Van der Corput's method for exponential sums and the Divisor Problem

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1 Introduction

The Dirichlet Divisor Problem, named after the German mathematician Peter Gustav Lejeune Dirichlet, is a classical problem in number theory. It concerns the distribution of the number of divisors of positive integers, which plays a crucial role in various areas of mathematics, including analytic number theory and algebraic geometry.

The problem can be succinctly stated as follows: given an integer n, let the divisor function $d(n)$ be the number of integer divisors of n (including 1 and n itself). For example, 24 has divisors $d_i = 1, 2, 3, 4, 6, 8, 12, 24$. Thus, $d(24) = 8$.

Let $D(x) = \sum_{n \leq x} d(n)$, so that $D(x)$ sums the numbers of divisors of integers smaller than x. Thus, the average number of divisors for any integer $\leq x$ is $\frac{1}{x}D(x)$.

In 1849, Dirichlet proposed this question and developed a method now known as the hyperbola method to prove that $D(x) = \sum_{n \leq x} d(n) = x \log x + x(2\gamma - 1) + O(\sqrt{x})$, where γ is the Euler's constant. His proof will be shown later in this paper. The Dirichlet Divisor Problem asks for the best estimate of the error term.

Despite its seemingly elementary nature, the Dirichlet Divisor Problem remains one of the most challenging unsolved problems in number theory. Progress has been made in certain special cases and under various assumptions, but a complete understanding of its behavior remains elusive. It is conjectured that $\theta = 1/4 + \epsilon$. In 1916, G. H. Hardy showed that $\inf \theta \geq \frac{1}{4}$. In other words, the above bound becomes false if $\frac{1}{4}$ is replaced by any smaller value.

In this paper, we will introduce Van der Corput's method of exponential sums, which provides a systematic way to bound exponential sums by exploiting the oscillatory behavior of the complex exponential function. We will describe the van der Corput's method, as well as its applications to the Dirichlet Divisor Problem.

2 Notation

In this section, we define some notations that we will use in this thesis. 1. $\lfloor x \rfloor$: $\lfloor x \rfloor$ is defined to be the largest integer that does not exceed x.
Examples: $\lfloor 1.5 \rfloor = 1$, $\lfloor 2 \rfloor = 2$
2. $\{x\}$: $\{x\}$ is defined to be the fractional part of x. $\{x\} = x - \lfloor x \rfloor$.
Notice: For any x, $\{x\} < 1$.
Examples: $\{1.8\} = 0.8$, $\{2\} = 0$
3. Big O notation: Let f, the function to be estimated, be a real or complex valued function and let g, the comparison function, be a real valued function. Let both functions be defined on some unbounded subset of the positive real numbers, and let $g(x)$ be strictly positive for all large enough values of x.
If the absolute value of $f$is at most a positive constant multiple of $g(x)$, i.e., if there exists a positive integer N and a positive real constant c such that $|f(x)| \leq c(g(x))\ \forall x \geq N$, we write $f(x) = O(g(x)$.
Equivalently, we can write $f(x) \ll g(x)$.
Examples:

- A constant function is : $O(1)$
- A linear function is : $O(n)$
- A logarithmic function is "order $\log(n)$": If $f(n) = \log_a n$ and $g(n) = \log_b n$, then $O(f(n)) = O(g(n))$; all log functions grow in the same manner in terms of Big-O.

4. Small o notation: The symbol $o(g(x))$ represents the set of functions $f(x)$ that grow slower than $g(x)$ as x approaches a certain limit. Formally, for two functions $f(x)$ and $g(x)$, we say that $f(x)$ is in the set $o(g(x))$ if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 0,$$

for some limit point c in the domain of $f(x)$ and $g(x)$.
5. Let f be a a real function. We say $f \approx g$ if and only if $f(x) \ll g(x)$ and $g(x) \ll f(x)$.
6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be in $C^1(\mathbb{R})$ if it is continuously differentiable, i.e., both f and its derivative f' exist and are continuous on $\mathbb{R}$.
Similarly, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be in $C^2(\mathbb{R})$ if it is twice continuously differentiable, i.e., both f and its first two derivatives f' and f'' exist and are continuous on $\mathbb{R}$.
7. An integrable function $f : \mathbb{R} \to \mathbb{R}$ is said to be in $L^1(\mathbb{R})$ if the integral of the absolute

value of $f(x)$ over the entire real line exists and is finite. In the other words, a function f belongs to $L^1(\mathbb{R})$ if

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

3 Dirichlet's Method

Theorem 1.1: For all $N \geq 1$, $\sum_{n \leq N} d(n) = N \log N + N(2\gamma - 1) + O(\sqrt{N})$.

Proof. Dirichlet's method turns the problem of solving for $D(N)$ into the problem of counting the lattice points in a bounded region.
Notice that geometrically, $d(n)$ counts the number of lattice points (points with integer coordinates) on the parabola $xy = n$.

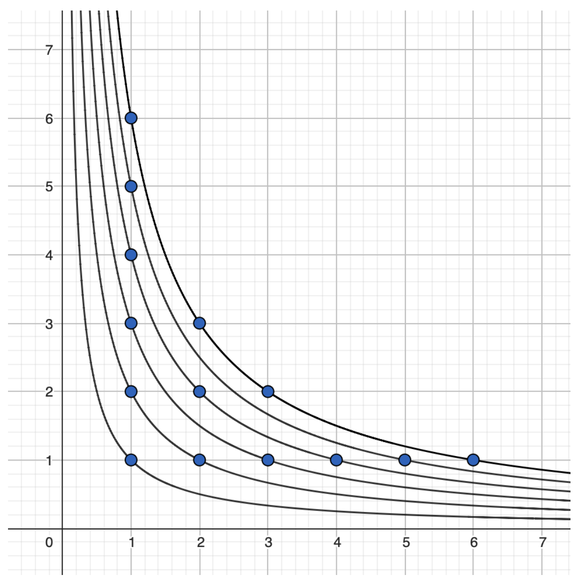
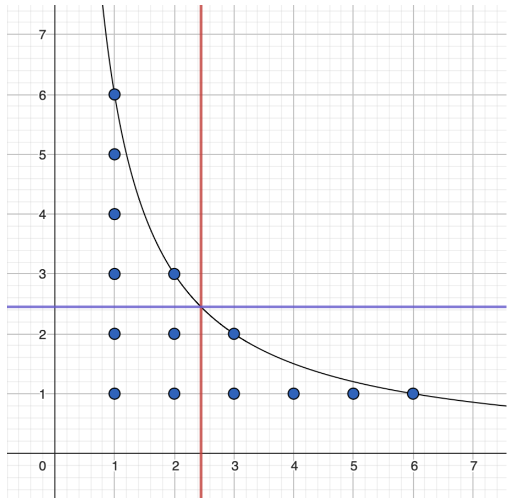


