Series Rearrangements Using Density Functions

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1 Introduction

The Riemann rearrangement theorem was proved in 1853 and published posthumously in 1866. When it was presented to me in my undergraduate real analysis course, I became interested in how to reorder sums to achieve precise values using constructive methods, ideally ones that I could personally compute. The Riemman Hypothesis states that for any conditionally convergent series and any number in the extended real line, there exists some rearrangement so that the series converges to the number [1]. My initial "investigation" primarily focused on the properties of the alternating harmonic series. One of the nicer properties is that its sum can be controlled purely by changing the ratio of positive to negative numbers. Once I started playing around with other series I made the realization that has been made many times before me; this nice property doesn't apply to many other series.

I then began examining the alternating $1/\sqrt{n}$ series, which was far touchier. Even slight changes in density immediately caused divergence. Ultimately, I leveraged its monotonicity to use it much in the way the Cauchey-MacLaurin integral test [2] does; I treated it as an integral and used that to get a feel for how quickly its sums moved towards zero. Using this approach I found that any change in the ratio of positive to negative numbers away from 1:1 would cause divergence; I needed a more precise notion of density than a ratio. By using smaller changes in density, I found that by treating any monotonically decreasing sequence as an integral, rather than a series, I could use this to combine it with another sequence that had similarly nice features to force the difference of the two series to converge. Much to my surprise I found while working through my proof that the series converges linearly with respect to one of the variables I had built into the combination; the proof shows that this is not a profound result however.

Considering how old both the techniques of the Riemann rearrangement theorem and the integral test are, it seems unlikely that I am the first to stumble upon such a technique. I have not been able to find other techniques along these lines though. However, I did stumble upon it almost completely independently of whatever previous work there was, and I hope that at the least my proof might have some interest to the reader as another perspective upon the problem.

2 The Difference of Two Series

We will begin by laying down the foundation of this paper; that we can force the difference of two series defined by functions that monotonically decrease to zero to converge by carefully choosing the "density" of the two series with respect to each other. Density is put in quotations as what we will be using is slightly more nuanced than a ratio but it amounts to a very similar thing; it will be performed using $p(x, y)$ whereas normal density would just be $k * x$. The y variable for p will be used later on, but not in this section. Throughout this paper we will be using the relevant definitions and basic theorems from Apostol's book on mathematical Analysis [3].

Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ and $g : \mathbf{R}_+ \to \mathbf{R}_+$ where $\mathbf{R}$ is understood to be $\{x|x >= 0\}$. We require that f, g have the following properties:

- f, g are monotone decreasing.
- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$
- $\int_0^x f(t)dt$ and $\int_0^x g(t)dt$ exist and are finite $\forall x \geq 0$
- $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} g(n) = \infty$

We will define $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$.

Remark. *F, G are strictly increasing and $F(\mathbf{R}_+) = G(\mathbf{R}_+) = \mathbf{R}_+$, so they both have inverse functions defined on $\mathbf{R}_+$.*

Remark. *By $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} g(n) = \infty$, we have*

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} G(x) = \infty$$

Let us define $S \subseteq \mathbf{R}_+ \times \mathbf{R}$ as $S = \{(x, y)|F(x)+y \geq 0\}$. Let $p(x, y) : S \to \mathbf{R}_+$ be $p(x, y) = G^{-1}(F(x) + y)$.

Lemma 2.1. *$p(x, y)$ is well defined, and p is monotone increasing with respect to both x and y, and $\lim_{x\to\infty} p(x, y) = \infty \ \forall y \in \mathbf{R}$.*

Proof. Note by $(x, y) \in S$ $F(x) + y \geq 0 \Rightarrow F(x) + y \in \mathbf{R}_+$. Thus, $p(x, y) = G^{-1}(F(x) + y)$ is well defined. Note F and G are differentiable by the Fundamental Theorem of Calculus. Furthermore, clearly by $\lim_{x\to\infty} f(x) = 0$, $\sum_{n=1}^{\infty} f(n) = \infty$, and f monotone decreasing, we can conclude

$$f(x) > 0 \,\forall\, x \in \mathbf{R}_+$$

Similarly for g. Then we find that $F'(x) > 0$ and $G'(x) > 0$. Then we may conclude that $F^{-1}(x)$ is differentiable, and that $(F^{-1})'(x) > 0$. Similarly, $G^{-1}(x)$ is differentiable, and $(G^{-1})'(x) > 0$.

