INVOLVE : (2020)
dx.doi.org/10.2140/involve.2020..101

Uniqueness of a three-dimensional stochastic differential equation

Carl Mueller and Giang Truong

(Communicated by Amarjit Singh Budhiraja)

In order to extend the study of the uniqueness property of multidimensional systems of stochastic differential equations, we look at the following three-dimensional system of equations, of which the two-dimensional case has been well studied: $dX_t = Y_t\,dt$, $dY_t = Z_t\,dt$, $dZ_t = |X_t|^\alpha\,dB_t$. We prove that if $(X_0, Y_0, Z_0) \neq (0, 0, 0)$ and $\frac{3}{4} < \alpha < 1$, then the system of equations has a unique solution in the strong sense.

1. Introduction and main results

The uniqueness of ordinary differential equations (ODEs) has been extensively studied; see for example [Hartman 1964]. In particular, if $F(u)$ is Lipschitz continuous, then

$$u'(t) = F(u(t)), \qquad u(0) = u_0,$$

has a unique solution for all $t \geq 0$. In the case above, F, $u(t)$, and u_0 take values in $\mathbb{R}^d$, $d \geq 1$. The stochastic differential equation (SDE) realm, on the contrary, has different criteria for uniqueness of solutions; see for example [Protter 1990]. One of the most well-known results regarding strong uniqueness of SDEs is due to [Watanabe and Yamada 1971]. The result states that if $f(x)$ is locally Hölder continuous with index $\alpha \in \left[\frac{1}{2}, 1\right]$ and with linear growth, then

$$dX = f(X)\,dW, \qquad X_0 = x_0,$$

has a unique strong solution for all times $t \geq 0$. Yamada and Watanabe's theory essentially focuses on one-dimensional SDEs.

MSC2010: primary 60H10; secondary 34F05.
Keywords: white noise, stochastic differential equations, uniqueness.
Mueller was supported by a Simons grant.

One motivation for studying higher-dimensional SDEs comes from the wave equation

$$\begin{aligned}\partial_t^2 u &= \Delta u,\\ u(0,x) &= u_0(x),\\ \partial_t u(0,x) &= u_1(x).\end{aligned}$$

In this equation, we have

$$\partial_t^2 u = \partial_x^2 u = \Delta u.$$

If we let

$$v = \partial_t u,$$

then we can rewrite the wave equation as the following system of equations:

$$\begin{aligned}\partial_t u &= v,\\ \partial_t v &= \Delta u.\end{aligned}$$

The original wave equation includes no noise. However, many physical systems are affected by noise. Hence, a modification of the wave equation which includes white noise is also studied:

$$\begin{aligned}\partial_t^2 u &= \Delta u + f(u)\dot{W},\\ u(0,x) &= u_0(x),\\ \partial_t u(0,x) &= u_1(x).\end{aligned} \tag{1}$$

Note $x \in \mathbb{R}$ and $\dot{W} = \dot{W}(t,x)$ is white noise.

One well-known point is that Lipschitz continuity is sufficient for the uniqueness of SDEs. Thus, many mathematicians have studied whether Hölder continuity can still ensure the uniqueness property of SDEs. Gomez, Lee, Mueller, Neuman, and Salins [Gomez et al. 2017] studied the uniqueness property of the following two-dimensional model of SDEs:

$$\begin{aligned}dX &= Y\,dt,\\ dY &= |X|^\alpha\,dB,\\ (X_0,Y_0) &= (x_0,y_0).\end{aligned} \tag{2}$$

The results focused on $f(x) = |x|^\alpha$ since it is a prototype of an equation with Hölder continuous coefficients. Moreover, (2) is a version of (1) when we drop the dependence on x, which allows us to study the modified wave equation with more simplicity. Notice that if we take the differential dY of the first derivative of X, which is Y in the system of equations, it resembles the second derivative in time in the stochastic wave equation.

It was proven in [Gomez et al. 2017] that if $\alpha > \frac{1}{2}$ and $(x_0, y_0) \neq (0, 0)$, then (2) has a unique solution in the strong sense up to the time τ at which (X_t, Y_t) first hits the origin $(0, 0)$.

