Estimates on constants related to Minkowski dimension

Filippo Iulianelli

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Abstract

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In this paper we develop the theory of fractal dimension, introducing several definitions of and concepts related to the Minkowski and Hausdorff dimensions of a set. After reporting some results from [1], we provide some bounds on constants that in turn determine bounds on intersections of sets of different dimensions.
Issues related to the shortcomings of this approach are discussed, in particular the fact that all the theorems hold up to a factor of ϵ in the exponent, and how this introduces significant limitations to the scope of this paper.

1 Introduction

The study of fractal geometry is usually ascribed to Mandelbrot, who coined the term fractal in 1975 [5], although, as it's often the case in mathematics, the idea of fractals and fractal dimension actually emerged from the work of previous mathematicians. Some of the most famous names are Weierstrass, Cantor, Hausdorff, Fatou, Julia.

In pop culture, the idea of fractals is associated to beautiful self-similar shapes such as the Mandelbrot set or the growth patterns of cauliflowers. There are, however, notions of fractals that are not restricted to strictly self-similar set. These notions usually rely on some sort of "statistical similarity" or "scale invariance" of a set, and can be defined rigorously, as we shall see.

It is important to note that these generalized versions of self-similarity are not pointless abstraction, but can be found everywhere in the world around us: the shape of the delta of a river, or the "jagged-ness" of the coast of an island [4], even in the behavior of prices in the stock market [1] can all be analyzed with these tools.

The first notion of non-integer dimension was proposed by Hausdorff in 1918 [3]. This definition, known as Hausdorff dimension, extends the usual notions of dimension to allow for non-integer values, and is still widely used. According to [7], the Hausdorff dimension is considered more "robust", and is treated as a "standard".
As one would expect, the Hausdorff dimension of "usual sets", such as lines, planes, or spheres, is exactly what one would expect. More generally, the Hausdorff dimension of an n-dimensional smooth manifold exists and is n.

There are other definitions of dimension that one can use to analyze sets. In this paper we are going to present two of them: the Minkowski dimension, and the discrete Hausdorff dimension.
The former is a very useful tool in concrete cases, as it lends itself well to computations, but it loses some of the nice properties that the Hausdorff dimension has.
As we shall see, this definition tries to capture the fact that if a d-dimensional object is scaled by a factor of δ, its volume will scale roughly as:

$$V \sim \delta^d$$

In particular, we will see that the number of sets of diameter at most δ needed to cover a d-dimensional set grows like $\sim C \cdot \delta^{-s}$ as δ goes to zero, where C is a positive constant. Much of the work in the last sections of this paper will revolve around C.

The discrete Hausdorff dimension, instead, defines a notion of dimension for sequences of finite point sets. This is very useful for practical applications, because we can approximate unknown sets by sampling a countable (in theory) or finite (in practice) amount of points.

For example, suppose we were trying to associate a notion of dimension to the graph of some real function. We could then take a time series of the graph by sampling a point every Δx. By reducing this parameter we can increase the size of our sample set and approximate the graph. This way we can compute a quantity associated to the time series known as *r-dimensional energy integral*, where r is a non-negative real number. Given a set P_n of n points in $\mathbb{R}^d$, we define this quantity as:

$$I_r(P_n) := n^{-2} \sum_{p \neq p' \in P_n} |p - p'|^{-r} \tag{1}$$

Studying how this quantity behaves as we increase the size of our sample set allows one to define some notion of dimension of the sequence of samples, which in turn provides information on the dimension of the underlying graph.

In the applied case when one is trying to analyze a practical situation, say the price of a stock with respect to time, one does not have access to the full graph, and it is impossible to take arbitrarily large samples. One then has to take finite samples and try to infer their asymptotic behavior.

One more use for this tool, which will be fundamental for our purposes is that it can be used to characterize the the validity of approximating a set by a lower-dimensional one. For example, one can show that if a function's graph has dimension strictly greater than one, there is no good approximation by smooth functions in a precise sense that will be defined later.

The aim of this paper is to study this topic by focusing on finding bounds on intersections of sets of different dimension. In particular, this will be achieved by building upon the work of [1]. There, one can find bounds on such intersection. These bounds, however, depend on some constants that in turn depend on the sets under scrutiny. This paper will provide bounds for those constants in terms of the dimensions of the sets involved.

2 The notion of dimension

There are several properties that characterize the dimension of a "space". The most intuitive one, is that the dimension of a space is the minimum number of real parameters needed to describe a point in the space. Clearly this definition cannot be used to describe fractional dimensions (what does it mean for a space to be parametrized by 1.5 parameters?). Furthermore, this kind of definition could run into other issues: for example, there exists a bijection between $\mathbb{R}$ and $\mathbb{R}^2$, so $\mathbb{R}^2$ could technically be parametrized by a single real number. This contradicts our intuition that $\mathbb{R}^2$ should be two-dimensional (in fact, it is the main example of a two dimensional space). More care is necessary when establishing this kind of definition.