Then we may conclude that $p(x, y)$ is differentiable with respect to x and y, and by the chain rule

$$\frac{\partial p}{\partial x}(x, y) = (G^{-1}(F(x) + y))' = G^{-1\prime}(F(x) + y) * F'(x) > 0$$

Thus p(x,y) is strictly increasing with respect to x.

Note that $lim_{x\to\infty} G(x) = \infty$ and G strictly increasing means that $lim_{x\to\infty} G^{-1}(x) = \infty$. By $lim_{x\to\infty} F(x) + y = \infty$ we may conclude that $\lim_{x\to\infty} p(x, y) = \infty$. □

Theorem 2.2. $\lim_{x\to\infty} \sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,0)\rfloor} g(n)$ *exists and is finite.*

Proof. We will show that the series is Cauchey by using the fact that the series differ from their respective integral by no more than the first term of the series.

Let $h(x) = \sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,0)\rfloor} g(n)$ where $h : \mathbf{R}_+ \to \mathbf{R}$. Choose $m > k > 0$ where $p(m, 0) > p(k, 0) + 2$. Such m exist by Lemma 1.1

Then note

$$h(m) - h(k) = \sum_{n=\lfloor k\rfloor}^{n=\lfloor m\rfloor} f(n) - \sum_{n=\lfloor p(k,0)\rfloor}^{n=\lfloor p(m,0)\rfloor} g(n) - f(\lfloor k\rfloor) + g(\lfloor p(k, 0)\rfloor)$$

Note then that by $f(x) > 0$ and $g(x) > 0$

$$h(m) - h(k) \le \int_{k-1}^{m+1} f(t)dt - \int_{p(k,0)+1}^{p(m,0)-1} g(t)dt + g(\lfloor p(k, 0)\rfloor)$$

We can expand this and find the right side equals

$$\int_{k}^{m} f(t)dt + \int_{k-1}^{k} f(t)dt + \int_{m}^{m+1} f(t)dt - \int_{p(k,0)+1}^{p(m,0)-1} g(t)dt + g(\lfloor p(k, 0)\rfloor)$$

By f monotone decreasing and $k < m$, we can conclude this is less than or equal to

$$\int_{k}^{m} f(t)dt + \int_{k-1}^{k} f(k-1)dt + \int_{m}^{m+1} f(k-1)dt - \int_{p(k,0)+1}^{p(m,0)-1} g(t)dt + g(\lfloor p(k, 0)\rfloor)$$

which is equal to

$$\int_{k}^{m} f(t)dt + 2f(k-1) - \int_{p(k,0)}^{p(m,0)} g(t)dt + \int_{p(k,0)}^{p(k,0)+1} g(t)dt + \int_{p(m,0)-1}^{p(m,0)} g(t)dt + g(\lfloor p(k, 0)\rfloor)$$

Once again, we may use that $g(x)$ is monotone decreasing to bound above using that the previous equation is

$$\le \int_{k}^{m} f(t)dt - \int_{p(k,0)}^{p(m,0)} g(t)dt + 2f(k-1) + 3g(p(k, 0) - 1)$$

We then apply the definitions of G, F, p to say that

$$h(m) - h(k) \le F(m) - F(k) - G(G^{-1}(F(m))) + G(G^{-1}(F(k))) + 2f(k-1) + 3g(p(k, 0) - 1)$$

therefore,

$$h(m) - h(k) \le 2f(k - 1) + 3g(p(k, 0) - 1)$$

We will now work on bounding it from below. Note that

$$h(m) - h(k) \geq \int_{k+1}^{m-1} f(t)dt - \int_{p(k,0)-1}^{p(m,0)+1} g(t)dt - f(\lfloor k \rfloor)$$