Since much is still unknown about higher-dimensional SDEs, we wish to continue the study of the uniqueness property of (2) in the three-dimensional case, which is

$$\begin{aligned} dX &= Y\,dt, \\ dY &= Z\,dt, \\ dZ &= |X|^{\alpha} dB, \\ (X_0, Y_0, Z_0) &= (x_0, y_0, z_0). \end{aligned} \tag{3}$$

Theorem 1. *If* $\frac{3}{4} < \alpha < 1$ *and* $(X_0, Y_0, Z_0) \neq 0$, *then* (3) *has a unique strong solution, up to the time* τ *at which the solution* (X_t, Y_t, Z_t) *first hits the value* $(0, 0, 0)$ *or blows up.*

Moreover, we say the solution (X_t, Y_t, Z_t) blows up in finite time, with positive probability, if there is a random time $\tau < \infty$ such that

$$P\Big(\lim_{t \uparrow \tau} |(X_t, Y_t, Z_t)|_{l^\infty} = \infty\Big) > 0.$$

2. Proof of Theorem 1

Let (X_t^i, Y_t^i, Z_t^i), $i = 1, 2$, be two solutions to (3) with $(x_0, y_0, z_0) \neq (0, 0, 0)$; in other words, (X_t^i, Y_t^i, Z_t^i), $i = 1, 2$, have the same initial condition $(X_0, Y_0, Z_0) \neq (0, 0, 0)$. Note that since (X_t^i, Y_t^i, Z_t^i), $i = 1, 2$, have the same initial conditions, from now on, we use X_0, Y_0, Z_0 instead of $X_0^{i,n}, Y_0^{i,n}, Z_0^{i,n}$.

Let τ be the first time t that either (X_t^1, Y_t^1, Z_t^1) or (X_t^2, Y_t^2, Z_t^2) hits the origin $(0, 0, 0)$ or blows up. We let τ be infinity if there is no such time.

First, if $X_0 \neq 0$, then (3) will have Lipschitz continuity up to the time that $X_t = 0$, and thus enables uniqueness to hold. Suppose after a certain amount of time X_t hits zero, where Lipschitz continuity no longer holds; then due to the strong Markov property, we begin the process again with $X_0 = 0$. So our goal is to prove pathwise uniqueness between excursions of X up to the time τ starting from $X_0 = 0$.

For any fixed n, let τ_n be the first time that either

$$|(X_t^1, Y_t^1, Z_t^1)|_{l^\infty} \wedge |(X_t^2, Y_t^2, Z_t^2)|_{l^\infty} \leq 2^{-n}$$

or

$$|(X_t^1, Y_t^1, Z_t^1)|_{l^\infty} \vee |(X_t^2, Y_t^2, Z_t^2)|_{l^\infty} \geq 2^{n}.$$

Here, the infinity norm (also known as the L_∞-norm, l_∞-norm, max norm, or uniform norm) of a vector $\vec{v}$ is denoted by $|\vec{v}|_{l_\infty}$ and is defined as the maximum of the absolute values of its components,

$$|\vec{v}|_{l_\infty} = \max\{|v_i| : i = 1, 2, \ldots, n\},$$

and

$$a \wedge b = \min(a, b),$$
$$a \vee b = \max(a, b).$$

If there is no such time, we let τ_n be infinity. Note that $\lim_{n\uparrow\infty}(\tau_n) = \tau$.

Now, for each fixed n, we will show uniqueness up to time τ_n in the system of equations

$$\begin{aligned} dX_t^{i,n} &= Y_t^{i,n}\,dt, \\ dY_t^{i,n} &= Z_t^{i,n}\,dt, \\ dZ_t^{i,n} &= |X_t^{i,n}|^\alpha\,\mathbf{1}_{[0,\tau_n]}(t)\,dB_t, \\ (X_0, Y_0, Z_0) &= (x_0, y_0, z_0). \end{aligned} \tag{4}$$

In other words, after the time τ_n, we have $dZ_t^{i,n} = 0$, which makes $Z_t^{i,n}$ become constant. Specifically, given $m, n \in N$, we need

$$(X_t^n, Y_t^n, Z_t^n) = (X_t^m, Y_t^m, Z_t^m)$$

for all $t \leq \tau_n \wedge \tau_m$.

Now, before continuing the proof of uniqueness, by the method of contradiction, we show that the times that X_t hits zero do not accumulate before the time τ_n, almost surely.

For each n, let A_n be the event on which the times that $X_t^{i,n} = 0$, $i = 1$ or $i = 2$, accumulate before τ_n, and assume $P(A_n) > 0$. Then, on A_n, suppose σ_n is an accumulation point of the times t at which $X_t^{i,n} = 0$; i.e, there exists a sequence of times $\rho_{1,n} < \rho_{2,n} < \cdots$ that converges to σ_n, and $X_{\rho_{k,n}}^{i,n} = 0$. Hence, on A_n, $\lim_{k\to\infty} \rho_{k,n} = \sigma_n$.