In this paper we will abandon the idea of "number of parameters" altogether, and utilize alternative definitions of dimension.
In particular, we will use another characteristic of dimension as our starting point.
Denote by μ_d the d-dimensional Lebesgue measure. One can think of μ_d simply as the "volume function" that maps certain subsets of euclidean spaces (known as measurable sets) to their volume.
A property that one can easily check is that the volume of a d-dimensional ball of radius r scales as r^d: in fact $\mu_d(B_d(r)) = \omega_d \, r^d$, where ω_d is the volume of the unit ball in d dimensions. It can be checked that this holds true for any measurable subset E of $\mathbb{R}^d$:

$$\mu(\lambda E) = \lambda^d \mu(E) \tag{2}$$

Where λ is a positive real number and $\lambda E = \{\lambda x, x \in E\}$.
The key idea here is that a d-dimensional set $E \subseteq \mathbb{R}^d$ scales as $diam(E)^d$.

The most straightforward way to elaborate on this is the Minkowski dimension, which will be the topic of the next section.

3 Minkowski dimension

This section closely follows Ch. 2 of [2].
One way of formalizing the notion of scaling described above that of Minkowski dimension.
There are various ways of defining this concept. Here we report two definitions, show that they coincide and explain why both are useful.
We will start with the heuristics.

Let us introduce a handy definition, that will recur throughout the paper;

Definition 3.1 (δ-cover) *Given $E \subseteq \mathbb{R}^d$, we say that the (at most countable) collection of sets $\{U_i\}_i \subseteq \mathbb{R}^d$ form a δ-cover of E if $diam(U_i) \leq \delta$ and $E \subseteq \cup_i U_i$*

From this point and throughout this paper we will assume that $E \subseteq [0,1]^d$. This assumption doesn't really impact the main ideas presented here, but it does simplify some proofs.

Suppose we wished to wished to find a δ-cover of E. Given any δ, define $N_\delta(E)$ to be the minimal number of sets of diameter at most δ needed to cover E.
According to 2 we see that, for an s dimensional set, we must have $N_\delta(E) \sim C \cdot \delta^{-s}$.
We can obtain an expression for s by taking the logarithm of both sides:

$$\ln(N_\delta(E)) \sim \ln(C) - s\ln(\delta)$$

Obtaining

$$s \sim -\frac{\ln(N_\delta(E))}{\ln(\delta)} + \frac{\ln(C)}{\ln(\delta)} \tag{3}$$

In the limit as $\delta \to 0$, the second term is suppressed and we get:

$$s \sim -\frac{\ln(N_\delta(E))}{\ln(\delta)} \tag{4}$$

To define this notion rigorously, we need to acknowledge that the limit in (4) may not exist. We do this by noting that the limit supremum and inferior always exist.

Definition 3.2 (Minkowski dimension) *Given $E \subseteq \mathbb{R}^d$, we define its upper and lower dimensions as:*

$$\begin{aligned} dim_B(E) &= \lim_{\delta \to 0} \sup \frac{ln(N_\delta(E))}{ln(\delta^{-1})} \\ dim_B(E) &= \lim_{\delta \to 0} \inf \frac{ln(N_\delta(E))}{ln(\delta^{-1})} \end{aligned} \tag{5}$$

Where $N_\delta(E)$ is defined as above.
When the two limits above are equal, we can define the Minkowski dimension of E:

$$dim_B(E) = \lim_{\delta \to 0} \frac{ln(N_\delta(E))}{ln(\delta^{-1})} \tag{6}$$

We can also take a different approach, known as box counting. Consider the collection of cubes in $\mathbb{R}^d$ of the form

$$Q^\delta_{(m_1,...,m_d)} = [m_1\delta, (m_1+1)\delta] \times ... \times [m_d\delta, (m_d+1)\delta]$$

For $m_i \in \mathbb{Z}$ and $i \in \{1, ..., d\}$. This is known as a δ-grid of $\mathbb{R}^d$
Denote by $N'_\delta(E)$ the number of cubes (of side length δ) as defined above that contain at least a point of E. We then have the following proposition:

Proposition 1 *The limits in 6 are unchanged if $N_\delta(E)$ is replaced by $N'_\delta(E)$. In particular, When the lim sup and lim inf coincide, the two definitions yield the same dimension.*

Proof: First of all, note that the diameter of a cube is given by the length of its longest diagonals: $\text{diam}(Q^\delta_{(m_1,...,m_d)}) = \sqrt{\delta^2 + ... + \delta^2} = \delta\sqrt{d}$.
Since the set of cubes of the form $Q^\delta_{(m_1,...,m_d)}$ that contain at least a point of E is a cover of E by sets of diameter at most $\delta\sqrt{d}$, we have:

$$N_{\delta\sqrt{d}} \leq N'_\delta \tag{7}$$

Similarly, can cover a set E of diameter at most δ with cubes from a $\delta - grid$ in the following way: pick any point $p \in E$. A δ-grid covers E, p must be contained in some $Q^\delta_{(m_1,...,m_d)}$. Since $\text{diam}(E) \leq \delta$, E must be covered by $Q^\delta_{(m_1,...,m_d)}$ and its adjacent cubes:

$$E \subset \cup_{(i_1,...,i_d) \in \{0,1,-1\}^d} Q_{(m_{\pm i_1},...,m_{\pm i_d})}$$

Note that union runs over 3^d cubes.
We just showed that every set of diameter at most δ can be covered by 3^d δ-cubes. This means that, if N_δ sets of diameter at most δ are necessary to cover E, then E can always be covered by at most $3^d N_\delta$ δ-cubes (one for each set). In symbols:

$$N'_\delta \leq 3^d N_\delta \tag{8}$$

Putting 7 and 8 together and taking the logarithm we get:

$$\log(N_{\delta\sqrt{d}}) \leq \log(N'_\delta) \leq d \cdot \log(3) + \log(N_\delta)$$

Divide by $-\log(\delta)$ to obtain:

$$-\frac{\log(N_{\delta\sqrt{d}})}{\log(\delta)} \leq -\frac{\log(N'_\delta)}{\log(\delta)} \leq -\frac{d \cdot \log(3) + \log(N_\delta)}{\log(\delta)}$$

If we now take the lim sup (lim inf) of these two inequalities as $\delta \to 0$ we see that the left and right sides both approach the upper (lower) Minkowski dimension.