Figure 1: lattice points under xy=6

Each lattice point will be on some parabola $xy = n$. Thus, $D(N)$ counts the lattice points in the first quadrant that are on or below the parabola $xy = N$.
This region under $xy = N$ can be divided into three parts: a square of length $\sqrt{N}$ and two symmetrical regions separated by the square.
In the square, there are $\lfloor \sqrt{N} \rfloor^2$ lattice points.
In the region bounded by $xy = N$ and $y = \sqrt{N}$, there are $\lfloor \frac{N}{n} \rfloor - \lfloor \sqrt{N} \rfloor$ on the each line $x = n$.
Adding the number of lattice points on each line, the number of lattice points contained in this region is

$$\sum_{n=1}^{\lfloor \sqrt{N} \rfloor} \lfloor \frac{N}{n} \rfloor - \lfloor \sqrt{N} \rfloor.$$



The number of lattice points in the symmetrical regions are identical. Combining these three parts , we have

$$\begin{aligned}\sum_{n=1}^{N} d(n) &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}(\lfloor\frac{N}{n}\rfloor - \lfloor\sqrt{N}\rfloor) + \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\lfloor\frac{N}{n}\rfloor - 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\lfloor\sqrt{N}\rfloor + \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\lfloor\frac{N}{n}\rfloor - 2\lfloor\sqrt{N}\rfloor\sum_{n=1}^{\lfloor\sqrt{N}\rfloor} 1 + \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\lfloor\frac{N}{n}\rfloor - 2\lfloor\sqrt{N}\rfloor^2 + \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\lfloor\frac{N}{n}\rfloor - \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}(\frac{N}{n} - \{\frac{N}{n}\}) - (\sqrt{N} - \{\sqrt{N}\})^2 \\ &= 2N\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{n} - 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\{\frac{N}{n}\} - N + 2\sqrt{N}\{\sqrt{N}\} - \{\sqrt{N}\}^2.\end{aligned} \tag{1}$$

Noticing that the fractional part is always ≤ 1:

$$2\sum_{n=1}^{\sqrt{N}}\{\frac{N}{n}\} \leq 2\sum_{n=1}^{\sqrt{N}} 1 = 2\sqrt{N}.$$

Therefore,

$$2\sum_{n=1}^{\sqrt{N}}\{\frac{N}{n}\} = O(\sqrt{N}).$$

Similarly,

$$2\sqrt{N}\{\sqrt{N}\} \leq 2\sqrt{N},$$

which gives us

$$2\sqrt{N}\{\sqrt{N}\} = O(\sqrt{N}).$$

Finally,

$$\{\sqrt{N}\}^2 < 1.$$

Thus, we have $\{\sqrt{N}\}^2 = O(1)$, which is absorbed by $O(\sqrt{N})$.
Combining these terms, we obtain

$$\sum_{n=1}^{N} d(n) = 2N\sum_{n=1}^{\sqrt{N}}\frac{1}{n} - N + O(\sqrt{N}) \tag{2}$$

Theorem 1.2 (Partial Sums of Harmonic Series):
For large N,

$$\sum_{1}^{N}\frac{1}{n} \approx \ln N + \gamma,$$

where $\gamma = \lim_{x\to\infty}(\sum_1^N \frac{1}{n} - lnN)$ is the *Euler constant* (Euler, 1735).
It is approximately equal to 0.57721.

By **Theorem 1.2**,

$$\begin{aligned}\int_1^{\sqrt{N}}\frac{1}{n}dn &= \sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{n} + \gamma + O(\frac{1}{\sqrt{N}})\\ \sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{n} &= \int_1^{\sqrt{N}}\frac{1}{n}dn - \gamma + O(\frac{1}{\sqrt{N}})\\ \sum_{n=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{n} &= \ln\sqrt{N} - \gamma + O(\frac{1}{\sqrt{N}})\end{aligned} \tag{3}$$

By (2), we have

$$
\begin{aligned}
\sum_{n=1}^{N} d(n) &= 2N(\ln\sqrt{N} - \gamma + O(\frac{1}{\sqrt{N}})) - N + O(\sqrt{N}) \\
&= 2N\ln\sqrt{N} - 2N\gamma + O(\frac{N}{\sqrt{N}}) - N + O(\sqrt{N}) \qquad (4) \\
&= 2N\ln N^{\frac{1}{2}} - 2N\gamma - N + O(\sqrt{N}) \\
&= N\ln N - N(2\gamma+1) + O(\sqrt{N}).
\end{aligned}
$$

This completes the proof. □

4 Van der Corput's Method

Johannes van der Corput was a Dutch mathematician who worked in the field of analytic number theory. He introduced the method of exponential sums which was a new method for making number-theoretic estimates. This theory is typically relevant to the Dirichlet divisor problem and to the circle problem. In 1922, van der Corput obtained that the remainder term in the Dirichlet divisor problem has order$\ll_\epsilon 33/100+\epsilon$
We first start by defining exponential sums.

Definition 1 (Exponential sum)**.** *Let* $A = \{x_1, x_2, ..., x_N\} \in \mathbb{R}$*. An exponential sum is a sum of the form*

$$\sum_{x\in A} e(x) = \sum_{x\in A} e^{2\pi i x},$$

where we introduce the standard notation $e(f(x)) = e^{2\pi i f(x)}$*.*

Notice: trivially, $|\sum_{x\in A} e(x)| \leq N$, with equality whenever the terms are all equal.
We will make use of the Poisson summation formula to study trigonometric sums.

