A Primer to Linear Integral Equations of the Second Kind

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Abstract

This honors thesis is a discussion of linear integral equations of the second kind. Much of the content draws upon an article from Mathematics Magazine by Glenn Ledder and we cover the theoretical background associated to these equations. We also go over some of their applications to various population models.

1 Introduction

In this paper, we present linear integral equations of the second kind. We focus on their analytical features, in particular the study of their solutions. This includes methodology proving their existence and uniqueness, as well as estimating their sizes. We also discuss applications of these integral equations to population modeling problems.

A linear integral equations of the second kind has the general form

$$y(x) = f(x) + \int_a^{b(x)} k(x,s)\, y(s)\, ds, \tag{1}$$

where $f = f(x)$ and $k = k(x,s)$ are known functions, $y = y(x)$ is the unknown function, a is a real constant and $b = b(x)$ is either x or a real constant $c > a$. In terms of terminology, the function k is usually called the *kernel* of the integral equation. Moreover, when $b(x) = x$ the equation is called a **linear Volterra integral equation**, while when $b(x) = c \in \mathbb{R}$ the equation is called a **linear Fredholm integral equation**. Given that these equations have many properties in common, we will work onward with the generic upper bound b. As it is usually the case, these equations are called *homogeneous* when $f \equiv 0$.

2 Prerequisites

In here, we introduce some of the basic assumptions and elements that we will be working with in establishing the theoretical results for our integral equations, as well as their practical applications discussed in this paper. We begin with:

Definition 2.1 *Let $k : [a, b] \times [a, b] \to \mathbb{R}$ be a continuous function. Then*

$$M_v := \max_{(x,s)\in[a,b]\times[a,b]} |k(x, s)|$$

and

$$M_f := \max_{x\in[a,b]} \int_a^b |k(x, s)| \, ds.$$

Both quantities are well-defined nonnegative real numbers. For M_v, this is due to Weierstrass theorem applied for the continuous function k on the compact $[a, b] \times [a, b]$. For M_f, this follows from the fact that a continuous function of several variables is also continuous in each of its variables. We will use M_v in connection to Volterra integral equations, while M_f will be relied upon when studying Fredholm integral equations.

It is to see that

$$0 \leq \int_a^b |k(x, s)| \, ds \leq (b - a)M_v, \qquad \forall\ x \in [a, b],$$

and, consequently,

$$M_f \leq (b - a)M_v.$$

Next, we introduce sequences of functions associated to the two integral equations.

Definition 2.2 *Let $f : [a, b] \to \mathbb{R}$ and $k : [a, b] \times [a, b] \to \mathbb{R}$ be both continuous functions. If $y_0 : [a, b] \to \mathbb{R}$ is an arbitrary continuous function, consider the sequence of functions $(y_n)_{n\geq 0} : [a, b] \to \mathbb{R}$ defined iteratively by*

$$y_n(x) := f(x) + \int_a^b k(x, s)y_{n-1}(s) \, ds, \qquad \forall\ n \geq 1,\ x \in [a, b].$$

These are called Volterra or Fredholm sequences depending on the particular formula for b.

An important tool used in working with Volterra and Fredholm sequences is that their homogeneous versions are uniformly convergent to the zero function, with an additional assumption in the case of Fredholm sequences. This is the content of the following two lemmas. The first one addresses Volterra homogeneous sequences.

Lemma 2.3 *If $y_0 : [a, b] \to \mathbb{R}$ is an arbitrary continuous function and*

$$y_n(x) = \int_a^x k(x, s) y_{n-1}(s)\, ds, \qquad \forall\ n \geq 1,\ x \in [a, b],$$

then

$$|y_n(x)| \leq \frac{cM_v^n (x-a)^n}{n!}, \qquad \forall\ n \geq 1,\ x \in [a, b],$$

where

$$c = \max_{x \in [a,b]} |y_0(x)|.$$

As an immediate consequence,

$$|y_n(x)| \leq \frac{cM_v^n (b-a)^n}{n!}, \qquad \forall\ n \geq 1,\ x \in [a, b],$$

and, hence, $y_n \to 0$ uniformly on $[a, b]$ as $n \to \infty$.