While these two definitions yield the same dimensions, they have different applications. The first definition is well-suited for theoretical analysis, as it provides a minimal value without the need to construct it explicitly.
The box counting definition, on the other hand, is more useful for numerical computations: finding a minimal covers by sets of a given diameter is a highly non-trivial problem, while counting how many boxes a given set intersects is more straightforward.

3.1 Other useful definitions of Minkowski dimension

We give here two more definitions of Minkowski dimension that will be used in this paper. We will not prove equivalence between the various definitions, but we will prove some useful bounds between them that will be used later in the paper, and that can be used to prove equivalence. This section follows Ch. 5 of [6].

Definition 3.3 (Packing number) *The* packing number *of E, $P_\delta(E)$, is defined to be the largest number of disjoint open balls of radius δ with centers in E. We call such a collection of balls a δ-packing of E.*

We have the following inequality:

$$N_{2\delta}(E) \leq P_\delta(E) \leq N_{\frac{\delta}{2}}(E) \tag{9}$$

Proof: The first inequality follows from noticing that if we take a δ-packing of E and double the radius of the balls, we get a 2δ-cover of E. If that weren't the case -i.e. if a point $x \in E$ wasn't contained in a ball of radius 2δ, then $B_\delta(x)$ would be a ball with center in E, that is disjoint from the other, contradicting the definition of $P_\delta(E)$.

The proof for the RHS is similar, see [6].

Corollary 1 *$P_\delta(E)$ and $N_\delta(E)$ give rise to the same upper and lower dimensions.*

We are now going to define one more quantity, the Minkowski content of a set, that can not only be used to define another equivalent definition of Minkowski dimension, but will also give us some useful bounds that will be explained in section 6.
First we need a preliminary definition:

Definition 3.4 (δ-neighborhood of a set) *We define the δ-neighborhood of a set E to be*

$$E(\delta) = \{x \in \mathbb{R}^d : d(x, E) < \delta\}$$

Where $d(x, E) = \inf\{d(x, y) : y \in E\}$ is the distance between a point and a set.

We can then define the s dimensional Minkowski content:

Definition 3.5 (Minkowski content) *The upper and lower Minkowski contents of E are defined as:*

$$\begin{aligned} \overline{M}^s(E) &= \limsup_{\delta \to 0} (2\delta)^{s-d} \mu_d(E(\delta)) \\ \underline{M}^s(E) &= \liminf_{\delta \to 0} (2\delta)^{s-d} \mu_d(E(\delta)) \end{aligned} \tag{10}$$

One can then define the upper and lower Minkowski dimensions using this concept:

$$\begin{aligned} \overline{dim}_B(E) &= \inf\{s : \overline{M}^s(E) = 0\} = \sup\{s : \overline{M}^s(E) > 0\} \\ \underline{dim}_B(E) &= \inf\{s : \underline{M}^s(E) = 0\} = \sup\{s : \underline{M}^s(E) > 0\} \end{aligned}$$

a These definitions are equivalent to all the ones we've shown before, altough we will not show that here. We will instead prove some useful bounds.

Proposition 2 *We have the following inequality*

$$P_\delta(E)\Omega(d)\delta^d \leq \mu_d(E(\delta)) \leq N_\delta(E)\Omega(d)(2\delta)^d \tag{11}$$

Proof: the fact that any δ-packing of E is contained in the δ-neighborhood of E proves the first inequality: the LHS is exactly the d-dimensional Lebesgue measure of the packing, so the result follows by the monotonicity of μ_d.

The second inequality can be proved by noting that if we replace the sets in a δ-cover with balls of radius $\delta/2$ that contain the original sets we still cover E. If the radius is then increased to 2δ we are guaranteed that $E(\delta)$ is also covered. The 2δ bound can be easily made sharper, but this will suffice for our purposes. If we call these balls with radius 2δ B_i, $1 \leq i \leq n$ we get:

$$E(\delta) \subseteq \cup_{i=1}^n B_i$$

By monotonicity and subadditivity of μ_d we get:

$$\mu_d(E(\delta)) \leq \sum_{i=1}^n \mu_d(B_i) \leq N_\delta(E)\Omega(d)(2\delta)^d$$

Thus proving the inequality.

3.2 Lipshitz functions

Here we briefly discuss the role that Lipshitz functions play in dimension theory. These functions are important as they preserve the Minkowski dimension of sets.

Definition 3.6 *A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be Lipshitz, or Lipshitz continuous, if there exists a positive real number L such that, for all $x, y \in \mathbb{R}^n$ we have:*

$$|f(x) - f(y)| \leq L|x - y|$$

L is known as the Lipshitz constant of f.