Definition 2 (Fourier transform)**.** *The Fourier transform of* f *is the function* $\hat{f} : \mathbb{R} \to \mathbb{C}$ *given by*

$$\hat{f}(\theta) = \int_{-\infty}^{\infty} f(t)e(-\theta t) \qquad (f \in L^1(\mathbb{R})).$$

Theorem 2.1: Let $f \in L^1(\mathbb{R})$. Assume that the series

$$\varphi(t) = \sum_{n\in\mathbb{Z}} f(n+t) \qquad (5)$$

converges for all t and its sum defines a function of bounded variation on [0,1] which is continuous at 0. Then we have

$$\lim_{N\to\infty} \sum_{|\nu|\le N} \hat{f}(\nu) = \sum_{n\in\mathbb{Z}} f(n). \tag{6}$$

Theorem 2.2: Let $f \in C^1[a,b]$ be such that $f'(t)$ is monotone and of constant sign on $[a,b]$. Write

$$m := \inf_{a<t<b} |f'(t)|.$$

Then we have

$$|\int_a^b e(f(t))dt| \le 2/\pi m. \tag{7}$$

Proof. Without loss of generality we may assume f' is non-increasing on $[a,b]$.

$$\frac{de(f(t))}{dt} = e(f(t))2\pi f'(t).$$

Thus,

$$\begin{aligned}
\left|2\pi \int_a^b e(f(t))dt\right| &= \left|\int_a^b \frac{1}{f'(t)} de(f(t))\right| \\
&= \left|\frac{e(f(t))}{f'(t)}\Big|_a^b - \int_a^b e(f(t))d\frac{1}{f'(t)}\right| \\
&= \left|\frac{e(f(b))}{f'(b)} - \frac{e(f(a))}{f'(a)} - \int_a^b e(f(t))d\frac{1}{f'(t)}\right| \\
&\le \left|\frac{e(f(b))}{f'(b)}\right| + \left|\frac{e(f(a))}{f'(a)}\right| + \left|\int_a^b e(f(t))d\frac{1}{f'(t)}\right| \\
&\le 2/m + \int_a^b d\frac{1}{f'(t)} \qquad \text{(as } |e(f(t))| \le 1) \\
&\le 2/m + \frac{1}{f'(b)} + \frac{1}{f'(a)} \\
&= 4/m
\end{aligned} \tag{8}$$

□

Theorem 2.3: Let $f \in C^2[a,b]$ be such that $f''(t)$ has constant sign on $[a,b]$. Write

$$r := \inf_{a<t<b} |f''(t)|.$$

Then we have

$$\left|\int_a^b e(f(t))dt\right| \le 4\sqrt{2/\pi r}. \tag{9}$$

Proof. Let us suppose, $f''(t) \leq -r \leq 0$ for $a < t < b$. Then $f'(t)$ vanishes **at most** once on $[a, b]$, say at t=c, i.e.$f'(c) = 0$.
Then, we can separate $[a, b]$ into three intervals and write

$$I := \int_a^b e(f(t))dt = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b} = I_1 + I_2 + I_3,$$

where the positive parameter δ satisfies $a + \delta \leq c \leq b - \delta$. By Fundamental Theorem of Calculus, we have

$$|f'(t) - f'(c)| = |f'(t)| = \left| \int_c^t f''(\nu)d\nu \right| \geq r|t - c| \geq r\delta$$

for $t \in [a, c - \delta] \cup [c - \delta, b]$. On $[a, c - \delta]$ and $[c + \delta, b]$ respectively, $f'(t)$ is decreasing and of constant sign. Thus we can apply Theorem 2.2 and obtain

$$|I_1| + |I_3| \leq 4/\pi r\delta.$$

Since trivially, $|I_2| \leq 2\delta$, it follows that

$$|I| \leq 2\delta + 4/\pi r\delta.$$

By choosing $\delta = \sqrt{2/\pi r}$, we have the stated result.
If with this choice of δ we have either $c < a + \delta$ or $c > b - \delta$, say $c < a + \delta$, then we can write:

$$|I| \leq \left| \int_a^c \right| + \left| \int_c^{c+\delta} \right| + \left| \int_{c+\delta}^b \right| \leq 2\delta + 2/\pi r\delta,$$

so that the stated upper bound remains valid.
If $f'(t)$ does not vanish on $[a, b]$, then $f'(t)$ is decreasing and of a constant sign on $[a, b]$. Without loss of generality, suppose $f'(t) > 0$. Then, $\inf_{a<t<b} f'(t) = f'(b) > 0$. For t$\in$ $[a, b - \delta]$,

$$|f'(t) - f'(b)| = f'(t) - f'(b) = \left| \int_t^b f''(\nu)d\nu \right| \geq r|t - b| \geq r\delta$$

Thus, $f'(t) > r\delta$. By theorem 2.2, $\int_a^{b-\delta} e(f(t))dt \leq 4/\pi r\delta$. Trivially, $| \int_{b-\delta}^b | \leq \delta$.
Thus,

$$|I| \leq \left| \int_a^{b-\delta} \right| + \left| \int_{b-\delta}^b \right| \leq 4/\pi r\delta + \delta$$

□

Lemma 2.4: Let M, N, ν be integers.

$$\sum_{M<\nu\leq N} e(-\nu t) \ll min(N-M, \frac{1}{||t||}).$$

Proof. (Proof of lemma)

$$\begin{aligned}
\sum_{M<\nu\leq N} e(-\nu t) &= e(-(M+1)t)(1+e(-t)+...+e(-t(-N+1))) \\
&= e(-(M+1)t)\frac{1-e(-t(N+2))}{1-e(-t)} \\
&\text{(Geometric series with N+2 terms and ratio } r=e(-t) \\
&= e(-(M+1)t)\frac{e(\frac{Nt}{2})[(e(-\frac{Nt}{2})-e(\frac{Nt}{2})]}{e(-\frac{t}{2})-e(\frac{t}{2})} \\
&= e(-(M+1)t+\frac{Nt}{2})\frac{e(-\frac{Nt}{2})-e(\frac{Nt}{2})}{e(-\frac{t}{2})-e(\frac{t}{2})} \\
&= e(-(M+1)t+\frac{Nt}{2})\frac{sin(\pi Nt)}{sin(\pi t))} \\
&\leq \left|e(-(M+1)t+\frac{Nt}{2})\right|\frac{|sin(\pi Nt)|}{sin(\pi t))} \\
&\leq \frac{1}{sin(\pi t))} \qquad \text{(using the trivial upper bound)}
\end{aligned} \tag{10}$$