We have $X_t^{i,n}$ is almost surely continuous, and that $X_{\rho_{k,n}}^{i,n} = 0$ on A_n, so

$$\lim_{\rho_{k,n}\to\sigma_n} X_{\rho_{k,n}}^{i,n} = X_{\sigma_n}^{i,n} = 0$$

on A_n.

Note that $dX_t^{i,n} = Y_t^{i,n}\,dt$, and $Y_t^{i,n}$ is almost surely continuous. So if $Y_{\sigma_n}^{i,n} \neq 0$ on A_n, then there exists a random interval $[\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$ of positive length for which $X_t^{i,n} \neq 0$ on $[\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$. This contradicts the hypothesis of $\rho_{k,n}$ converging to σ_n.

If $Y_{\sigma_n}^{i,n} = 0$, there are two cases,

$$\sigma_n \geq \tau_n \quad \text{and} \quad \sigma_n < \tau_n.$$

If $\sigma_n \geq \tau_n$, then it means that the times at which $X_t^{i,n}$ hit zero do not accumulate before τ_n.

If $\sigma_n < \tau_n$, then with $X^{i,n}_{\sigma_n} = 0$ and $Y^{i,n}_{\sigma_n} = 0$, we have $|Z^{i,n}_{\sigma_n}| > 2^{-n}$. Now suppose $Z_{\sigma^{i,n}_n} > 2^{-n}$, as the case $Z^{i,n}_{\sigma} < 2^{-n}$ is approached similarly due to symmetry.

If $Z^{i,n}_{\sigma_n} > 2^{-n}$, since Z_t is almost surely continuous, there exists a time interval $[\sigma_n(\omega) - \epsilon'(\omega), \sigma_n(\omega)]$ on which $Z^{i,n}_t > 2^{-n}/2$. Thus for all $t \in [\sigma_n(\omega) - \epsilon'(\omega), \sigma_n(\omega)]$ we almost surely have

$$Y^{i,n}_t = Y^{i,n}_{\sigma_n} - \int_t^{\sigma_n} Z^{i,n}_s \, ds < -\frac{2^{-n}}{2}(\sigma_n - t).$$

Hence, integrating over $X^{i,n}_t$ for $t \in [\sigma_n - \epsilon', \sigma_n]$, we have

$$X^{i,n}_t = X^{i,n}_{\sigma_n} - \int_t^{\sigma_n} Y^{i,n}_s \, ds = -\int_t^{\sigma_n} Y^{i,n}_s \, ds > \int_t^{\sigma_n} \frac{2^{-n}}{2}(\sigma_n - s)\, ds$$
$$= \frac{2^{-n}}{2}\Big(\sigma_n s - \frac{s^2}{2}\Big)\Big|_t^{\sigma_n} = \frac{2^{-n}}{4}(\sigma_n - t)^2 > 0$$

as $t < \sigma_n$. Hence, this contradicts the hypothesis of $\rho_{k,n}$ converging to σ_n on A_n. So in conclusion, $P(A_n) = 0$, which means the times at which X_t hits zero do not accumulate before the time τ_n.

One more point we need to address before continuing with the proof of uniqueness is the existence of solutions. This problem is resolved in Theorems 21.7 and 21.8 and Lemma 21.17 of [Kallenberg 2002], which prove that for all times $t \geq 0$, solutions of multidimensional SDEs exist with probability 1 provided the coefficients are continuous and bounded.

Specifically, in our problem, since $\alpha \in (0, 1)$, the coefficients of system (3) are continuous and bounded by $(2^n)^\alpha \vee 2^n = 2^n$ up to the τ_n for each n, which satisfies the condition stated in the existence theory in [Kallenberg 2002]. Hence, existence of solution holds for all $t \leq \tau_n$ for all n. Therefore, up to the time $\tau = \sup \tau_n$, existence of solutions is ensured.