Through similar logic, we quickly can conclude that

$$h(m) - h(k) \geq \int_{k}^{m} f(t)dt - 2f(k-1) - \int_{p(k,0)}^{p(m,0)} g(t)dt - 2g(p(k,0)-1) - f(k-1)$$

and so

$$h(m) - h(k) \geq -3f(k-1) - 2g(p(k,0)-1)$$

Then we conclude that $\mid h(m) - h(k) \mid \leq 3f(k-1) + 3g(p(k,0)-1)$. Note that $\lim_{k\to\infty} p(k,0) = \infty$, and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. Then $\exists M$ such that for $k > M$, $f(k-1) < \frac{\epsilon}{12}$ and $g(p(k,0)-1) < \frac{\epsilon}{12}$. Then

$$\mid h(m) - h(k) \mid \leq \frac{\epsilon}{2}$$

Then for m_1 and m_2 such that they fulfill the requirements on m for $k > M$, we have

$$\mid h(m_1) - h(m_2) \mid \leq \mid h(m_1) - h(k) \mid + \mid h(m_2) - h(k) \mid \leq \epsilon$$

Then $h(x)$ is Cauchy, so it converges. □

This proof means that for any two functions, we now have a clear way to order a convergent infinite series composed of terms from the two functions. We will see in the next section how using $p(x,y)$ instead of fixing $y = 0$ lets us obtain more re-orderings and a useful tool to compute these sums easily as long as we have some λ where we know the sum using $p(x,\lambda)$.

3 Linearity with respect to density

We will begin by defining a function $d : \mathbf{R} \to \mathbf{R}$ as

$$d(y) = \lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,y) \rfloor} g(n)$$

By Theorem 2.2 d(0) is well defined and finite. We may now easily state and prove our next theorem, which is that d is linear with slope -1.

Lemma 3.1. *$d(y)$ exists and is finite for all finite y.*

Proof. Let us define $l(x) : \mathbf{R} \to \mathbf{R}$ as $l(x) = \{ {}^{2y-2yx \,:\, x\in[0,1]}_{0 \,:\, x\in(1,\infty)} \}$. Note that $f + l$ fulfills all of the necessary requirements. Let us call $L(x) = \int_0^\infty f(x) + l(x)dx$. Finally let $d'(y) = \lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} (f+l)(n) - \sum_{n=1}^{\lfloor p'(x,y) \rfloor} g(n)$ for p' appropriate. Now note that $F(x) - L(x) = y$ for $x \geq 1$. Note that $d'(0)$ exists and $p'(x,0) = p(x,y)$, so $d'(0) = d(y)$ and thus we are done. □

Theorem 3.2. $d(y) = d(0) - y$

Proof. This proof will largely hinge upon the fact that we have defined $p(x, y) = G^{-1}(F(x) + y)$, and so $F(x) - G(p(x, y)) = F(x) - G(G^{-1}(F(x) + y)) = F(x) - F(x) - y = -y$. Everything else amounts to managing the error terms for estimating series with integrals. If $y > 0$, Note

$$d(y) - d(0) = \lim_{x\to\infty} \sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,y)\rfloor} g(n) - \lim_{x\to\infty} \sum_{n=1}^{\lfloor x\rfloor} f(n) + \sum_{n=1}^{\lfloor p(x,0)\rfloor} g(n)$$

If we simplify, we may use Lemma 1.1 and Lemma 3.1 to conclude

$$d(y) - d(0) = \lim_{x\to\infty} - \sum_{\lfloor 1+p(x,0)\rfloor}^{\lfloor p(x,y)\rfloor} g(n)$$