For $t \in [-\frac{1}{2}, \frac{1}{2}]$, $|sin\pi t| \geq 2|t|$.Let $||t||$ denote the distance from t to the closest integer. Then, $||t|| \leq 1/2$.Thus, $sin(\pi t) \geq 2||t||$, i.e., $\frac{1}{sin(\pi t))} \leq 2||t||$. Hence, $\frac{1}{sin(\pi t))} \ll \frac{1}{||t||}$. When t gets close to an integer, $\frac{1}{||t||}$ gets large. In that case, we use the other trivial bound, $\sum_{M<\nu\leq N} e(-\nu t) \ll N-M$.
Thus, we obtain the stated result. ☐

Theorem 2.5: Let $f \in C^1[a,b]$ such that $f'(t)$ is monotone on $[a,b]$. Set

$$\alpha := \inf_{a<t<b} f'(t), \quad \beta := \sup_{a<t<b} f'(t).$$

Then, for each $\epsilon > 0$, we have

$$\sum_{a<n\leq b} e(f(n)) = \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} \int_a^b e(f(t)-\nu t)dt + O_\epsilon(\log(\beta-\alpha+2)). \tag{11}$$

Proof. Let ϵ be fixed. If $f(t)$ is replaced by $g(t) = f(t)+kt$ for any $k \in \mathbb{Z}$, formula (11) is invariant.

$$\alpha' := \inf_{a<t<b} g'(t) = \inf_{a<t<b} f'(t) + k = \alpha + k.$$

Similarly,

$$\beta' := \sup_{a<t<b} g'(t) = \sup_{a<t<b} f'(t) + k = \beta + k.$$

Thus, we can assume that $-1 \leq \alpha - \epsilon < 0$ without loss of generality.

We could equally well restrict ourselves to the case when a and b are of the form $m + \frac{1}{2}$ for some $m \in \mathbb{Z}$.

For any b, exists some integer m such that $|b - m| \leq \frac{1}{2}$. If we replace b by $m + \frac{1}{2}$, then the left hand side of (11) becomes

$$\sum_{a<n\leq m+1/2} e(f(n)).$$

Thus, the error term on the left hand side is at most $O(1)$, if $b < m$.

By Lemma 2.4, $\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t) \ll min(\beta - \alpha + 2, \frac{1}{||t||})$ since $\alpha - \epsilon$ and $\beta + \epsilon$ are not necessarily integers. Without loss of generality, let us assume $m \leq b < m + 1/2$.

If we replace b by $m + \frac{1}{2}$, the error resulted in the right hand side of (11) is

$$\begin{aligned}
&\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} \int_b^{m+\frac{1}{2}} e(f(t) - \nu t)dt \\
&= \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} \int_b^{m+\frac{1}{2}} e(f(t))e(-\nu t)dt \\
&\leq \left| \int_b^{m+\frac{1}{2}} \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(f(t))e(-\nu t)dt \right| \\
&\leq \int_b^{m+\frac{1}{2}} \left| \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(f(t))e(-\nu t) \right| dt \\
&\leq \int_b^{m+\frac{1}{2}} \left| \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t) \right| dt \\
&\leq \int_m^{m+\frac{1}{2}} \left| \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t) \right| dt \qquad \text{(since } m \leq b \text{ by assumption)} \\
&\ll \int_m^{m+\frac{1}{\beta-\alpha+2}} (\beta - \alpha + 2)dt + \int_{m+\frac{1}{\beta-\alpha+2}}^{m+\frac{1}{2}} \frac{1}{||t||}dt \\
&= 1 + \log(\frac{1}{2}) + \log(\beta - \alpha + 2)
\end{aligned}$$

By similar argument, if we replace a with some integer Thus, the error on the right hand side is $O(log(\beta - \alpha + 2))$.

Finally, since$f'(t)$ is monotone on $[a, b]$, we can suppose f' is decreasing on $[a, b]$ without loss of generality. Let us set

$$F(t) = \begin{cases} e(f(t)), & \text{if } a < t \leq b \\ 0, & \text{otherwise} \end{cases}$$

Let $\varphi := \sum_{n \in \mathbb{Z}} F(n+t)$. φ is continuous at 0 since $a, b \notin \mathbb{Z}$. Moreover, φ has bounded variation on $[0,1]$. The Poisson formula (6) then implies

$$\sum_{a<n\leq b} e(f(n)) = \sum_{|\nu|\leq N} \hat{F}(\nu) + o(1) \qquad (N \to \infty)$$

with

$$\hat{F}(\nu) = \int_a^b e(f(t) - \nu t)dt.$$

Taking account of $-1 \leq \alpha - \epsilon < 0$, it remains to show that we have

$$\sum_{\substack{|\nu \leq N| \\ \nu \notin [0, \beta+\epsilon]}} \hat{F}(\nu) = O_\epsilon(log(\beta + 2)). \tag{12}$$

Since

$$\frac{d}{dt} e(f(t) - \nu t) = e(f(t) - \nu t) 2\pi i (f'(t) - \nu)$$

,

$$\begin{aligned} 2\pi i \hat{F}(\nu) &= \int_a^b \frac{d\{e(f(t) - \nu t)\}}{f'(t) - \nu} \\ &= \left[\frac{e(f(t) - \nu t)}{f'(t) - \nu}\right]_a^b - \int_a^b e(f(t) - \nu t) d\{\frac{1}{f'(t) - \nu}\} \\ &= \frac{e(f(b) - \nu b)}{f'(b) - \nu} - \frac{e(f(a) - \nu a)}{f'(a) - \nu} - \int_a^b e(f(t) - \nu t) d\{\frac{1}{f'(t) - \nu}\} \end{aligned} \tag{13}$$