Proposition 3 (Lipshitz functions and Minkowski dimension) *For any set $E \subset [0,1]^d$ we have*

$$\begin{aligned} \overline{dim}_B f(E) &\leq \overline{dim}_B E \\ \underline{dim}_B f(E) &\leq \underline{dim}_B E \end{aligned} \tag{12}$$

Proof: If $\{U_i\}_{i=1}^{N_\delta(E)}$ is a δ-cover of E, then $\{f(U_i)\}_{i=1}^{N_\delta(E)}$ is an $L \cdot \delta$-cover of E.
Therefore $N_{L\delta}(f(E)) \leq N_\delta(E)$. Taking the log of both sides and dividing by $\log(\delta^{-1})$:

$$\frac{N_{L\delta}(f(E))}{\log(\delta^{-1})} \leq \frac{N_\delta(E)}{\log(\delta^{-1})}$$

Recognizing that $\log(\delta^{-1}) = \log((L\delta)^{-1}) + \log(L)$ we finally get:

$$\frac{N_{L\delta}(f(E))}{\log((L\delta)^{-1}) + \log(L)} \leq \frac{N_\delta(E)}{\log(\delta^{-1})}$$

One can then take the upper and lower limits to obtain the desired result. The $\log(L)$ term at the denominator of the LHS will disappear in the limit.

4 Hausdorff dimension

We define here the notion of Hausdorff dimension, which is one popular way of assigning dimension of a subset of $\mathbb{R}^d$, even when there is no sensible way of assigning an integer dimension to it.
We start by defining the s-dimensional Hausdorff measure of a set $E \subseteq \mathbb{R}^d$, which has the nice property of naturally extending the (integer-dimensional) Lebesgue measure.

First of all, we define the δ-Hausdorff content:

Definition 4.1 *The δ-Hausdorff content of a set E is defined as:*

$$H^s_\delta(E) := \inf\{\sum_{i=1}^{\infty} diam(U_i)^s : U_i \text{ is a } \delta\text{-cover of } E\} \tag{13}$$

Note that a δ'-cover of E is also a δ-cover of E, as long as $\delta' < \delta$, so $H^s_\delta(E)$ is non-increasing (and non-negative), and thus the limit as δ goes to 0 exists, leading us to our next definition

Definition 4.2 (s-dimensional Hausdorff measure) *Let E be a subset of $\mathbb{R}^d$ and let $H^s_\delta(E)$ be defined as above. Then we defined the s-dimensional Hausdorff measure of E as:*

$$H^s(E) := \lim_{\delta \to 0} H^s_\delta(E) \tag{14}$$

It can be shown that H^s is indeed a measure (in the measure-theoretic sense) and that it coincides with the Lebesgue measure (up to a constant factor depending on s) when s is an integer.

We will now see that, for every set E, there exists a unique value of s such that $H^s(E)$ is finite.
Let U_i be a δ-cover of E, and suppose $t > r \geq 0$.
Then we have:

$$\sum_i \mathrm{diam}(U_i)^t = \sum_i \mathrm{diam}(U_i)^{t-r}\mathrm{diam}(U_i)^r \leq \delta^{t-r} \sum_i \mathrm{diam}(U_i)^s$$

It can be shown that the inequality still holds when taking the infimum over δ-covers, to obtain:

$$H^t_\delta(E) \leq \delta^{t-r} H^r_\delta(E) \tag{15}$$

Note that (15) implies that if $r \neq t$ then $H^t_\delta(E)$ and $H^r_\delta(E)$ cannot be both finite and non-zero. More concretely, suppose that $H^r_\delta(E) < \infty$ and that $r < t$, then (15) must hold for any δ. By taking the limit as $\delta \to 0$, we see that the only way to satisfy (15) is if $H^r_\delta(E) = 0$.
Similarly, if $0 < H^t_\delta(E)$, the only way to satisfy (15) is if $H^r_\delta(E) = \infty$.
We summarize the above paragraph in the following definition and theorem.

Theorem 1 (Uniqueness of Hausdorff dimension) *For any given $E \subseteq \mathbb{R}^d$, there exists at most one real number $s > 0$, such that $0 < H^s(E) < 0$.*
In particular, if $0 < H^s(E) < 0$, and t and r are real numbers satisfying $0 < r < s < t$, we have $H^t(E) = 0$, $H^r(E) = \infty$,

Definition 4.3 (Hausdorff dimension) *When such a value s exists such that $0 < H^s(E) < 0$ we say that the set E has Hausdorff dimension s.*

Theorem 1 shouldn't come as a surprise. After all, the same is true with the usual Lebesgue measure: for example, if μ_d is the d-dimensional Lebesgue measure and Q_2 is the two dimensional unit cube (unit square), we have $\mu_1(Q_2) = \infty$, $\mu_2(Q_2) = 1$, and $\mu_3(Q_2) = 0$.
This captures the idea that a unit square has "unit area", "zero volume", and "infinite length".

While Hausdorff dimension is a very powerful tool to analyze and characterize subsets of $\mathbb{R}/d$, the following theorem tells us that we need a different tool to deal with countable sequences of finite sets.

Theorem 2 (Hausdorff dimension of countable sets) *If $S \subset \mathbb{R}^d$ is countable, then it has Hausdorff dimension zero.*

One idea that will allow us to analyze the dimension of countable sets is a modification of the Hausdorff dimension known as Discrete Hausdorff dimension. This will be the subject of the next section.