From now on, $X^{i,n}_t$ means $X^{1,n}_t$ and $X^{2,n}_t$. We define $Y^{i,n}_t$ and $Z^{i,n}_t$ similarly. Back to the proof of uniqueness, we have, for all $t \in [0, \tau_n]$,

$$|Z^{i,n}_t| \vee |Y^{i,n}_t| \leq 2^n.$$

So

$$|Y^{i,n}_t| = \left| Y_0 + \int_0^t Z^{i,n}_s \, ds \right| \leq |Y_0| + \int_0^t |Z^{i,n}_s| \, ds \leq 2^n + 2^n t.$$

Therefore,

$$|X^{i,n}_t| \leq |X_0 + \int_0^t Y^{i,n}_s \, ds| \leq \int_0^t |Y^{i,n}_s| \, ds \leq \int_0^t (2^n + 2^n s)\, ds = 2^n\left(t + \frac{t^2}{2}\right).$$

Now let

$$t_{0,n} = \frac{2^{-2n}}{2}. \tag{5}$$

Then

$$t_{0,n}+\frac{t_{0,n}^2}{2}=\frac{2^{-2n}}{2}+\frac{2^{-2n}}{8}\le\frac{2^{-2n}}{2}+\frac{2^{-2n}}{2}=2^{-2n}.$$

So

$$2^n\left(t_{0,n}+\frac{t_{0,n}^2}{2}\right)\le 2^n\cdot 2^{-2n}=2^{-n}.$$

Since the quadratic function $2^n(t+t^2/2)$ is increasing when $t\ge 0$, we have

$$|X_t^{i,n}|\le 2^n\left(t+\frac{t^2}{2}\right)\le 2^{-n}$$

for all $t\in[0,t_{0,n}]$.

Since $|X_t^{1,n}|$ and $|X_t^{2,n}|$ belong in $[0,2^{-n}]$ for $t\in[0,t_{0,n}]$, based on the definition of τ_n above, either

$$|Y_0|\ge 2^{-n} \tag{6}$$

or

$$|Z_0|\ge 2^{-n} \tag{7}$$

for each fixed n. This is due to the fact that the solutions have the same initial condition (X_0,Y_0,Z_0) and for each time $t\in[0,t_{n,0}]$, either

$$|Y_t^{i,n}|\ge 2^{-n}\quad\text{or}\quad |Z_t^{i,n}|\ge 2^{-n}.$$

First, we deal with $Y_0>0$. Due to symmetry, we can deal with the case $Y_0<0$ with similar methods and thus omit the proof.

Now, with $Y_0>0$, we look at other subcases based on $Z_0^{i,n}$.

Case I: $Y_0>0,\ |Z_0|\le 2^{-n}$. If $|Z_0|\le 2^{-n}$, then (6) takes place. We are looking at the case $Y_0>0$, and thus $Y_0>2^{-n}$. Also, note that $dZ_t^{i,n}=0$ for all $t>\tau_n$ and $|Z_t^{i,n}|\le 2^n$ for all $t\in[0,\tau_n]$. Hence, $|Z_t^{i,n}|\le 2^n$ for all t, which means $Z_t^{i,n}\ge -2^n$ for all t. Next, we have

$$Y_t^{i,n}=Y_0+\int_0^t Z_s\,ds\ge 2^{-n}-\int_0^t 2^n\,ds=2^{-n}-2^nt.$$

If

$$0<t<t_{0,n}=\frac{2^{-2n}}{2},$$

where $t_{0,n}$ is defined as in (5), then

$$2^nt<\frac{2^{-n}}{2}.$$

Thus

$$2^{-n}-2^nt>\frac{2^{-n}}{2}$$

for all $t\in[0,t_{0,n}]$. In other words, $Y_t^{i,n}>2^{-n}/2$ for $t\in[0,t_{0,n}]$. So, for all $t\in[0,t_{0,n}]$, we have

$$Y_t^{i,n}\ge\frac{2^{-n}}{2}.$$

Hence

$$X_t^{i,n} \geq X_0^{i,n} + \int_0^t \frac{2^{-n}}{2}\, ds \geq \frac{2^{-n}}{2} t. \tag{8}$$

Furthermore, based on (8) and $|X_t^{i,n}| \leq 2^{-n}$ for all $t \in [0, t_{0,n}]$, it leads to $t_{0,n} \leq 2$, otherwise $X_t^{i,n} > 2^{-n}$, which means that $t > t_{0,n}$, a contradiction.

Note that

$$\begin{aligned} X_t^{i,n} &= X_0 + \int_0^t Y_s^{i,n}\, ds, \\ Y_s^{i,n} &= Y_0 + \int_0^s Z_k^{i,n}\, dk, \\ Z_k^{i,n} &= Z_0 + \int_0^k |X_r^{i,n}|^\alpha\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r. \end{aligned}$$

Thus

$$\begin{aligned} X_t^{i,n} &= X_0 + Y_0 t + \int_0^t \int_0^s \left(Z_0 + \int_0^k |X_r^{i,n}|^\alpha\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r \right) dk\, ds \\ &= X_0 + Y_0 t + \int_0^t \int_0^s Z_0\, dk\, ds + \int_0^t \int_0^s \int_0^k |X_r^{i,n}|^\alpha\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r\, dk\, ds \\ &= X_0 + Y_0 t + Z_0 \frac{t^2}{2} + \int_0^t \int_0^s \int_0^k |X_r^{i,n}|^\alpha\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r\, dk\, ds. \end{aligned}$$