Proof. The argument is by induction over n. For the base case, $n = 1$, we have

$$|y_1(x)| \leq \int_a^x |k(x, s)||y_0(s)|\, ds \leq cM_v \int_a^x ds = cM_v(x - a), \qquad \forall\ x \in [a, b].$$

If we assume

$$|y_n(x)| \leq \frac{cM_v^n (x-a)^n}{n!}, \qquad \forall\ x \in [a, b],$$

then

$$\begin{aligned} |y_{n+1}(x)| &\leq \int_a^x |k(x, s)||y_n(s)|\, ds \ \leq \frac{cM_v^{n+1}}{n!} \int_a^x (s-a)^n\, ds \\ &= \frac{cM_v^{n+1} (x-a)^{n+1}}{(n+1)!}, \qquad \forall\ x \in [a, b]. \end{aligned}$$

This concludes the proof.

The second lemma deals with Fredholm homogeneous sequences.

Lemma 2.4 *If $y_0 : [a,b] \to \mathbb{R}$ is an arbitrary continuous function and*

$$y_n(x) = \int_a^b k(x,s) y_{n-1}(s)\, ds, \qquad \forall\ n \geq 1,\ x \in [a,b],$$

then

$$|y_n(x)| \leq c M_f^n, \qquad \forall\ n \geq 1,\ x \in [a,b],$$

where c is as in the previous lemma. If $M_f < 1$, then $y_n \to 0$ uniformly on $[a,b]$ as $n \to \infty$.

Proof. We rely again on induction over n. When $n = 1$ we have

$$|y_1(x)| \leq \int_a^b |k(x,s)||y_0(s)|\, ds \leq c \int_a^b |k(x,s)|\, ds \leq c M_f, \qquad \forall\ x \in [a,b].$$

If we assume

$$|y_n(x)| \leq c M_f^n, \qquad \forall\ x \in [a,b],$$

then

$$\begin{aligned} |y_{n+1}(x)| &\leq \int_a^b |k(x,s)||y_n(s)|\, ds \ \leq c M_f^n \int_a^b |k(x,s)|\, ds \\ &\leq c M_f^{n+1}, \qquad \forall\ x \in [a,b]. \end{aligned}$$

This finishes the argument.

We conclude this section by discussing the optimality of these estimates for two actual Volterra/Fredholm homogeneous sequences. We take:

$$[a,b] = [0,1], \qquad k(x,s) = \lambda x s^2, \qquad y_0 \equiv 1.$$

Computation of a few terms for both sequences allows one to formulate an induction hypothesis in regards to a general formula in terms of n, which is then proven in a direct manner. The results are as follows. The Volterra homogeneous sequence has the general formula

$$y_n(x) = \frac{\lambda^n x^{4n}}{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4n-1)}$$

and the Fredholm homogeneous sequence is given by

$$y_n(x) = \frac{\lambda^n x}{3 \cdot 4^{n-1}}.$$

On the other hand, using Definition 2.1, we deduce

$$M_v = |\lambda| \quad \text{and} \quad M_f = \frac{|\lambda|}{3}.$$

Thus, according to Lemma 2.3 and Lemma 2.4, we should have

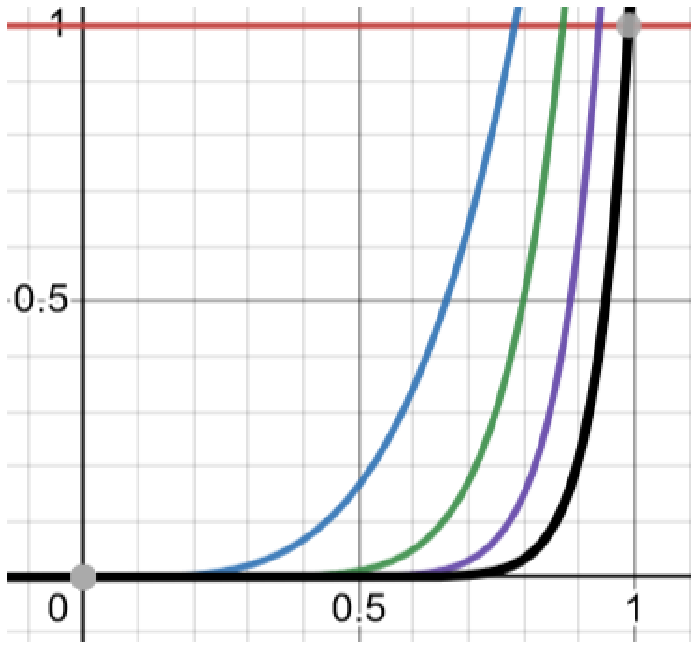
$$|y_n(x)| \leq \frac{|\lambda|^n x^n}{n!}$$