5 Discrete Hausdorff dimension

We now introduce a very important tool in the development of the idea of discrete Hausdorff dimension: the energy integral.

Definition 5.1 (Energy integral) *Given a finite set $P_n \subset [0,1]^d$ with $|P_n| = n \in \mathbb{Z}^+$, we define the discrete r-energy of P_n as:*

$$I_r(P_n) := n^{-2} \sum_{p \neq p'} |p - p'|^{-r} \tag{16}$$

This quantity is referred to as the "energy" of a finite point set, as it mimics the electric potential energy of a finite set of point particles with identical charges, in the case when $r = 2$.

Definition 5.2 (Time series) *A time series is a collection P of sets $P_n \in [0,1]^d$ with $|P_n| = n$.*
If all P_n are subsets of a set $E \subset [0,1]^d$ we say that P is a time series of E.

Definition 5.3 (Discrete Hausdorff dimension) *Given a time series $P = \{P_n\}, n \in \mathbb{Z}^+$, we defined it's discrete Hausdorff dimension $dim_{H_D}(P)$ as:*

$$dim_{H_D}(P) := \sup\{r \in [0,d] : \sup_n I_r(P_n) < \infty\} \tag{17}$$

Now that we have our basic ideas set up, we are almost ready to report some results from [1], which we will then build upon.

Before we introduce the results, let us take a small digression to discuss an issue in the definition of N_δ.

5.1 Freedom in choosing $N_\delta(E)$

When proving the equality of various definitions of Minkowski dimension we used the fact that we could multiply whatever definition of $N_\delta(E)$ we chose by a constant, say k, as the latter would be suppressed by a $\log(\delta^{-1})$ term:

$$\limsup_{\delta \to 0} \frac{\log(k \cdot N_\delta(E))}{\log(\delta^{-1})} = \limsup_{\delta \to 0} \frac{\log(k)}{\log(\delta^{-1})} + \frac{\log(N_\delta(E))}{\log(\delta^{-1})} = \limsup_{\delta \to 0} \frac{\log(N_\delta(E))}{\log(\delta^{-1})}$$

As the first term goes to zero in the limit.

There is no reason to require that k be a constant: it may as well be a function of δ (call it $k(\delta)$), as long as it grows slow enough:

$$\limsup_{\delta\to 0} \frac{\log(k(\delta))}{\log(\delta^{-1})} = 0 \tag{18}$$

Remark 1 *$k(\delta)$ must go to zero or infinity slower than any polynomial.*

Proof: $\forall\epsilon > 0$ there must exist a $\Delta > 0$ such that $0 < \delta < \Delta$ implies $k(\delta) < C_\epsilon \delta^{-\epsilon}$, where C_ϵ is a constant. If we considered the lower Minkowski dimension we would get a similar inequality: $k(\delta) > c\delta^\epsilon$.

This requirement is to be expected: if $k(\delta)$ behaved like a polynomial, it would affect the exponent in $N_\delta \sim C \cdot \delta^{-s}$.
We see that the property that gives the Minkowski dimension its flexibility also gives rise to some issues with its definition: it can't resolve subpolynomial behavior. This issue introduces a significant limitation to the results of this paper:

Remark 2 *All the theorems below are true up to a factor of (arbitrarily small) ϵ in the exponent, and up to a multiplicative constant C_ϵ. Note that this will indeed prevent us from drawing definitive conclusions about the main goals of this paper. This will be described clearly once we get to those results.*

6 Results from fractals paper

Here we report the results from [1]. We start by stating and proving some lemmas from the paper that are necessary to obtain our results.

The first lemma is the most fundamental one, and it underlines all the theorems that follow.

Lemma 1 *If a set $E \subseteq \mathbb{R}^d$ has upper Minkowski dimension s, there must exist a constant C_E depending only on E such that $N_\delta(E) \leq C_E \delta^{-s}$.*

Note that, as mentioned in Remark 2, this lemma holds up to a factor of ϵ in the exponent, and a multiplicative constant, which we may absorb into a single constant $C_{E,\epsilon}$. In stating this theorem we have omitted it, but we will keep track of it in later proofs.

The following lemma is a consequence of the first, and it quantifies the amount of "clustering" in a time series of a set. In particular, it serves as a bridge between the Minkowski dimension of a set and the properties of its time series.

Lemma 2 *Let $P = \{P_n\}$ be a time series of E, with $\overline{dim}_B(E) = s$. Then $|\{(p, p') : |p - p'| \leq \delta\}| \geq C_E^{-1}\delta^s n^2$*

The point of defining the discrete Hausdorff dimension is that it allows us to compute the notion of dimension of a set by simply sampling points. Theorem 4 of [1] provides a connection between the Minkowski measure of a set and the DHD of any of its discret subsets in the following sense:

Theorem 3 (Lower bound for $I_r(P_n)$ - Thm 4 of FRACTALS) *Let P be a family of point sets contained in a subset $E \subset [0,1]^d$ of upper Minkowski dimension s .*
Then, $dim_{H_D} P = r \leq s$. If, instead, we have $r > s$, we have the following quantitative lower bound:

$$I_r(P_n) \geq \frac{s}{r-s} \left(C_E^{-1}\right)^{\frac{r}{s}} n^{\frac{r}{s}-1} - \frac{\left(C_E^{-1}\right) r}{r-s} + \frac{1}{n} \tag{19}$$

Where $P_n \in P$ and C_E is as in (2).