Hence

$$(X_t^{1,n} - X_t^{2,n})^2 = \left(\int_0^t \int_0^s \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n]}(r)\, dB_r\, dk\, ds \right)^2.$$

Apply the Cauchy-Schwarz inequality twice, we get

$$\begin{aligned} (X_t^{1,n} - X_t^{2,n})^2 &\leq t \int_0^t \left(\int_0^s \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n]}(r)\, dB_r\, dk \right)^2 ds \\ &\leq t \int_0^t s \int_0^s \left(\int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n]}(r)\, dB_r \right)^2 dk\, ds. \end{aligned}$$

Thus

$$E[(X_t^{1,n} - X_t^{2,n})^2] \leq t E \int_0^s s \int_0^s \left(\int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n]}(r)\, dB_r \right)^2 dk\, ds.$$

By Itô's isometry,

$$\begin{aligned} E[(X_t^{1,n} - X_t^{2,n})^2] &\leq t E \int_0^t s \int_0^s \int_0^k ((|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n]}(r))^2\, dr\, dk\, ds \\ &\leq t E \int_0^t t \int_0^s \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2\, dr\, dk\, ds \end{aligned}$$

$$\leq t^2 E \int_0^t \int_0^s \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr \, dk \, ds$$
$$\leq t^2 E \int_0^t \int_0^t \int_0^t (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr \, dk \, ds$$
$$= t^4 E \int_0^t (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr.$$

Now we apply the mean value theorem for the function $f(x) = x^\alpha$, $0 < \alpha < 1$, and $a < b$:

$$b^\alpha - a^\alpha = \alpha c^{\alpha-1}(b-a) \leq \alpha a^{\alpha-1}(b-a)$$

for $c \in (a, b)$. Then for $r \in [0, t_{0,n}]$, where t_0 is determined in (5), we apply (8):

$$\big| |X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha \big| \leq \alpha \left(\frac{2^{-n}}{2} r \right)^{\alpha-1} \big| |X_r^{1,n}| - |X_r^{2,n}| \big|. \tag{9}$$

Now let

$$D_t = E[(|X_t^{1,n}| - |X_t^{2,n}|)^2].$$

Since $t_{0,n} \leq 2$, we have for all $t \in [0, t_{0,n}]$

$$D_t \leq E[(X_t^{1,n} - X_t^{2,n})^2] \leq C_n \int_0^t r^{2\alpha-2} D_r \, dr.$$

for some C_n depending on n. Since $\alpha > \frac{3}{4}$, we have $r^{2\alpha-2}$ is integrable on $[0, t_{0,n}]$. At this stage we apply Gronwall's lemma:

Lemma. *Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable on the interior I^o of I (the interval I without the endpoint a and possibly b) and satisfies the differential inequality*

$$u^{'}(t) \leq \beta(t) u(t), \quad t \in I^o,$$

then u is bounded by the solution of the corresponding differential equation $v^{'}(t) = \beta(t) v(t)$:

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s) \, ds \right)$$

for all $t \in I$.

Hence, with $D_0 = 0$, we have $D_t = 0$ for all $t \in [0, t_{0,n}]$. Therefore, (3) has unique strong solution in $[0, t_{0,n}]$.

Since $\alpha \geq \frac{3}{4}$, we have $2\alpha - 2 \geq -1$. Hence, $r^{2\alpha-2}$ is integrable on $[0, t_{0,n}]$ Note that in this case since $X_t \geq 2^{-n}$, $\eta \leq 1$. Applying Gronwall's lemma, with $D_0 = 0$, we have $D_t = 0$ for all $t \in [0, t_{0,n}]$. Therefore, (3) has a unique solution in strong sense up to time $t_{0,n}$.

Note that for all $t \in [0, t_{0,n}]$, we have $Y_t^{i,n} > 2^{-n}/2 > 0$. Hence $X_t^{i,n}$ is strictly increasing, which, leads to $X_{t_{0,n}}$ strictly positive. Therefore, by the strong Markov property, we have uniqueness until the time X next hits zero.