As a, b are of the form $m + \frac{1}{2}$, with m$\in \mathbb{Z}$,

$$\begin{aligned} e(f(b) - \nu b) &= e(f(b)) e(-\nu b) \\ &= e(f(b)) e(-\nu m - \frac{1}{2}\nu) \\ &= e(f(b)) e^{-2\pi i m \nu} e^{-\pi i \nu} \\ &= e(f(b)) \cdot 1 \cdot (e^{-\pi i})^\nu \\ &= e(f(b))(-1)^\nu \\ \text{Similarly, } e(f(a) - \nu a) &= e(f(a))(-1)^\nu. \end{aligned} \tag{14}$$

Moreover, by assumption, $f'(t)$ is decreasing on $[a,b]$. Thus, $f'(b) = \inf_{a<t<b} f'(t) = \alpha$ and $f'(a) = \sup_{a<t<b} f'(t) = \beta$.
We can rewrite (12),

$$\begin{aligned} 2\pi i \hat{F}(\nu) &= (-1)^{\nu} \frac{e(f(b))}{\alpha-\nu} + (-1)^{\nu+1} \frac{e(f(a))}{\beta-\nu} - \int_a^b e(f(t)-\nu t) d\{\frac{1}{f'(t)-\nu}\} \\ &= (-1)^{\nu} \frac{e(f(b))}{\alpha-\nu} + (-1)^{\nu+1} \frac{e(f(a))}{\beta-\nu} + O(\frac{1}{\alpha-\nu} - \frac{1}{\beta-\nu}) \end{aligned} \tag{15}$$

The contribution to (12) of the main terms is $O_\epsilon(1)$.
Moreover, that of the error term is

$$\begin{aligned} &\sum_{\nu \notin [0,\beta+\epsilon]} \frac{1}{\alpha-\nu} - \frac{1}{\beta-\nu} \\ &= \sum_{\nu \notin [0,\beta+\epsilon]} \frac{\beta-\alpha}{(\alpha-\nu)(\beta-\nu)} \\ &\ll \sum_{\nu \notin [0,\beta+\epsilon]} \frac{\beta+1}{\nu(\nu-\beta)} \qquad \text{(as } -1+\epsilon \le \alpha < \epsilon) \\ &= \sum_{\nu \le -1} \frac{\beta+1}{\nu(\nu-\beta)} + \sum_{\nu > \beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\ &= \sum_{\nu \in [1,\infty]} \frac{\beta+1}{-\nu(-\nu-\beta)} + \sum_{\nu > \beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\ &= \sum_{\nu \in [1,\infty]} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\nu > \beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\ &\le \Big[\sum_{1 \le \nu \le \beta+1} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\beta+\epsilon < \nu \le 2\beta} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\nu > 2\beta} \frac{\beta+1}{\nu(\nu+\beta)} \Big] \\ &+ \Big[\sum_{\beta+\epsilon < \nu \le 2\beta} \frac{\beta+1}{\nu(\nu-\beta)} + \sum_{\nu > 2\beta} \frac{\beta+1}{\nu(\nu-\beta)} \Big] \\ &\le \sum_{1 \le \nu \le \beta+1} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\beta+\epsilon < \nu \le 2\beta} \Big[\frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \Big] + \sum_{\nu > 2\beta} \Big[\frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \Big]. \end{aligned}$$

The first term is

$$\le \sum_{1 \le \nu \le \beta+1} \frac{\beta+1}{\nu(1+\beta)} = \sum_{1 \le \nu \le \beta+1} \frac{1}{\nu}.$$

The second term is

$$
\begin{aligned}
&\sum_{\beta+\epsilon<\nu\leq 2\beta} \frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \\
&\leq \sum_{\beta+\epsilon<\nu\leq 2\beta} \frac{\beta+1}{(\beta+1)(2\beta+1)} + \frac{\beta+1}{(\beta+1)(\nu-\beta)} \\
&= \sum_{\beta+\epsilon<\nu\leq 2\beta} \frac{1}{2\beta+1} + \frac{1}{\nu-\beta} \\
&= O(1) + \sum_{\beta+\epsilon<\nu\leq 2\beta} \frac{1}{\nu-\beta}
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&= (\beta+1)\sum_{\nu>2\beta} \frac{2\nu}{\nu(\nu^2-\beta^2)} \\
&\leq (\beta+1)\sum_{\nu>2\beta} \frac{2}{\nu^2-(\frac{\nu}{2})^2} \\
&= \frac{8}{3}(\beta+1)\sum_{\nu>2\beta} \frac{1}{\nu^2}.
\end{aligned}
$$

Combing these terms, we obtain that the error term is

$$
\begin{aligned}
&\ll_\epsilon \sum_{1\leq\nu\leq\beta+1} \frac{1}{\nu} + \sum_{\beta+\epsilon<\nu\leq 2\beta} \frac{1}{\nu-\beta} + \frac{8}{3}(\beta+1)\sum_{\nu>2\beta} \frac{1}{\nu^2} \\
&\ll_\epsilon \int_1^{\beta+1} \frac{1}{\nu} + \int_{\beta+\epsilon}^{2\beta} \frac{1}{\nu-\beta} + \frac{8}{3}(\beta+1)\int_{2\beta}^{\infty} \frac{1}{\nu^2} \\
&= \log|\nu|\Big|_1^{\beta+1} + \log|\nu-\beta|\Big|_{\beta+\epsilon}^{2\beta} + \frac{8}{3}(\beta+1)\frac{1}{\nu}\Big|_{2\beta}^{\infty} \\
&= \log(\beta+1) + \log(\beta) - \log(\epsilon) + \frac{8}{3}\cdot\frac{\beta+1}{2\beta} \\
&\ll \log(\beta+2).
\end{aligned}
$$