Proof: see appendix.

A direct application of Thm 3. is that it constrains our ability to approximate a fractal set with smooth surfaces. This is explain concretely by Thm 8. from [1].

Theorem 4 (Intersection of sets with different dimensions) *Let $P = \{P_n\}$ be a time series with $dim_{H_D}(E) = s$ and let E be a subset of $[0,1]^d$ with $\overline{dim}_B(E) = r > s$. Then, for every $\epsilon > 0$, there exists a constant C_ϵ such that:*

$$|P_n \cap E| \leq C_\epsilon n^{\frac{2}{1+r/s}+\epsilon} \tag{20}$$

In Thm 4. P is the set we are trying to approximate with E. What it means concretely is that if the dimension of P is greater than that of E, we will not be able to approximate it well.
For a concrete example example, suppose we wished to approximate a set $P \subset \mathbb{R}^2$ with DHD greater than one by a smooth line (which has UMD equal to one). What Thm 4. tells us is that, no matter how well we approximate a finite subset of P, if we try to add more points most of them will lie outside of the approximating curve.

Note the presence of ϵ at the exponent, and of the multiplicative constant C_ϵ.
Proof: This proof follows that of [1], while accounting for constants that in that paper are lumped in to C_ϵ. In this case, C_ϵ is comes from the constant C_E in Lemma 1, which then ripples through the proofs.

Let $P'_m = P_n \cap E$ $m = |P'_m|$.
Then we have:

$$\begin{aligned} I_r(P'_m) &= m^{-2} \sum_{\substack{p \neq p' \\ p,p' \in E}} |p-p'|^{-r} \leq m^{-2} \sum_{p \neq p'} |p-p'|^{-r} = m^{-2} n^2 I_r(P_n) \\ &\leq C_r m^{-2} n^2 \end{aligned} \tag{21}$$

Where C_s is the constant for which $I_s(P_n) \leq C_s$. This exists by hypothesis, as $r > s$.
We now apply Thm 3:

$$I_r(P'_m) \geq C_m \frac{s}{r-s} \left(C_E^{-1}\right)^{\frac{r}{s}} m^{\frac{r}{s}-1} \tag{22}$$

Where C_m is a constant that guarantees that:

$$\frac{s}{r-s} \left(C_E^{-1}\right)^{\frac{r}{s}} m^{\frac{r}{s}-1} \geq -\frac{\left(C_E^{-1}\right) r}{r-s} + \frac{1}{m}$$

Note that C_m can be taken to be arbitrarily close to 1 as n goes to infinity.

Combining this with (21) we obtain:

$$C_s m^{-2} n^2 \geq C_m \frac{s}{r-s} \left(C_E^{-1}\right)^{\frac{r}{s}} m^{\frac{r}{s}-1}$$

We can use this to find a bound for m:

$$m \leq \left(\frac{C_S C_E^{\frac{r}{s}}}{sC_m}(r-s)\right)^{\frac{1}{\frac{r}{s}+1}} n^{\frac{2}{\frac{r}{s}+1}}$$

However, recall that C_E depends on ϵ, which we have no control over, so we will have to say

$$m \lesssim_\epsilon \left(\frac{C_S C_E^{\frac{r}{s}}}{sC_m}(r-s)\right)^{\frac{1}{\frac{r}{s}+1}} n^{\frac{2}{\frac{r}{s}+1}} \tag{23}$$

7 Results

In this section we report the main results of the paper. These results are estimates on the value of the constant C_E on sets satisfying some given properties. In particular, the bounds are given by the Hausdorff measure, and the upper and lower Minkowski contents of the set E.

As explained before, bounding this constant allows one to make the statement of theorem 4 more precise.
Notation: in what follows we will often omit the reference to the set E. For example, we will write N_δ in lieu of $N_\delta(E)$

We are now ready to state our results. We will start with a bound in the case the Hausdorff measure of E can be computed.

Proposition 4 (Lower bound for C_E) *Suppose $dim_H(E) = \overline{dim}_B(E) = s$. Then, For any $\epsilon > 0$ There exist $\Delta > 0$ such that $\forall \delta, 0 < \delta < \Delta$ we have $N_\delta > (V - \epsilon)\delta^{-s}$.*
Where we defined $V := H^s(E) \geq H^s_\delta(E)$.

Proof:
$H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$ and $H^s_\delta(E)$ is non decreasing as δ decreases so, given ϵ, there exists Δ such that $\forall\delta$, $0<\delta<\Delta$ we have $V - H^s_\delta(E) < \epsilon \Rightarrow$.

$$H^s_\delta(E) > V - \epsilon \tag{24}$$

By definition, we have

$$H^s_\delta(E) \leq \sum_i |U_i|^s \text{ for any } \{U\}_i \text{ that is a } \delta\text{-cover of E}$$

In particular, we may choose $\{U\}_i$ to be a δ-cover with minimal N_δ (which we know must be finite, as E is bounded). Then:

$$H^s_\delta(E) \leq N_\delta \delta^s$$

Combining this with 24 we get:

$$N_\delta \delta^s > V - \epsilon \Rightarrow N_\delta > (V-\epsilon)\delta^{-s} \tag{25}$$

This issue with this theorem is that computing the Hausdorff measure of a set is usually difficult. Furthermore, if we are assuming $dim_H(E) = \overline{dim}_B(E) = s$ we also have $\underline{dim}_B(E) = s$ too, so we may as well try to get a bound out of that. There are two propertes of the LMD that make it a more appealing starting point for a lower bound for N_δ. The first reasonn is that the LDM is usually easier to compute that the Hausdorff dimension. The second reason is that the LDM is directly related to N_δ, while the Hausdorff measure takes into account countable δ-covers as well, so one cannot expect it to directly characterize N_delta.