for the Volterra sequence and

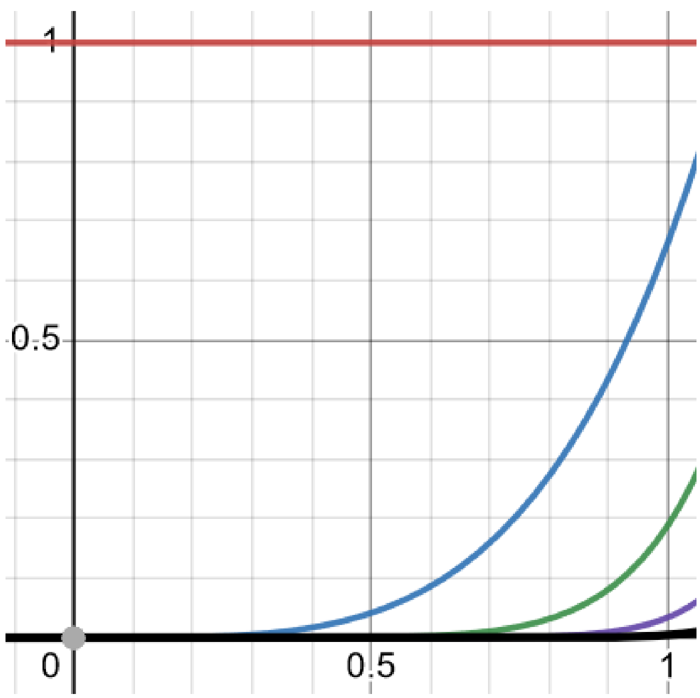
$$|y_n(x)| \leq \frac{|\lambda|^n}{3^n}$$

for the Fredholm sequence. By comparing these bounds with the above formulas, we first notice that the Volterra sequence decays faster than what Lemma 2.3 predicts. This is because x^{4n} decays considerably faster than x^n on $[0, 1]$ and the denominator in the formula for the sequence is significantly larger than $n!$ (i.e., by a factor of 3^n). For the Fredholm sequence the situation is somewhat similar, in the sense that it actually decays like $(|\lambda|/4)^n$, rather than like $(|\lambda|/3)^n$ as Lemma 2.4 suggests. Another important remark is that Lemma 2.4 is also more restrictive than the actual formula in terms of values for λ which are amenable to uniform convergence. Lemma 2.4 predicts $|\lambda| < 3$, while the actual formula accommodates $|\lambda| < 4$.

To have a visual idea about the decay of the two sequences, on the next page, we included the graphs of the first five iterates: $y_0 \equiv 1$ (red graph), y_1 (blue graph), y_2 (green graph), y_3 (purple graph), and y_4 (black graph). For the Volterra homogeneous sequence, we use the values $\lambda = 8$ and $\lambda = 2$. For the Fredholm homogeneous sequence, we use the values $\lambda = 2$ and $\lambda = 0.5$.