Case II: $Y_0 > 0,\ Z_0 < -2^{-n}$. Since Z_0 starts negative, $Y_t^{i,n}$ decreases for an amount of time. Since Y_0 is positive, let say $Y_0 = \beta > 0$. Note that for all t, we have $|Z_t^{i,n}| < 2^n$, which means $Z_t^{i,n} > -2^n$. First, we have

$$Y_t^{i,n} = \beta + \int_0^t Z_s\, ds \geq \beta - \int_0^t 2^n\, ds = \beta - 2^n t.$$

Let

$$t_{0,n}^{'} = \frac{\beta}{2^{n+1}}. \tag{10}$$

If

$$0 < t < t_{0,n}^{'} = \frac{\beta}{2^{n+1}},$$

then

$$2^n t < \frac{\beta}{2};$$

thus

$$\beta - 2^n t > \frac{\beta}{2}$$

for all $t \in [0, t_{0,n}{'}]$. In other words, $Y_t^{i,n} > \beta/2$ for all $t \in [0, t_{0,n}^{'}]$.

So, for all $t \in [0, t_{0,n} \wedge t_{0,n}^{'}]$, where $t_{0,n}$ and $t_{0,n}^{'}$ are determined in (5) and (10) respectively, we have

$$Y_t^{i,n} \geq \frac{\beta}{2}$$

Hence

$$X_t^{i,n} \geq X_0^{i,n} + \int_0^t \frac{\beta}{2}\, ds \geq \frac{\beta}{2} t.$$

Applying the same method (9) above, we use the mean value theorem for the new lower bound of $X_t^{i,n}$:

$$\big| |X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha \big| \leq \alpha \left(\frac{\beta}{2} r\right)^{\alpha-1} \big| |X_r^{1,n}| - |X_r^{2,n}| \big|.$$

Hence

$$D_t \leq E[(X_t^{1,n} - X_t^{2,n})^2] \leq C_n \int_0^t r^{2\alpha-2} D_r\, dr.$$

Again, applying Gronwall's lemma, with $D_0 = 0$, we have $D_t = 0$ for all $t \in [0, t_{0,n} \wedge t_{0,n}^{'}]$. Therefore, (3) has a unique strong solution in $[0, t_{0,n} \wedge t_{0,n}^{'}]$. As in the previous cases, we have $Y_t^{i,n} > \beta/2 > 0$ for all $t \in [t_{0,n} \wedge t_{0,n}^{'}]$, which makes $X_t^{i,n}$ strictly increasing. So $X_{t_{0,n}} \wedge t_{0,n}^{'}$ is strictly positive. Thus, by the strong Markov property, we have uniqueness until the next time X hits zero.

Case III: $Y_0 > 0,\ Z_0 > 2^{-n}$. Now we let T_n be the first time that either $Z_t^{1,n}$ or $Z_t^{2,n}$ hits the value $2^{-n}/2$. Since both $Z_t^{1,n}$ and $Z_t^{2,n}$ are continuous, we have $T_n > 0$ with probability 1. So we now prove uniqueness up to the time $t_{0,n} \wedge T_n$, where $t_{0,n}$ is defined in (5).

Then for all t in $[0, t_{0,n} \wedge T_n]$, we have

$$Z_t^{i,n} \geq \frac{2^{-n}}{2};$$

therefore

$$Y_t^{i,n} \geq Y_0 + \int_0^t \frac{2^{-n}}{2}\, ds \geq \frac{2^{-n}}{2} t \tag{11}$$

since $Y_0^{i,n} \geq 0$.

Based on (11), for $t \in [0, t_{0,n} \wedge T_n]$

$$X_t^{i,n} \geq X_0 + \int_0^t \frac{2^{-n}}{2} s\, ds \geq \frac{2^{-n}}{4} t^2. \tag{12}$$

Now we define

$$X_t^{i,n} = \widetilde{X}_t^{i,n}, \quad Y_t^{i,n} = \widetilde{Y}_t^{i,n}, \quad Z_t^{i,n} = \widetilde{Z}_t^{i,n}$$

for $i = 1, 2$ and for $t \leq \tau_n \wedge\ T_n \wedge t_{0,n}$, where $t_{0,n}$ is defined as in (5) above.

Thus the following system of equations holds up to the stopping time $\tau_n \wedge T_n \wedge t_{0,n}$:

$$\begin{aligned} d\widetilde{X}_t^{i,n} &= \widetilde{Y}_t^{i,n}\, dt \\ d\widetilde{Y}_t^{i,n} &= \widetilde{Z}_t^{i,n}\, dt \\ d\widetilde{Z}_t^{i,n} &= |\widetilde{X}_t^{i,n}|^\alpha\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}(t)\, dB_t, \end{aligned} \tag{13}$$

with $(\widetilde{X}_0^{i,n}, \widetilde{Y}_0^{i,n}, \widetilde{Z}_0^{i,n}) = (X_0, Y_0, Z_0)$ for $i = 1, 2$. Furthermore, using (13), $\widetilde{X}_t^{i,n}$, $\widetilde{Y}_t^{i,n}$, and $\widetilde{Z}_t^{i,n}$ can be defined for all times.