It turns out that indeed we can use the LDM and its related formalism to bound to N_δ, as explained in the next theorem:

Theorem 5 *For Δ sufficiently small we have:*

$$N_\delta \gtrsim_{s,d} \delta^{-s} \cdot (\underline{M}^s - \epsilon)$$

Where the constant implicit in $\gtrsim_{s,d}$ is equal to $(2^s \cdot \Omega(d))^{-1}$

Proof: Use the right part of (11):

$$\mu_d(E\,(\delta)) \leq N_\delta \cdot \Omega(d) \cdot (2\delta)^n$$

Multiply both sides by $(2\delta)^{s-d}$ and obtain:

$$\mu_d(E\,(\delta)) \cdot (2\delta)^{s-d} \leq N_\delta \cdot \Omega(d) \cdot (2\delta)^d \cdot (2\delta)^{s-d}$$

$$\mu_d(E\,(\delta)) \cdot (2\delta)^{s-d} \leq N_\delta \cdot (2\delta)^s \cdot \Omega(d)$$

Once again, recognize the LHS to be as in the definition of $A(\delta)$. Apply the least upper bound proposition: for Δ sufficiently small:

$$(\underline{M}^s - \epsilon) \leq N_\delta \cdot \delta^s \cdot 2^s \cdot \Omega(d)$$

Rearranging:

$$\begin{aligned} N_\delta &\geq \delta^{-s} \cdot (\underline{M}^s - \epsilon) \cdot \frac{1}{2^s \Omega(d)} \\ N_\delta &\gtrsim_{s,d} \delta^{-s} \cdot (\underline{M}^s - \epsilon) \end{aligned} \tag{26}$$

We also provide a converse theorem:

Theorem 6 *For Δ small enough we have*

$$N_\delta \lesssim_d \delta^{-s} \cdot (\overline{M}^s(S) + \epsilon)$$

Where $\overline{M}^s(S)$ is the s-dimensional upper Minkowski content of S, and where the constant implicit in $\lesssim_d$ is equal to $\frac{2^d}{\Omega(d)}$.

Proof:
We restate equations (9) and (11) for reference:

$$N_{2\delta}(S) \leq P_\delta(S) \tag{27}$$

$$P_\delta(S) \cdot \Omega(d) \cdot \delta^d \leq \mu_d(S(\delta)) \tag{28}$$

Start from (11) with δ replaced by $\frac{\delta}{2}$:

$$P_{\frac{\delta}{2}}(S) \cdot \Omega(d) \left(\frac{\delta}{2}\right)^d \leq \mu_d\left(S\left(\frac{\delta}{2}\right)\right)$$

Use (9) and obtain:

$$N_\delta(S) \cdot \Omega(d) \left(\frac{\delta}{2}\right)^d \leq \mu_d\left(S\left(\frac{\delta}{2}\right)\right)$$

Multiply both sides by $\left(\frac{\delta}{2}\right)^{s-d} = \delta^{s-d}$:

$$N_\delta(S) \cdot \Omega(d)\delta^d \cdot \delta^{s-d} \cdot 2^{-d} \leq \mu_d\left(S\left(\frac{\delta}{2}\right)\right) \cdot \delta^{s-d}$$

Simplifying, and recognizing that the RHS is just like in the definition of Minkowski content :

$$2^{-d} \cdot \Omega(d) N_\delta(S) \cdot \delta^s \leq \overline{M}^s(S) + \epsilon$$

Finally, solving for $N_\delta(S)$:

$$\begin{aligned} N_\delta(S) &\leq \delta^{-s} \cdot \left(\overline{M}^s(S) + \epsilon\right) \cdot \frac{2^d}{\Omega(d)} \\ N_\delta(S) &\lesssim_d \delta^{-s} \cdot \left(\overline{M}^s(S) + \epsilon\right) \end{aligned} \tag{29}$$

8 Conclusions

In this paper we reviewed the theory of Minkowski and Hausdorff dimension, providing several different definitions and showing their equality and different use cases. After summarizing some results from [1], we showed how one can find some bounds for the constants involved in the theorems using the Minkowski content of the set under examination. The procedure used has a major shortcoming, in that the bounds only hold up to a factor of ϵ in the exponent. In order to ensure control over this factor, one has to introduce another multiplicative constant, depending on ϵ. This then leaves the overall constant undetermined, and further study is required to determine whether one can find a bound on the new constant.

9 Appendix A

Here we report the proofs of the theorems in [1] that we mentioned above. The proofs are almost identical to the paper's, with some details filled in.
Proof of Lemma 2.
Cover E with $N_\delta(E)$ balls of radius $\delta \Rightarrow N_\delta(E) \leq C_E^{-1}\delta^{-s}$.
We can then partition E as follows: let $B_1, ..., B_{N_\delta}$ be the balls mentioned above, and define $E_i' := E \cap B_i$.
To ensure that the sets are disjoint, define

$$E_i := \bigcap_{j=i+1}^{n} E_i - E_j$$

Where "$-$" indicates set difference. This guarantees that the set E_i form a partition. The fact that this was a minimal cover guarantees that none of the E_i is empty.