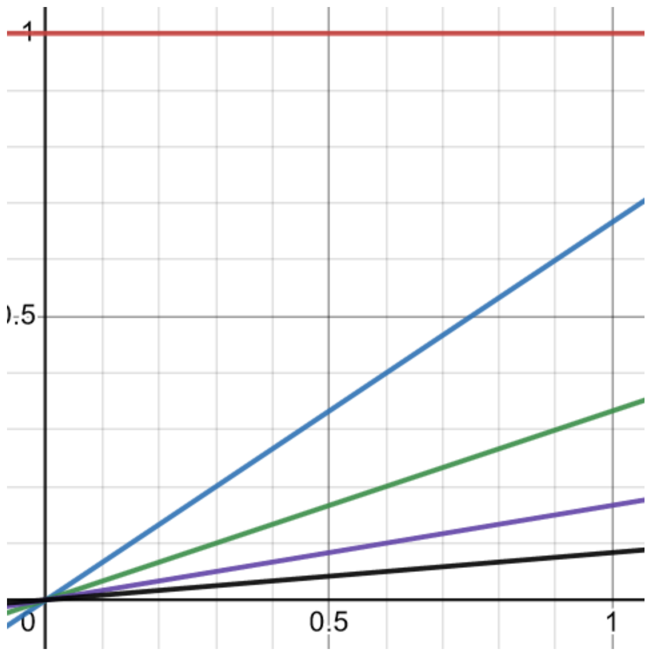


(a) $y_0 = 1$, $k(x,s) = 8xs^2$, $a = 0$, $b = 1$

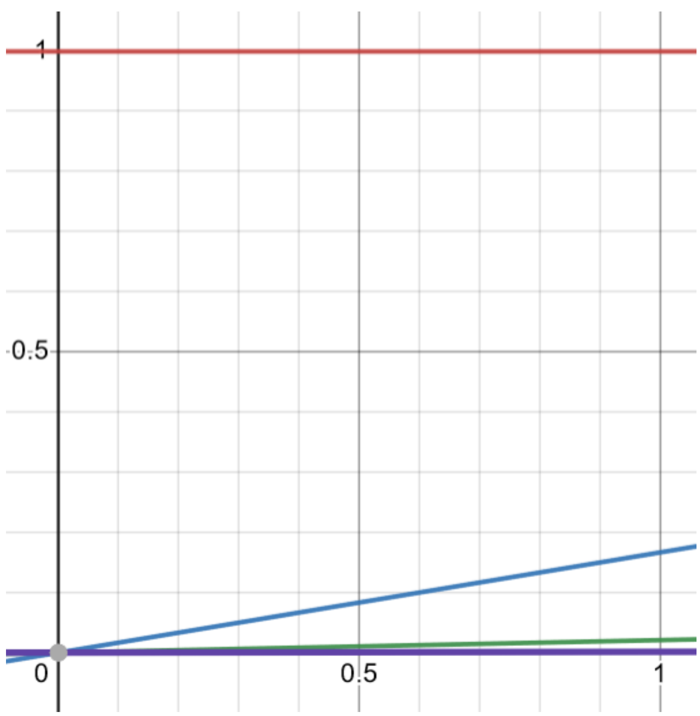


(b) $y_0 = 1$, $k(x,s) = 2xs^2$, $a = 0$, $b = 1$

Figure 1: Homogeneous Volterra Iteration



(a) $y_0 = 1$, $k(x,s) = 2xs^2$, $a = 0$, $b = 1$



(b) $y_0 = 1$, $k(x,s) = 0.5xs^2$, $a = 0$, $b = 1$

Figure 2: Homogeneous Fredholm Iteration

3 Uniqueness and Existence Theorems

Having established these lemmas, we can put the to use by showing the existence and uniqueness of continuous solutions. This first theorem deals with the both the existence and uniqueness of such a solution in the homogeneous case for both Volterra and Fredholm equations.

Theorem 3.1 *(for homogeneous equations)*

- $y = 0$ *is the unique solution to* $y(x) = \int_a^x k(x,s)y(s)\,\mathrm{d}s$
- *If* $M_f < 1$, $y = 0$ *is the unique solution to* $y(x) = \int_a^b k(x,s)y(s)\,\mathrm{d}s$

Proof. $y = 0$ is clearly a solution for $y(x) = \int_a^x k(x,s)y(s)\,\mathrm{d}s$. Now let $y_0 = \phi$ where ϕ is also a solution. Now using the iteration for Volterra/Fredholm sequences, $y_1(x) = \int_a^x k(x,s)\phi\,\mathrm{d}s = \phi$ and clearly, $y_n = \phi$ for all n. Therefore, $\lim_{n\to\infty} y_n(x) = \phi$. Now applying Lemma 2.3 in the Volterra case and Lemma 2.4 in the Fredholm case, $\lim_{n\to\infty} y_n(x) = 0$ (as long as $M_f < 1$ for the Fredholm case). Therefore, $\phi = 0$.

Using the above theorem, we can achieve a similar result to prove the uniqueness of solutions in the non-homogeneous case.

