma^2+4\gamma_1}{(s-1)^2} + \frac{4\gamma^3+16\gamma\gamma_1+4\gamma_2}{(s-1)} + \cdots\\ &\equiv \frac{1}{(s-1)^4} + \frac{a_1}{(s-1)^3} + \frac{a_2}{(s-1)^2} + \frac{a_3}{(s-1)} + \cdots.\end{aligned}$$

Then, we can compute

$$\begin{aligned}\mathrm{Res}_{w=1}\left\{\frac{\zeta(w)^4}{\zeta(2w)}H_{kl}(w)\frac{x^w}{w}\right\} &= x\log(x)^3\left(\frac{H_{kl}(1)}{\pi^2}\right)\\ &+ x\log(x)^2\left(\frac{12H_{kl}(1)\gamma}{\pi^2} + \frac{c_1H_{kl}(1)}{2} + \frac{3(H_{kl}'(1)-H_{kl}(1))}{\pi^2}\right) + O_{k,l}(x\log(x)).\end{aligned}$$

By residue theorem, for $b > 1/2$,

$$\mathrm{Res}_{w=1}\left\{\frac{\zeta(w)^4}{\zeta(2w)}H_{kl}(w)\frac{x^w}{w}\right\} = \frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{c-iT}\right)\left(\frac{\zeta(w)^4}{\zeta(2w)}H_{kl}(w)\frac{x^w}{w}\right)\,dw.$$

Note that

$$H_{kl}(1) = \prod_{p|kl}\left(p^{-1} + (p-1)\frac{(1-\delta_p(k)\delta_p(l))p^{-1} + (1+\delta_p(k))(1+\delta_p(l))}{(p+1)}\right).$$

Take $b = (1/2) + 1/\log(x)$. If $|\mathrm{Im}\{w\}| \le T$ and $b \le \mathrm{Re}\{w\} \le c$, then

$$\begin{aligned}|H_{kl}(w)| &= \prod_{p|kl}\left|p^{-s} + (1-p^{-s})\frac{(1-\delta_p(k)\delta_p(l))p^{-s} + (1+\delta_p(k))(1+\delta_p(l))}{(1+p^{-s})}\right|\\ &< \prod_{p|kl}\left(p^{-1/2} + (1+p^{-1/2})\frac{|1-\delta_p(k)\delta_p(l)|p^{-1/2} + (1+\delta_p(k))(1+\delta_p(l))}{(1+p^{-2})}\right) \equiv C_{kl}.\end{aligned}$$

Then, we have that

$$\left|\int_{c+iT}^{b+iT}\frac{\zeta(w)^4}{\zeta(2w)}H_{kl}(w)\frac{x^w}{w}\,dw\right| \ll \frac{T^{2/3}C_{kl}x^c\log(x)}{T} = \frac{C_{kl}x^c\log(x)}{T^{1/3}}.$$

And the bottom edge has the same bound. If we consider the left edge, then

$$\begin{aligned}\left|\int_{b-iT}^{b+iT} \frac{\zeta(w)^4}{\zeta(2w)} H_{kl}(w) \frac{x^w}{w}\, dw\right| &= \left|\int_{-T}^{T} \frac{\zeta(b+it)^4}{\zeta(b+it)} H_{kl}(b+it) \frac{x^{b+it}}{b+it}\, dt\right| \\ &\ll C_{kl} x^{(1/2)+\log(x)} \log(x) \int_{-T}^{T} \left|\frac{t^{2/3}}{b+it}\right| dt \\ &\ll C_{kl} x^{1/2} \log(x) T^{2/3}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(w)\frac{x^w}{w}\, dw &= \mathrm{Res}_{w=1}\left\{\frac{\zeta(w)^4}{\zeta(2w)} H_{kl}(w)\frac{x^w}{w}\right\} \\ &\quad + O\left(\frac{C_{kl}x^c\log(x)}{T^{1/3}}\right) + O\left(C_{kl}x^{1/2}\log(x)T^{2/3}\right).\end{aligned}$$