This completes the proof. □

Theorem 2.6(van der Corput): Let $f \in C^2[a,b]$, such that

$$
|f''(t)| \approx \lambda > 0 \qquad (a < t < b).
$$

Then we have

$$\sum_{a<n\leq b} e(f(n)) \ll (b-a+1)\lambda^{1/2} + \lambda^{-1/2}. \tag{16}$$

Proof. If $\lambda > 1$, then (19) is satisfied trivially since

$$\begin{aligned}
|\sum_{a<n\leq b} e(f(n))| &\leq \sum_{a<n\leq b} |e(f(n))| \\
&\leq b-a+1 \\
&\ll (b-a+1)\lambda^{1/2} + \lambda^{-1/2}.
\end{aligned}$$

Thus, we can assume $\lambda \leq 1$.
Let $\alpha := \inf_{a<t<b} f'(t)$, $\beta := \sup_{a<t<b} f'(t)$
By Theorem 2.5,

$$\begin{aligned}
\sum_{a<n\leq b} e(f(n)) &\ll \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} \int_a^b e(f(t)-\nu t)dt + \log(\beta-\alpha+2) \\
&\ll (\beta-\alpha+1) \max_{\alpha+\epsilon<\nu<\beta+\epsilon} \left| \int_a^b e(f(t)-\nu t)dt \right| + \log(\beta-\alpha+2).
\end{aligned} \tag{17}$$

Since $f''(t) \approx \lambda > 0$, $f''(t)$ has constant sign on $[a,b]$.
There exists some constant c such that $\inf_{a<t<b} |f''(t)| = c\lambda$.

$$|\frac{d^2}{dt^2}(f(t)-vt)| = |f''(t)|.$$

Applying Theorem 2.3, we have

$$|\int_a^b e(f(t)-\nu t)dt| \leq 4\sqrt{2/\pi c\lambda} \ll \lambda^{-1/2}.$$

The upper bound in (20) is thus

$$\ll (\beta-\alpha+1)\lambda^{-1/2} + \log(\beta-\alpha+2).$$

The condition on f'' forces f to be monotone. We may assume f' is increasing on $[a,b]$. By Mean Value Theorem,

$$\beta-\alpha = \left| \int_a^b f''(t)dt \right| \approx \lambda(b-a).$$

The previous bound is thus

$$\begin{aligned}
&\ll \lambda(b-a)\lambda^{1/2} + \lambda^{-1/2} + log(\lambda(b-a)+2)\\
&\ll \lambda(b-a)\lambda^{1/2} + \lambda^{-1/2} + 1 + \lambda(b-a)\\
&\ll \lambda(b-a)\lambda^{1/2} + \lambda^{-1/2}
\end{aligned}$$

□

We establish a variant of Theorem 2.6 for a function of class C^3.

Theorem 2.7: Let $f \in C^3[a,b]$, with $b-a \geq 1$. Suppose that

$$|f'''(t)| \approx \lambda > 0 \qquad (a < t < b).$$

Then

$$\sum_{a<n\leq b} e(f(n)) \ll (b-a)\lambda^{1/6} + (b-a)^{1/2}\lambda^{-1/6} \tag{18}$$

To prove Theorem 2.7, we first prove the following lemma.

Lemma 2.8: Let f be a real function defined on $[a,b]$. For any integer q with $1 \leq q \leq b-a$, we have

$$\left|\sum_{a<n\leq b} e(f(n))\right| \leq \frac{2(b-a)}{\sqrt{q}} + 2\left\{\frac{(b-a)}{q}\sum_{r=1}^{q-1}\left|\sum_{a<n\leq b-r} e(f(n+r)-f(n))\right|\right\}^{1/2}.$$

Proof of Lemma. Let

$$F(t) = \begin{cases} e(f(t)), & \text{if } a < t \leq b \\ 0, & \text{otherwise} \end{cases}$$

Set $S := \sum_{n\in\mathbb{Z}} F(n)$ be the sum to be estimated. For any fixed m,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} F(n+m) &= \sum_{a<n+m\leq b} e(f(n+m))\\
&= \sum_{a<n'\leq b} e(f(n'))\\
&= \sum_{n\in\mathbb{Z}} F(n)
\end{aligned}$$

Thus, we have

$$S = \frac{1}{q}\sum_{m=1}^{q}\sum_{n\in\mathbb{Z}} F(n+m).$$

Interchanging the order of summation, we obtain

$$S = \frac{1}{q} \sum_{n \in \mathbb{Z}} \sum_{m=1}^{q} F(n+m).$$

Cauchy-Schwarz Inequality for Complex Sums: For all complex numbers $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_n$, the following inequality holds:

$$\left| \sum_{i=1}^{n} z_i \overline{w_i} \right|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2 \tag{19}$$

where $\overline{w_i}$ denotes the complex conjugate of w_i.
Apply the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} |S|^2 &= \frac{1}{q^2} \left(\left| \sum_{n \in \mathbb{Z}} \sum_{m=1}^{q} F(n+m) \right| \right)^2 \\ &\leq \frac{1}{q^2} \sum_{n \in \mathbb{Z}}{}' 1 \cdot \sum_{n \in \mathbb{Z}}{}' \left| \sum_{m=1}^{q} F(n+m) \right|^2 \\ &= \frac{1}{q^2} \sum_{n \in \mathbb{Z}}{}' 1 \cdot \sum_{n \in \mathbb{Z}}{}' \sum_{m,\, m'=1}^{q} F(n+m) \overline{F(n+m')} \end{aligned} \tag{20}$$

where $'$ indicates that the summation is restricted to integers n with $a < n+m \leq b$ for at least one m such that $1 \leq m \leq q$. Thus, $\sum_{n \in \mathbb{Z}}{}' 1$ does not exceed $b - a + q \leq 2(a-b)$. We can rewrite the inner sum and obtain

$$\begin{aligned} &\sum_{m,\, m'=1}^{q} F(n+m) \overline{F(n+m')} \\ &= \sum_{m=1}^{q} F(n+m) \overline{F(n+m)} + \sum_{1 \leq m < m' \leq q} \left(F(n+m) \overline{F(n+m')} + F(n+m') \overline{F(n+m)} \right) \end{aligned} \tag{21}$$