Then we apply $g(n)$ monotone decreasing to change into integrals and see that

$$d(y) - d(0) \leq - \lim_{x\to\infty} \int_{p(x,0)+1}^{p(x,y)-1} g(t)dt$$

Once again by $g(n)$ monotone decreasing we may add back on the edges, canceling them out with the constant $g(p(x, 0))$, and convert the main portion of the integral into

$$d(y) - d(0) \leq - \lim_{x\to\infty} G(p(x, y)) - G(p(x, 0)) + 2g(p(x, 0))$$

But by using the definition of $p(x)$, we have

$$d(y) - d(0) \leq - \lim_{x\to\infty} G(G^{-1}(F(x) + y)) - G(G^{-1}(F(x))) + 2g(p(x, 0))$$

Then we find that

$$d(y) - d(0) \leq -y - \lim_{x\to\infty} 2g(p(x, 0)) = -y$$

Now we will bound by the other side and say

$$d(y) - d(0) \geq - \lim_{x\to\infty} \int_{p(x,0)-1}^{p(x,y)+1} g(t)dt$$

so then using the previous steps, we have

$$d(y) - d(0) \geq - \lim_{x\to\infty} G(G^{-1}(F(x) + y)) - G(G^{-1}(F(x))) - 2g(p(x, 0) - 1) = -y$$

Then we have $-y \leq d(y) - d(0) \leq -y$, so $d(y) = d(0) - y$.

If $y = 0$, we immediately see that $d(y) = d(0) = d(0) - 0$.

If $y > 0$, we have that

$$d(y) - d(0) = \lim_{x\to\infty} \sum_{\lfloor p(x,y)+1 \rfloor}^{\lfloor p(x,0) \rfloor} g(n)$$

We can then see, applying the previously used logic, that

$$d(y) - d(0) \leq \lim_{x\to\infty} \int_{p(x,y)-1}^{p(x,0)+1}$$

$$d(y) - d(0) \leq \lim_{x\to\infty} G(p(x,0)) - G(p(x,y)) + 2g(p(x,y))$$

$$d(y) - d(0) \leq -y$$

Additionally, bounding in the other direction, we have

$$d(y) - d(0) \geq \lim_{x\to\infty} \int_{p(x,y)+1}^{p(x,0)-1}$$

$$d(y) - d(0) \geq \lim_{x\to\infty} G(p(x,0)) - G(p(x,y)) - 2g(p(x,y))$$

$$d(y) - d(0) \geq -y$$

Then we once again have that $-y \leq d(y) - d(0) \leq -y$, and so we are done with all three cases. □

Because $d(c) = b - c$ for some b, if we can calculate $d(c)$ for some value of c we are instantly able to calculate it for all values of c. If it is infeasible to calculate $d(c)$ precisely for any value, it becomes important to consider how many terms of the series one would have to calculate to obtain an accurate estimate. The following lemmas describe the rate at which the series converges do $d(c)$. I by no means claim that these are the best terms one can obtain; in fact I suspect that with use of better estimation tools such as Abel's identity as written in Apostol's *Introduction to Analytic Number Theory* [4] or some other similar estimate, better terms might be able to be obtained. Ultimately though this resolves down to that the series converges at least as quickly as the individual terms converge to zero.

Lemma 3.3. $d(C) - (\sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,C) \rfloor} g(n)) \leq (f(k-1) - f(\lfloor k \rfloor)) + g(\lfloor p(k,C) \rfloor)$

Proof. Let $h(x) = \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,C) \rfloor} g(n)$ where $h : \mathbf{R}_+ \to \mathbf{R}$. Let $m > k > 0$ so $p(m,C) > p(k,C)$. Note $h(\infty) = d(C)$

Then note

$$h(\infty) - h(k) = \lim_{m\to\infty} \sum_{n=\lfloor k \rfloor}^{m} f(n) - \sum_{n=\lfloor p(k,C) \rfloor}^{\lfloor p(m,C) \rfloor} g(n) - f(\lfloor k \rfloor) + g(\lfloor p(k,C) \rfloor)$$