Theorem 3.2 *(uniqueness of solutions)*

- $y(x) = f(x) + \int_a^x k(x,s)y(s)\,\mathrm{d}s$ *has at most one solution*
- *If* $M_f < 1$, *then* $y(x) = f(x) + \int_a^b k(x,s)y(s)\,\mathrm{d}s$ *has at most one solution*

Proof. Let ϕ_1 and ϕ_2 be solutions and let $z = \phi_1 - \phi_2$. Now $z(x) = \int_a^x k(x,s)(\phi_1(s) - \phi_2(s))\,\mathrm{d}s = \int_a^x k(x,s)z(s)\,\mathrm{d}s$. Since this integral equation is homogeneous, theorem 3.1 can be applied to z, yielding $z = 0$, so $\phi_1 = \phi_2$.

Having shown the uniqueness of solutions in both the homegenous and non-homogeneous cases, all that remains is the issue of existence, for which we apply the previous lemmas.

Theorem 3.3 *(existence of solutions) Let $y_0 = f$ and consider a Volterra/Fredholm sequence $\{y_n\}$ using $k(x, s)$.*

- *In the Volterra case, $\{y_n\}$ converges to the unique solution of $y(x) = f(x) + \int_a^x k(x, s)y(s)\,\mathrm{d}s$*
- *In the Fredholm case, if $M_f < 1$, then $\{y_n\}$ converges to the unique solution of $y(x) = f(x) + \int_a^b k(x, s)y(s)\,\mathrm{d}s$*

Proof. First define the sequence $\{E_n\}$: $E_0 = y_0$ and $E_n = y_n - y_{n-1}$. Importantly, the triangle inequality gives that for any positive integers n and m:

$$\begin{aligned} | y_{n+m} - y_n| &= |y_{n+m} - y_{n+m-1} + y_{n+m-1} - y_{n+m-2} + ...y_{n+1} - y_n| \\ &\leq |E_{n+m}| + |E_{n+m-1}| + ...|E_{n+1}| < \sum_{m=1}^{\infty} |E_{n+m}| \end{aligned}$$

This will help us prove that $\{y_n\}$ is a Cauchy series, which means that it converges to some function ϕ. Furthermore, the convergent function ϕ will be a solution to the integral equation because of the aforementioned uniform convergence:

$$\phi(x) = \lim_{n\to\infty} y_n(x) = \int_a^R k(x, s) \lim_{n\to\infty} y_{n-1}(s)\,\mathrm{d}s = \int_a^R k(x, s)\phi(s)\,\mathrm{d}s$$

Theorem 3.2 gives that $\lim_{n\to\infty} y_n = \phi$ will be the unique solution to the integral equations. Now all we need to show is that $g_n = \sum_{m=1}^{\infty} |E_{n+m}|$ exists and converges to zero. The above related inequality will then automatically show that $\{y_n\}$ is a Cauchy series.

To this end, we will use induction to first establish:

$$E_n = \int_a^R k(x, s)E_{n-1}(s)\,\mathrm{d}s$$

When n=1, we have

$$E_1 = y_1 - y_0 = f(x) + \int_a^R k(x,s) y_0(s)\,\mathrm{d}s - f(x) = \int_a^R k(x,s) E_0(s)\,\mathrm{d}s$$

If we assume

$$E_m = \int_a^R k(x,s) E_{m-1}(s)\,\mathrm{d}s \quad for \;\; m = 1....n-1$$

Then,

$$E_n = y_n - y_{n-1} = \int_a^R k(x,s)(y_{n-1}(s) - y_{n-2}(s))\,\mathrm{d}s = \int_a^R k(x,s) E_{n-1}(s)\,\mathrm{d}s$$

Now we can now use Lemmas 2.3and 2.4 on E_n.

In the Volterra case, define the real number $\beta = \sum_{m=1}^{\infty}[M_V^m (b-a)^m/m!] = e^{M_v(b-a)} - 1$. Now applying Lemma 2.3:

$$\begin{aligned} g_n = \sum_{m=1}^{\infty} |E_{n+m}| &\leq \sum_{m=1}^{\infty} cM_V^{n+m}(x-a)^{n+m}/(n+m)! \\ &< cM_V^n (x-a)^n/(n)! * \sum_{m=1}^{\infty} cM_V^m (x-a)^m/(m)! \leq \beta c M_V^n (b-a)^n/(n)! \end{aligned}$$

This computation is done first applying lemma, then by systematically replacing x with b and then using $n!m! < (m+n)!$. Given this bound for g_n, $\lim_{n\to\infty} g_n = 0$ and the proof is completed for the Volterra case.