Since

$$F(n+m') \overline{F(n+m)} = \overline{F(n+m) \overline{F(n+m')}},$$

the expression in (23) is equal to

$$q + 2\Re \left(\sum_{1 \leq m < m' \leq q} F(n+m) \overline{F(n+m')} \right).$$

Since $\Re(\alpha) \leq |\alpha|$, the second sum $\sum_{n\in\mathbb{Z}}{}' \sum_{m,\ m'=1}^{q} F(n+m)\overline{F(n+m')}$ is at most

$$2(b-a)q + 2\Bigg|\sum_{1\leq m<m'\leq q}\sum_{n\in\mathbb{Z}} F(n+m)\overline{F(n+m')}\Bigg|. \tag{22}$$

Let $m+n=\nu$, $m-m'=r$, then ν runs through $\mathbb{Z}$ and $r \in \{1,2,\ldots,q-1\}$. Let ν, r be fixed, then there are exactly $q-r$ solutions for $\{n,m,m'\}$, since $n=\nu-m$, $m'=m+r$ and

$$1 \leq m+r \leq q \ \Rightarrow\ m \in \{1,2,\ldots,q-r\}.$$

After perform the change of variable, we obtain that the expression in (24) is

$$\begin{aligned}
&\leq 2(b-a)q + 2\Bigg|\sum_{r=1}^{q-1}(q-r)\sum_{\nu\in\mathbb{Z}} F(\nu)\overline{F(\nu-r)}\Bigg| \\
&\leq 2q\Bigg\{(b-a) + \sum_{r=1}^{q-1}\Bigg|\sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)}\Bigg|\Bigg\}
\end{aligned}$$

Inserting this upper bound in (23), we have

$$\begin{aligned}
|S|^2 &\leq \frac{1}{q^2}\cdot 2(b-a)\cdot 2q\Bigg\{(b-a) + \sum_{r=1}^{q-1}\Bigg|\sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)}\Bigg|\Bigg\} \\
&= \frac{4(b-a)^2}{q} + \frac{4(b-a)}{q}\sum_{r=1}^{q-1}\Bigg|\sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)}\Bigg| \\
&= \frac{4(b-a)^2}{q} + \frac{4(b-a)}{q}\sum_{r=1}^{q-1}\Bigg|\sum_{a<n\leq b-r} e(f(n+r))e(-f(n))\Bigg|
\end{aligned} \tag{23}$$

Since $\forall a,b \in \mathbb{R}$, $2\sqrt{ab} \geq 0$, we have

$$a + 2\sqrt{ab} + b \geq a + b,$$

which indicates

$$(\sqrt{a}+\sqrt{b})^2 \geq (\sqrt{a+b})^2.$$

Thus,

$$\sqrt{a}+\sqrt{b} \geq \sqrt{a+b}.$$

Taking square root on both sides, we have the stated result. Applying it to (26), we have

$$
\begin{aligned}
|S| &\leq \sqrt{\frac{4(b-a)^2}{q} + \frac{4(b-a)}{q}\sum_{r=1}^{q-1}\Bigg|\sum_{a<n\leq b-r} e(f(n+r)-f(n))\Bigg|} \\
&\leq \frac{2(b-a)}{\sqrt{q}} + 2\Bigg\{\frac{(b-a)}{q}\sum_{r=1}^{q-1}\Bigg|\sum_{a<n\leq b-r} e(f(n+r)-f(n))\Bigg|\Bigg\}^{1/2}.
\end{aligned}
$$

□

Proof of Theorem 2.7. Let $g(x) := f(x+r) - f(x)$, then for $x \in (a, b-r)$,

$$g''(x) = f''(x+r) - f''(x).$$

Since f'' is differentiable, Taylor's theorem of order 1 states

$$f''(x) \approx f''(x_0) + f'''(x)(x - x_0).$$

Thus,

$$g''(x) \approx f'''(x+r)(x+r-x_0) - f'''(x)(x-x_0).$$

Since $|f'''(x)| \approx \lambda > 0$, we have $|g''(x)| \approx r\lambda$.
Let $L := b - a$. Applying Lemma 2.8, we have

$$
\begin{aligned}
\sum_{a<n\leq b} e(f(n)) &\ll Lq^{-1/2} + \Bigg\{Lq^{-1}\Bigg|\sum_{a<n\leq b-r} e(f(n+r)-f(n))\Bigg|\Bigg\}^{1/2} \\
&= Lq^{-1/2} + \Bigg\{Lq^{-1}\Bigg|\sum_{a<n\leq b-r} e(g(n))\Bigg|\Bigg\}^{1/2}.
\end{aligned}
\tag{24}
$$

Then, we can apply Theorem 2.6 on $\sum_{a<n\leq b-r} e(g(n))$ and get (24)

$$\ll Lq^{-1/2} + \Bigg\{Lq^{-1}\sum_{r=1}^{q-1}\Big(L(r\lambda)^{1/2} + (r\lambda)^{-1/2}\Big)\Bigg\}^{1/2}.$$

Through observation, we notice that this bound is

$$
\begin{aligned}
&\ll Lq^{-1/2} + \left\{ Lq^{-1}q\Big(L(q\lambda)^{1/2} + (q\lambda)^{-1/2} \Big) \right\}^{1/2} \\
&= Lq^{-1/2} + \left\{ L^2(q\lambda)^{1/2} + L(q\lambda)^{-1/2} \right\}^{1/2} \\
&\leq Lq^{-1/2} + L(q\lambda)^{1/4} + L^{1/2}(q\lambda)^{-1/4}.
\end{aligned}
$$

If λ satisfies $1 \leq \lambda^{-1/3} \leq L$, we could choose $q = \lfloor \lambda^{-1/3} \rfloor$. This makes the upper bound

$$
\begin{aligned}
&\ll L\lambda^{1/6} + L\lambda^{1/6} + L^{1/2}\lambda^{-1/6} \\
&\ll (b-a)\lambda^{1/6} + (b-a))^{1/2}\lambda^{-1/6},
\end{aligned}
$$

which is the desired result.
The estimate (18) is trivially valid when $\lambda > 1$ or $\lambda < L^{-3}$. □