Note then that by $f(x) > 0$ and $g(x) > 0$ and both monotone decreasing,

$$h(m) - h(k) \leq \lim_{m\to\infty} \int_{k-1}^{m} f(t)dt - \int_{p(k,C)}^{p(m,C)+1} g(t)dt - f(\lfloor k \rfloor) + g(\lfloor p(k,C) \rfloor)$$

We can expand this and find the right side equals

$$\lim_{m\to\infty} \int_{k}^{m} f(t)dt + \int_{k-1}^{k} f(t)dt - \int_{p(k,C)}^{p(m,C)} g(t)dt - \int_{p(m,C)}^{p(m,C+1)} g(t)dt - f(\lfloor k \rfloor) + g(\lfloor p(k,C) \rfloor)$$

By f monotone decreasing and $k < m$, we can conclude this is less than or equal to

$$\lim_{m\to\infty} \int_{k}^{m} f(t)dt + f(k-1) - \int_{p(k,C)}^{p(m,C)} g(t)dt - g(p(m,C)+1) - f(\lfloor k \rfloor) + g(\lfloor p(k,C) \rfloor)$$

which is equal to

$$\lim_{m\to\infty} \int_{k}^{m} f(t)dt + f(k-1) - \int_{p(k,C)}^{p(m,C)} g(t)dt - f(\lfloor k \rfloor) + g(\lfloor p(k,C) \rfloor)$$

We then apply the definitions of G, F, p rewrite this as

$$\lim_{m\to\infty} F(m) - F(k) - G(G^{-1}(F(m)+C)) + G(G^{-1}(F(k)+C)) + (f(k-1) - f(\lfloor k \rfloor)) + g(\lfloor p(k,C) \rfloor)$$

The first terms cancel out leaving us with,

$$h(\infty) - h(k) \leq (f(k-1) - f(\lfloor k \rfloor)) + g(\lfloor p(k,C) \rfloor)$$

$\square$

This lemma follows in exactly way the preceding lemma did, and so will be left unproven for the sake of space and the reader's time.

Lemma 3.4. *$d(C) - h(k) \geq -(g(p(k,C)-1) - g(\lfloor p(k,C) \rfloor)) - f(\lfloor k \rfloor)$*

The question arises whether the function $p(x, y)$ is at some level unique. In other words, does the series converge if and only if the "density" follows closely to p? It turns out that it does, and the next two corollaries provide the test for convergence using p.

Consider $\sum_{n=1}^{\infty} a_n$ where there exist some f, g, h such that f and g fulfill our previous requirements, and $h(x)$ is monotone increasing, and $\lim_{x\to\infty} h(x) = \infty$ such that $\sum_{n=1}^{\lfloor x \rfloor + \lfloor h(x) \rfloor} a_n = \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor h(x) \rfloor} g(n)$.

Corollary 3.4.1. *If $\lim_{x\to\infty} G(h(x)) - F(x) = C \in \mathbf{R}_\infty$, then $\sum_{n=1}^{\infty} a_n = d(C)$*

Proof. Note before we begin that $G(h(x)) - F(x)$ is effectively $p^{-1}(x, h(x))$, and we are testing whether that inverse converges.

Let $C \in \mathbf{R}$. Consider that we have $\lim_{x\to\infty} G(h(x)) - (F(x)+C) = 0$. Note that by definition of $G(x)$, $G(0) = 0$. Then by applying $G^{-1}$to both sides of the equation we get $\lim_{x\to\infty} h(x) - G^{-1}(F(x)+C) = \lim_{x\to\infty} h(x) - p(x,C) = 0$. Consider

$$\sum_{n=1}^{\infty} a_n - d(C) = 0$$

is equivalent to

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor h(x)\rfloor} g(n) - \sum_{n=1}^{\lfloor p(x,C)\rfloor} g(n) = 0$$

Note that $g(n)$ is monotone decreasing, and that

$$\exists M s.t. \forall x > M, |h(x) - p(x,C)| < 2$$

Then note that $||h(x)\rfloor - \lfloor p(x,C)\rfloor| \leq 2$ Then $\forall x > M$ we have

$$|\sum_{n=1}^{\lfloor h(x)\rfloor} g(n) - \sum_{n=1}^{\lfloor p(x,C)\rfloor} g(n)| \leq 2g(\min(\lfloor h(M)\rfloor, \lfloor p(M,C)\rfloor)$$