In the Fredholm case, we can simply apply Lemma 2.4 so long as $M_F < 1$:

$$g_n = \sum_{m=1}^{\infty} |E_{n+m}| \leq \sum_{m=1}^{\infty} cM_F^{n+m} = cM_F^{n+1} \sum_{m=1}^{\infty} cM_F^{m-1} = c(1-M_F)^{-1} M_F^{n+1}$$

Given this bound $\{g_n\}$ converges to 0 in the Fredholm case as well, completing the proof.

3.1 Non-Homogenous Examples

Given the constructive nature of theorem 3, we can construct an iterative sequence that will converge to the solution of any Volterra equation and Fredholm equation with $M_f < 1$. Using computers, it is simple to compute enough terms to achieve a given accuracy. However, this process will still be affected by how quickly the sequence converges. To demonstrate this process, we will use the following 2 examples to show the convergence of non-homogeneous linear integral equations.

$$y(x) = 1 - 2x^4/3 + \int_0^x 2xs^2 y(s)\,\mathrm{d}s$$

$$y(x) = 1 - 2x/3 + \int_0^1 2xs^2 y(s)\,\mathrm{d}s$$

In this example, $[a, b] = [0, 1]$ and $M_v = 2, M_f = 2/3$, so we can safely apply the theorem. Additionally, the kernal functions have been taken from the previous homogeneous examples and the non-homogeneous terms were chosen to ensure that the sequence converged to $y \equiv 1$. The following graphs once again show the first 5 iterations: $y_0 \equiv f$ (red graph), y_1 (blue graph), y_2 (green graph), y_3 (purple graph), and y_4 (black graph).