Now we show the error term obtained using van der Corput's method in the Dirichlet divisor problem is $O_\epsilon(x^{1/3+\epsilon})$. Theorem 2.8 (Voroni, 1903). For $x \geq 2$, we have

$$
\sum_{n \leq x} d(n) = x(\log x + 2\gamma - 1) + O(x^{1/3}\log x). \tag{25}
$$

Proof. The hyperbola method (Theorem 1.1) shows that the left-hand side of (25)

$$
= 2\sum_{n=1}^{\lfloor \sqrt{x} \rfloor} \lfloor \frac{x}{n} \rfloor - \lfloor \sqrt{x} \rfloor^2 \qquad \text{(see (1))}. \tag{26}
$$

Write $N := \lfloor \sqrt{x} \rfloor$. Let $B_1(t) = t - \frac{1}{2}$ denote the first Bernoulli function, then (26) becomes

$$
\begin{aligned}
2\sum_{n \leq N} \lfloor \frac{x}{n} \rfloor - N^2 &= 2\sum_{n \leq N} \left(\lfloor \frac{x}{n} \rfloor - \{ \frac{x}{n} \} \right) - N^2 \\
&= 2\sum_{n \leq N} \left(\frac{x}{n} - B_1\left(\frac{x}{n} \right) - \frac{1}{2} \cdot 2\sum_{n=1}^{N} 1 - N^2 \right. \\
&= 2\sum_{n \leq N} \left(\frac{x}{n} - B_1\left(\frac{x}{n} \right) \right) - N - N^2
\end{aligned}
$$

Using the harmonic series result to estimate $\sum_{x \leq N} \frac{1}{x}$, we can write the above expression in the form

$$
P(x) - 2R(x)
$$

with

$$P(x) := 2x(\log N + \gamma + \frac{1}{2N} + O(\frac{1}{x})) - N - N^2$$

and

$$R(x) := \sum_{n \leq N} B_1(\frac{x}{n}).$$

Writing $N = \sqrt{x} - \theta$, with $0 \leq \theta < 1$, we have

$$\begin{aligned}
P(x) &= 2x\left(\log(\sqrt{x} - \theta) + \gamma + \frac{1}{2(\sqrt{x} - \theta)} + O\left(\frac{1}{x}\right)\right) - (\sqrt{x} - \theta) - (\sqrt{x} - \theta)^2 \\
&= 2x\log(\sqrt{x} - \theta) + 2x\lambda - x + \frac{x}{\sqrt{x} - \theta} + O(1) - \sqrt{x} + \theta - x - \theta^2 + 2\sqrt{x}\theta \\
&= 2x\log(\sqrt{x}(1 - \frac{\theta}{\sqrt{x}})) + 2x\lambda - x + \frac{x}{\sqrt{x} - \theta} + O(1) - \sqrt{x} + \theta - x - \theta^2 + 2\sqrt{x}\theta \\
&= 2x\log(\sqrt{x}) + 2x\log(1 - \frac{\theta}{\sqrt{x}})) + 2x\lambda - x + \frac{x}{\sqrt{x} - \theta} + O(1) - \sqrt{x} + \theta - x - \theta^2 + 2\sqrt{x}\theta \\
&= xlog(x) + 2x\log(1 - \frac{\theta}{\sqrt{x}})) + 2x\lambda - 2x + \frac{x}{\sqrt{x} - \theta} - \sqrt{x} + 2\sqrt{x}\theta + O(1)
\end{aligned}$$

Then, we expand $log(1 - \frac{\theta}{\sqrt{x}})$ as a function of $\frac{\theta}{\sqrt{x}}$ up to order 1 and obtain

$$log(1 - \frac{\theta}{\sqrt{x}}) = -\frac{\theta}{\sqrt{x}}.$$

Thus, P(x) is

$$\begin{aligned}
&= x\log(x) - 2x\frac{\theta}{\sqrt{x}} + 2x\lambda - 2x + \frac{x}{\sqrt{x} - \theta} - \sqrt{x} + 2\sqrt{x}\theta + O(1) \\
&= x\log(x) - 2\sqrt{x}\theta + 2x\lambda - 2x + \frac{x}{\sqrt{x} - \theta} - \sqrt{x} + 2\sqrt{x}\theta + O(1) \\
&= x\log(x) + 2x\lambda - 2x + \frac{x}{\sqrt{x} - \theta} - \sqrt{x} + O(1) \\
&= x(\log x + 2\lambda - 1) + \frac{\sqrt{x}\theta}{\sqrt{x} - \theta} + O(1).
\end{aligned}$$

Since $0 \leq \theta < 1$,

$$\frac{\sqrt{x}\theta}{\sqrt{x} - \theta} \leq \frac{\sqrt{x}}{\sqrt{x} - 1} = O(1).$$

Thus, we obtain that

$$P(x) = x(\log x + 2\lambda - 1) + O(1).$$

Hence, to obtain the desired result, it suffices for us to prove

$$R(x) \ll x^{1/3} \log x. \tag{27}$$

To establish (27), we apply van der Corput's technique. To begin with, we expand $B_1(t)$ as a Fourier series, which gives us

$$B_1(t) = -\frac{1}{\pi} \sum_{j=1}^{\infty} \frac{\sin(2\pi jt)}{j}.$$

Integrate B_1, we get

$$B(t) := \int B_1(t)dt = \sum_{j=1}^{\infty} \frac{cos(2\pi jt)}{2\pi^2 j^2}.$$

Since

$$\left| \sum_{j=1}^{\infty} \frac{cos(2\pi jt)}{2\pi^2 j^2} \right| \leq \sum_{j=1}^{\infty} \left| \frac{cos(2\pi jt)}{2\pi^2 j^2} \right| \leq \sum_{j=1}^{\infty} \frac{1}{2\pi^2 j^2},$$

B(t) is absolutely convergent. □