Using Itô's isometry as above with $\widetilde{X}_t^{i,n}$, $\widetilde{Y}_t^{i,n}$, and $\widetilde{Z}_t^{i,n}$,

$$\begin{aligned} E[(\widetilde{X}_t^{1,n} - \widetilde{X}_t^{2,n})^2] &\leq tE \int_0^t s \int_0^s \int_0^k \big((|\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha)\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}(r)\big)^2\, dr\, dk\, ds \\ &\leq tE \int_0^t t \int_0^s \int_0^k (|\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha)^2\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}(r)\, dr\, dk\, ds \\ &\leq t^2 E \int_0^t \int_0^s \int_0^k (|\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha)^2\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}(r)\, dr\, dk\, ds \\ &\leq t^2 E \int_0^t \int_0^t \int_0^t (|\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha)^2\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}\, dr\, dk\, ds \\ &= t^4 E \int_0^t (|\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha)^2\, \mathbf{1}_{[0,\tau_n \wedge T_n \wedge t_{0,n}]}\, dr. \end{aligned}$$

Using (12) and the mean value theorem, for $r \in [0, \tau_n \wedge T_n \wedge t_{0,n}]$, we have

$$\big| |\widetilde{X}_r^{1,n}|^\alpha - |\widetilde{X}_r^{2,n}|^\alpha \big| \le \alpha \left(\frac{2^{-n}}{4} r^2 \right)^{\alpha-1} \big| |\widetilde{X}_r^{1,n}| - |\widetilde{X}_r^{2,n}| \big|.$$

Hence

$$E[(\widetilde{X}_t^{1,n} - \widetilde{X}_t^{2,n})^2] \le t^4 \alpha^2 \left(\frac{2^{-n}}{4} \right)^{2(\alpha-1)} E \int_0^t r^{4(\alpha-1)} (|\widetilde{X}_r^{1,n}| - |\widetilde{X}_r^{2,n}|)^2 \, dr,$$

so if we let

$$D_t = E[(|\widetilde{X}_t^{1,n}| - |\widetilde{X}_t^{2,n}|)^2],$$

then

$$D_t \le E[(\widetilde{X}_t^{1,n} - \widetilde{X}_t^{2,n})^2] \le C_n \int_0^t r^{4\alpha-4} D_r \, dr.$$

Again, applying Gronwall's lemma, with $D_0 = 0$, we have $D_t = 0$. Note that at the time $t_{0,n} \wedge T_n$, since we have $Z_t^{i,n} > 0$ for all $t \in [0, t_{0,n} \wedge T_n]$, and also $Y_0 > 0$, it leads to $Y_t^{i,n} > 0$ for all $t \in [0, t_{0,n} \wedge T_n]$. Thus $X_t^{i,n}$ is strictly increasing, which means $X_{t_{0,n} \wedge T_n}$ must be strictly greater than zero. Therefore, by the strong Markov property, we obtain uniqueness of the process until X next hits zero.

Case IV: $Y_0 = 0$. If $Y_0 = 0$, then based on the definition of τ_n, we have $|Z_0| > 2^{-n}$. We will first deal with the case $Z_0 > 2^{-n}$, and the case $Z_0 < 2^{-n}$ is approached the same way due to symmetry. As in Case III, let T_n be the first time that either $Z_t^{1,n}$ or $Z_t^{2,n}$ hits the value $2^{-n}/2$. Due to the continuity of $Z_t^{1,n}$ and $Z_t^{2,n}$, we have $T_n > 0$ with probability 1. So with for all $t \in [0, t_0 \wedge T_n]$, where t_0 is determined in (5), we have

$$Y_t^{i,n} \ge Y_0 + \int_0^t \frac{2^{-n}}{2} \, ds = \frac{2^{-n}}{2} t.$$

Then

$$X_t^{i,n} \ge X_0 + \int_0^t \frac{2^{-n}}{2} s \, ds \ge \frac{2^{-n}}{4} t^2.$$

We now apply the same method as in Case III by looking at $\widetilde{X}_t^{i,n}$, $\widetilde{Y}_t^{i,n}$, and $\widetilde{Z}_t^{i,n}$, which are defined as

$$X_t^{i,n} = \widetilde{X}_t^{i,n}, \quad Y_t^{i,n} = \widetilde{Y}_t^{i,n}, \quad Z_t^{i,n} = \widetilde{Z}_t^{i,n}$$

for $i = 1, 2$ and for $t \le \tau_n \wedge T_n \wedge t_{0,n}$, as $t_{0,n}$ defined as in (5) above.