$|d(kn)| \leq (kn)^\delta$, thus we take $\psi(n) = (kl)^\delta n^{2\delta}$. In addition,

$$\sum_{n=1}^{\infty}\frac{d(kn)d(ln)}{n^\sigma} = (kl)^\sigma \sum_{n=1}^{\infty}\frac{d(kn)d(ln)}{(kln)^\sigma} \leq (kl)^\sigma \sum_{n=1}^{\infty}\frac{d(n)^2}{n^\sigma} = (kl)^\sigma \frac{\zeta(\sigma)^4}{\zeta(2\sigma)} \ll \frac{1}{(\sigma-1)^4}$$

Choose $\alpha = 4$. Taking x to be half of an odd integer, the error terms from Perron's formula are

$$\begin{aligned} O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) &= O\left(\frac{x^c}{T(c-1)^4}\right), \\ O\left(\frac{\psi(2x)x^{1-\sigma}\log(x)}{T}\right) &= O\left(\frac{x^{1+2\delta}\log(x)}{T}\right), \\ O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right) &= O\left(\frac{x^{1+2\delta}}{T}\right).\end{aligned}$$

We will take $c = 1 + 1/\log(x)$. Then,

$$O\left(\frac{x^c}{T(c-1)^2}\right) = O\left(\frac{x\log(x)^4}{T}\right) \ll O\left(\frac{x^{1+\delta}\log(x)}{T}\right), \quad O\left(\frac{C_{kl}x^c\log(x)}{T^{1/3}}\right) = \left(\frac{C_{kl}x\log(x)}{T^{1/3}}\right).$$

In order to minimize error, we want

$$\frac{x\log(x)}{T^{1/3}} = x^{1/2}\log(x)T^{2/3} \implies T = x^{1/2}.$$

Then, by Perron's formula,

$$\sum_{n<x} d(kn)d(ln) = \mathrm{Res}_{w=1}\left\{\frac{\zeta(w)^4}{\zeta(2w)} H_{kl}(w)\frac{x^w}{w}\right\} + O(C_{kl}x^{5/6}\log(x)).$$

$\cdot /\!/ \cdot$

Proof of Corollary 3.5: By Corollary 2.5, $H_{kl}(1)$ simplifies to

$$H_{kl}(1) = \prod_{p|k}\left(1 + \delta_p(k)\left(\frac{p-1}{p+1}\right)\right)\prod_{p|l}\left(1 + \delta_p(l)\left(\frac{p-1}{p+1}\right)\right).$$

$\cdot /\!/ \cdot$

References

[Ber] B. C. Berndt, "Ramanujan's Notebooks I", Springer-Verlag, (1985).

[Br] R. Brown, "The Natural Density of Some Sets of Square-free Numbers."

[Ti] E. C. Titchmarsh, "The Theory of the Riemann Zeta Function." Second edition, published by Oxford University Press in 1986.

A Code

```
import math

def main():

    # Parameters and counts
    primes = [2, 3]      # List primes to filter out here
    N = 1000000          # Select number to count until
    square_free = 0      # Count for square frees
    sf_without_p = 0     # Count for square frees without the specified primes

    # Running the loop
    for i in range(N):

        # Variables for the loop
        square_free_add = 1             # Assume i is square free
        without_add = 1                 # Assume i is indivisible by specified primes
        previous_prime = 1              # Variable to keep track of previous prime
        cri = math.ceil(i ** (1 / 3))   # Highest factor we need to check is i^{1/3}

        while i > cri:
            next_prime = prime_factor(i)
            if next_prime == previous_prime:
                square_free_add = 0
                without_add = 0
                break
            if next_prime in primes:
                without_add = 0
            previous_prime = next_prime
            i /= previous_prime

        # Checking if remainder is equal to previous prime
        if i in primes:
            without_add = 0
        if i == previous_prime:
            square_free_add = 0
            without_add = 0

    # Print out results
    print(square_free)
    print(sf_without_p)

def prime_factor(n):
    if n in [1, 2, 3]:
        return n
    for i in range(2, math.ceil(math.sqrt(n)) + 1):
        if n % i == 0:
            return i
    return n
```