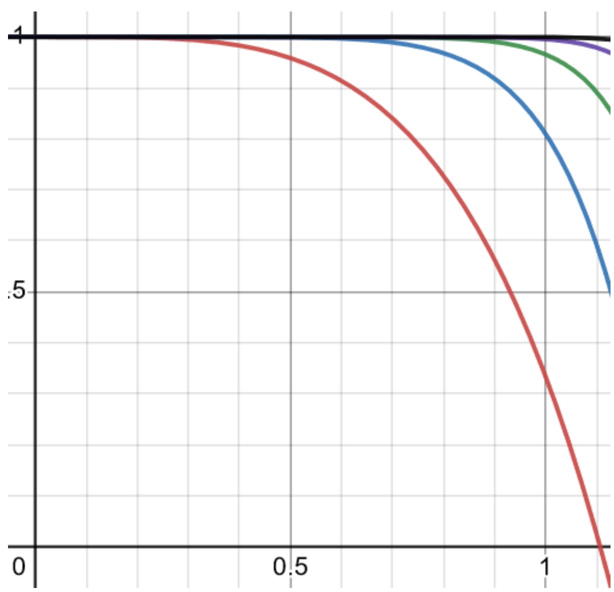


Figure 3: Non-Homogeneous Volterra Iteration

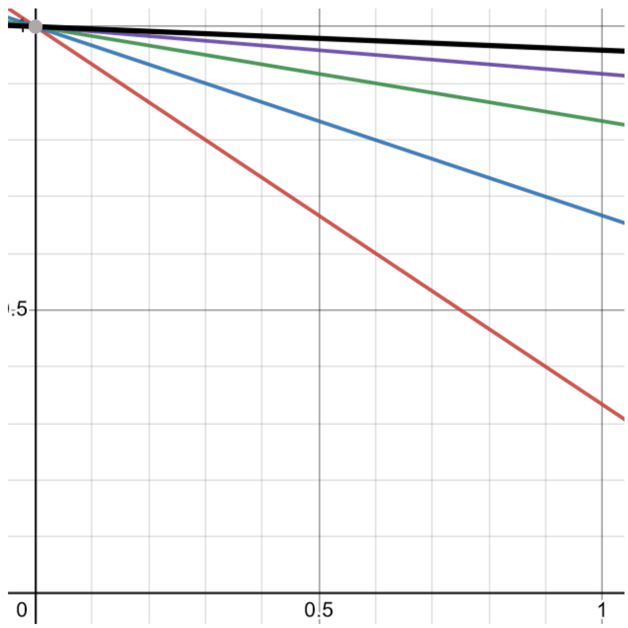


Figure 4: Non-Homogeneous Fredholm Iteration

4 Population Model

To illustrate an application of integral equations, we will showcase their use in a simple population model: a general single sex population model with unlimited resources and an immortal population. First, let $P(t)$ be a continuous function representing the population and let $P(0) = P_0$ be the initial population. As is generally the case when modelling large populations, $P(t)$ is an approximation of the population, not the actual population, which is an integer equation that is not continuous.

Next, let $B(t)$ be the net birth rate such that $B(t) = \mathrm{d}P/\mathrm{d}t$. In the case of a constant fertility rate r:

$$B(t) = rP(t), P(0) = P_0$$

In the rest of the model, we assume that the net birth rate is related to the current population by a positive, continuous, age-dependent fertility rate, $r(a)$ where a is the age. This function is assumed to be increasing up to age A and decreases thereafter.

4.1 Age Distribution

Aside from the total population, it is also important to keep track of the population's age distribution. As a result, let $Q(a,t)$ represent the population of individuals under age a at time t, and let $u(a,t)$ be the population density function:

$$u(a,t) = \frac{\mathrm{d}Q}{\mathrm{d}a}(a,t) \geq 0$$

Using this definition, the population of individuals between ages a_1 and a_2, P_{a_1,a_2} can be represented as:

$$P_{a_1,a_2}(t) = Q(a_2,t) - Q(a_1,t) = \int_{a_1}^{a_2} \frac{\mathrm{d}Q}{\mathrm{d}a}(a,t)\,\mathrm{d}a = \int_{a_1}^{a_2} u(a,t)\,\mathrm{d}a$$

$$P(t) = P_{0,\infty}(t) = \int_0^\infty u(a,t)\,\mathrm{d}a$$

This integral converges whenever the population is finite. Now let $u(a,0) = u_0(a)$ be the known initial age distribution. As a result:

$$P(0) = \int_0^\infty u_0(a)\,\mathrm{d}a$$

Furthermore, the birth rate at time t of individuals from ages a to $a + \mathrm{d}a$ is approximated by $r(a)u(a,t)\mathrm{d}a$, which gives the following equation for the net birth rate at time t:

$$B(t) = \int_0^\infty r(a)u(a,t)\,\mathrm{d}a$$

This integral collapses into $B(t) = rP(t)$ when r is constant. To show that this integral converges generally, we use the fact that r(a) increases until a point A and decreases thereafter:

$$0 \leq \int_0^\infty r(a)u(a,t)\,\mathrm{d}a = \int_0^A r(a)u(a,t)\,\mathrm{d}a + \int_A^\infty r(a)u(a,t)\,\mathrm{d}a \leq \\ \int_0^A r(a)u(a,t)\,\mathrm{d}a + r(A)P_{A,\infty}(t)$$

Once we solve for the birth rate, we can integrate over the birth rate to determine the population at any time:

$$P(t) = P(0) + \int_0^t B(s)\,\mathrm{d}s$$

4.2 Solving the Birthrate

Since our population is immortal:

$$Q(\infty, t) - Q(a + t, t) = Q(\infty, 0) - Q(a, 0)$$

Differentiating with respect to a yields

$$u(a + t, t) = u(a, 0)$$

or

$$u(a, t) = u(a - t, O) = u_0(a - t) \quad for \;\; a \geq t \geq O.$$

Moreover, given a < t, the number of individuals under age a at time t is simply the number of individuals born between t-a and t:

$$Q(a, t) = \int_{t-a}^{t} B(\tau)\, \mathrm{d}\tau for a < t$$

Differentiating with respect to a once again:

$$u(a, t) = B(t - a) for a < t$$

Now we can substitute back into our equation for B(t):

$$B(t) = \int_{t}^{\infty} r(a) u_0(a - t)\, \mathrm{d}a + \int_{0}^{t} r(a) B(t - a)\, \mathrm{d}a$$