Thus the following system of equations holds up to the stopping time $\tau_n \wedge T_n \wedge t_{0,n}$:

$$\begin{aligned} d\widetilde{X}_t^{i,n} &= \widetilde{Y}_t^{i,n} \, dt \\ d\widetilde{Y}_t^{i,n} &= \widetilde{Z}_t^{i,n} \, dt \\ d\widetilde{Z}_t^{i,n} &= |\widetilde{X}_t^{i,n}|^\alpha \, \mathbf{1}_{[0, \tau_n \wedge T_n \wedge t_{0,n}]}(t) \, dB_t, \end{aligned}$$

with $(\widetilde{X}_0^{i,n}, \widetilde{Y}_0^{i,n}, \widetilde{Z}_0^{i,n}) = (X_0, Y_0, Z_0)$ for $i = 1, 2$. Furthermore, using (13), $\widetilde{X}_t^{i,n}$, $\widetilde{Y}_t^{i,n}$, $\widetilde{Z}_t^{i,n}$ can be defined for all time.

Again, using the same strategy in Case III and the mean value theorem, we have

$$\big||\widetilde{X}_r^{1,n}|^\alpha - \widetilde{X}_r^{2,n}|^\alpha\big| \le \alpha\left(\frac{2^{-n}}{4}r^2\right)^{\alpha-1}\big||\widetilde{X}_r^{1,n}| - |\widetilde{X}_r^{2,n}|\big|.$$

Hence, if we let

$$D_t = E[(|\widetilde{X}_t^{1,n}| - |\widetilde{X}_t^{2,n}|)^2],$$

then

$$D_t \le E[(\widetilde{X}_t^{1,n} - \widetilde{X}_t^{2,n})^2] \le C_n \int_0^t r^{4\alpha-4} D_r\, dr.$$

Gronwall's lemma with $D_0 = 0$ yields $D_t = 0$, completing the proof of Theorem 1.

In this case, we also have $Y_t^{i,n} > 2^{-n}/2t > 0$ for all $t \in [0, t_{0,n} \wedge T_n]$. Hence $X_t^{i,n}$ is strictly increasing, which yields $X_{t_{0,n}\wedge T_n}$ strictly positive. So, by the strong Markov property, we have uniqueness up to the time X next hits zero.

Now with uniqueness proved, we actually can even strengthen the proof by showing that $\tau_n^1 = \tau_n^2$ for all n, where τ_n^1 and τ_n^2 respectively stand for the stopping times at the critical values for X_t^1 and X_t^2. Without loss of generality, suppose $\tau_n^2 > \tau_n^1$: So at the time τ_n^1, X_t^2 has not yet reached the critical values, which are 2^{-n} or 2^n, as stated above. But since we have uniqueness up to $\tau_n^1 \wedge \tau_n^2$, this implies X_t^1 has also not reached the critical value at the time τ_n^1, which is a contradiction to the definition of τ_n^1. Hence, τ_n^1 and τ_n^2 must be equal.

References

[Gomez et al. 2017] A. Gomez, J. J. Lee, C. Mueller, E. Neuman, and M. Salins, "On uniqueness and blowup properties for a class of second order SDEs", *Electron. J. Probab.* **22** (2017), art. id. 72. MR Zbl

[Hartman 1964] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR Zbl

[Kallenberg 2002] O. Kallenberg, *Foundations of modern probability*, 2nd ed., Springer, 2002. MR Zbl

[Protter 1990] P. Protter, *Stochastic integration and differential equations: a new approach*, Appl. Math. (NY) **21**, Springer, 1990. MR Zbl

[Watanabe and Yamada 1971] S. Watanabe and T. Yamada, "On the uniqueness of solutions of stochastic differential equations, II", *J. Math. Kyoto Univ.* **11** (1971), 553–563. MR Zbl

Received: 2019-10-03 Revised: 2020-02-18 Accepted: 2020-05-23

carl.e.mueller@rochester.edu *Department of Mathematics, University of Rochester, Rochester, NY, United States*

gtruong@u.rochester.edu *Department of Mathematics, University of Rochester, Rochester, NY, United States*