Note that if two points p and p' are contained in E_i for some i, they must also be in $|\{(p,p') : |p-p'| \leq \delta\}|$. Hence:

$$|\{(p,p') : |p-p'| \leq \delta\}| \geq \sum_i |E_i \cap P_n|^2$$

Where the power of two in the RHS comes from the fact that we are considering pairs of points. Apply the Cauchy-Schwartz inequality:

$$\sum_i |E_i \cap P_n|^2 = 1 \cdot \sum_i |E_i \cap P_n|^2 = \sum_i \left(N_\delta(E)^{-1/2}\right)^2 \cdot \sum_i |E_i \cap P_n|^2$$

$$\geq \left(\sum_i N_\delta(E)^{-1/2} \cdot |E_i \cap P_n|\right)^2 = N_\delta(E)^{-1}\left(\sum_i |E_i \cap P_n|\right)^2$$

Now one has to notice that $\sum_i |E_i \cap P_n| = |P_n| = n$ to conclude:

$$|\{(p,p') : |p-p'| \leq \delta\}| \geq C_E^{-1} \delta^s n^2$$

Proof of Thm. 3.
Recall the form of the energy integral:

$$I_r(P_n) = n^{-2} \sum_{p \neq p'} |p-p'|^{-r} \tag{30}$$

We want to use what we know about the upper Minkowski dimension of E to bound this expression. The only tools we have that relates the UMD of a set to its time series is Lemma 2, so we are going to re-express (30) in a form that will allow us to apply Lemma 2.

Start from noticing that:

$$|p-p'|^{-r} = r \int_0^\infty \mathbf{1}_{[0,\infty)}(\delta - |p-p'|)\delta^{-r-1}\, d\delta$$

Hence:

$$\begin{aligned} I_r(P_n) &= n^{-2} \sum_{p \neq p'} |p-p'|^{-r} \\ &= rn^{-2} \sum_{p \neq p'} \int_0^\infty \mathbf{1}_{[0,\infty)}(\delta - |p-p'|)\delta^{-r-1}\, d\delta \end{aligned} \tag{31}$$

The sum above is finite, so we can swap the sum and the integral:

$$I_r(P_n) = rn^{-2} \int_0^\infty \left(\sum_{p \neq p'} \mathbf{1}_{[0,\infty)}(\delta - |p-p'|) \right) \delta^{-r-1}\, d\delta \tag{32}$$

Note that the sum in parenthesis just counts the number of point pairs that are less than δ apart, excluding the n pairs with $p = p'$:

$$I_r(P_n) = rn^{-2} \int_0^\infty \left(|\{(p,p') : |p-p'| \leq \delta\}| - n \right) \delta^{-r-1}\, d\delta \tag{33}$$

Note that the expression above is exactly what we have in Lemma 2, which we are now going to apply: if δ, $\left(C_E^{-1} n\right)^{-1/s}$ we are only guaranteed the existence of the n pairs $p = p'$. and thus:

$$|\{(p,p') : |p-p'| \leq \delta\}| - n \geq 0$$

If, instead, $\delta \geq \left(C_E^{-1} n\right)^{-1/s}$ Lemma 2 tells us more:

$$|\{(p,p') : |p-p'| \leq \delta\}| - n \geq C_E^{-1} \delta^s n^2 - n \tag{34}$$

Applying (34) to (33) we get:

$$
\begin{aligned}
I_r(P_n) &\geq rn^{-2} \int_{\left(C_E^{-1} n\right)^{-1/s}}^{\infty} \left(C_E^{-1} \delta^s n^2 - n\right) \delta^{-r-1} \, d\delta \\
&\geq rC_E^{-1} \int_{\left(C_E^{-1} n\right)^{-1/s}}^{\infty} \delta^{s-r-1} d\delta \\
&\quad - rn^{-1} \int_{\left(C_E^{-1} n\right)^{-1/s}}^{\infty} \delta^{-r-1} d\delta
\end{aligned} \tag{35}
$$

These integrals are finite, because we are assuming that $r > s$, and can be evaluated to:

$$
= \frac{r}{s-r} \left(C_E^{-1}\right)^{\frac{r}{s}} n^{\frac{r}{s}-1} - \frac{rC_E^{-1}}{r-s} + \frac{1}{n} \tag{36}
$$

References

[1] Iosevich Chio, Fleischman. Fractal dimension, approximation and neural networks. *Unpublished*, 2021.

[2] K. J. Falconer. *Fractal geometry: Mathematical foundations and applications*. Wiley, 2014.

[3] Felix Hausdorff. Dimension und äußeres maß. *Mathematische Annalen*, 79, 1918.

[4] Benoit Mandelbrot. How long is the coast of britain? statistical self-similarity and fractional dimension. *Science*, 156, 1967.

[5] Benoit B. Mandelbrot. *Les objets fractals : forme, hasard et dimension / Benoit Mandelbrot*. Flammarion Paris, 1975.

[6] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*. Cambridge University Press, 1999.

[7] Terence Tao. 245c, notes 5: Hausdorff dimension (optional), May 2009.