We can now change the variables to make this into a Volterra equation, finishing the model and allowing us to use the previously discussed properties and iteration:

$$B(t) = f(t) + \int_{0}^{t} r(t - s) B(s)\, \mathrm{d}s$$

where

$$f(t) = \int_{0}^{\infty} r(t + s) u_0(s)\, \mathrm{d}s$$

5 Incorporating Death Rates

Adding to the previous model, we rework the problem to include an age dependent death rate, removing the previous immortality assumption. Let $u(a, t)$ for $a, t \geq 0$ be the age distribution function that we need to solve. Let $\mu(a, t) \geq 0$

be the age specific death rate and $\lambda(a,t)$ is the age specific fertility rate. We subject u(a,t) to the following conditions:

$$\frac{\partial u}{\partial a} + \frac{\partial u}{\partial t} = -\mu(a,t)u \tag{2}$$

$$u(0,t) = \int_0^\infty \lambda(a,t)u(a,t)\,\mathrm{d}a \tag{3}$$

$$u(a,0) = u_0(a)$$

5.1 Method of Characteristics

To solve for $u(a,t)$ for $a, t \geq 0$ we will employ the method of characteristics on the first condition [3]. Let $x = (a(s), t(s))$, $z = u(x(s))$, $p(s) = Du(x(s))$. Restructuring equation 2:

$$F(p,z,x) = p_1 + p_2 + \mu(x_1, x_2)z = 0$$

Therefore, the method of characteristics indicates:

$$\dot{x} = F_p = (1,1) \tag{4}$$

$$\dot{z} = p \cdot F_p = p_1 + p_2 = -\mu(x_1, x_2)z \tag{5}$$

Using equation 5 and integrating factors, the general solution to $u(x_1, x_2)$ with initial data c is:

$$u(x_1(s), x_2(s)) = c * exp(-\int_0^S \mu(x_1(\sigma), x_2(\sigma))\,\mathrm{d}\sigma)$$

From equation 4, we see that the characteristics of this differential equation are parallel to t=a. However, our conditions is only valid for $a, t \geq 0$, so the initial data will change based on whether $a \geq t$ or $a < t$. In the first case, let $\gamma = a - t$, so $x_1(s) = s + \gamma$ and $x_2(s) = s$. Furthermore, the initial data $z_0 = u(\gamma, 0) = u_0(a-t)$ and thus the solution to u for $a \geq t$ is:

$$u(a,t) = u_0(a-t) * exp(-\int_0^t \mu(\sigma + a - t, \sigma)\,\mathrm{d}\sigma)$$

Similarly, if $a < t$, let $\gamma = t - a$ and suppose there exists a function such that $B(t) = u(0,t)$. The initial data would then be $z_0 = u(0,\gamma) = B(t-a)$, and for x we have $x_1(s) = s$ and $x_2(s) = s + \gamma$. Therefore the solution to u for $a < t$ is:

$$u(a,t) = B(t-a) * exp(-\int_0^a \mu(\sigma, \sigma + t - a)\,\mathrm{d}\sigma)$$

Now solving for B(t), set

$$k(a,t) = exp(-\int_0^a \mu(\sigma, \sigma + t - a)\,\mathrm{d}\sigma)$$

$$m(a,t) = exp(-\int_0^t \mu(\sigma + a - t, \sigma)\,\mathrm{d}\sigma)$$

Now using equation 3, $B(t) = u(0,t) = \int_0^\infty \lambda(a,t)u(a,t)\,\mathrm{d}a$ and substituting for k and m:

$$B(t) = \int_0^t \lambda(a,t)B(t-a)k(a,t)\,\mathrm{d}a + \int_t^\infty \lambda(a,t)u_0(t-a)m(a,t)\,\mathrm{d}a$$

Rewriting this equation, we can reduce the problem of solving for u(a,t) to a Volterra equation:

$$B(t) = f(t) + \int_0^t K(a,t)B(t-a)\,\mathrm{d}a \tag{6}$$

Where $f(t) = \int_t^\infty \lambda(a,t)u_0(t-a)m(a,t)\,\mathrm{d}a$ and $K(a,t) = \lambda(a,t)k(a,t)$. Importantly, f(t) and K(a,t) depend solely on given data for the given population [3].

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