Note that by $h(x) \to \infty$, $p(x,C) \to \infty$, and $g(x) \to 0$, we have that

$$\lim_{M\to\infty} 2g(\min(\lfloor h(M)\rfloor, \lfloor p(M,C)\rfloor) = 0$$

Then we have that

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor h(x)\rfloor} g(n) - \sum_{n=1}^{\lfloor p(x,C)\rfloor} g(n) = 0$$

and thus we are done with the finite case.

Note if $C = \infty$, then we immediately have for each c finite, $\sum_{n=1}^{\infty} a_n \leq d(c)$. Then by $d(c) = d(0) - c$, $\sum_{n=1}^{\infty} a_n = -\infty$. Similarly we get that if $C = -\infty$, $\sum_{n=1}^{\infty} a_n = \infty$. □

Corollary 3.4.2. *If* $\lim_{x\to\infty} G(h(x)) - F(x)$ *does not converge, then neither does* $\sum_{n=1}^{\infty} a_n$

Proof. Let $B = \limsup G(h(x) - F(x)$. Let $C = \liminf G(h(x)) - F(x)$. Then there exist some $b_{n1\geq n} \to \infty$ and $c_{n1\geq n} \to \infty$ such that $G(h(b_n)) - F(b_n) \to B$ and $G(h(c_n)) - F(c_n) \to C$. By Corollary 2.2.2,

$$\lim_{m\to\infty} \sum_{n=1}^{\lfloor b_m\rfloor} f(n) - \sum_{n=1}^{\lfloor h(b_m)\rfloor} g(n) = d(B)$$

We also have that

$$\lim_{m\to\infty} \sum_{n=1}^{\lfloor c_m\rfloor} f(n) - \sum_{n=1}^{\lfloor h(c_m)\rfloor} g(n) = d(C)$$

By $\lim_{x\to\infty} G(h(x)) - F(x)$ not convergent, $B \neq C$ and by the properties of $d(y)$, $d(B) \neq d(C)$, and thus $\sum_{n=1}^{\infty} a_n$ does not converge. □

To illustrate the use of these theorems and corollaries, we will recompute the fact that if c is the ratio of positives to negatives in the alternating harmonic series converges to $ln(2\sqrt{1/c})$.

Corollary 3.4.3. *Let a_n be a series such that for a function $h(x) = cx$ for some c in $\mathbf{R}_+$ $\sum_{n=1}^{m} a_n = \sum_{n=1}^{m} \frac{1}{2n-1} - \sum_{n=1}^{h(m)} \frac{1}{2n}$ then*

$$\sum_{n=1}^{m} a_n = \log(2\sqrt{\frac{1}{c}})$$

Proof. First we need to compute $F(x)$ and $G(x)$ and use these to compute the C_0 for the alternating harmonic case, and the C_1 for this rearrangement.

Let's define $f(x) = 1/(2x-1)$ where $x \geq 1$ and 1 otherwise. Then $F(x) = ln(2x-1)/2 + Y$. Similarly, $g(x) = 1/(2x)$ and $G(x) = ln(2x)/2 + Z$ where Y and Z are constants.

Note for the alternating harmonic case, $h(x) = x$, so we take

$$\lim_{x\to\infty} G(h(x)) - F(x) = \lim_{x\to\infty} ln(2x)/2 + Z - ln(2x-1)/2 - Y = ln(1)/2 + (Z-Y) = Z - Y = C_0$$

For our $h(x) = cx$ we have

$$G(c) - F(x) = \lim_{x\to\infty} ln(2cx)/2 + Z - ln(2x-1)/2 - Y$$

this is equal to

$$\lim_{x\to\infty} ln(2cx/(2x-1))/2 + (Z-Y) = ln(c)/2 + Z - Y = C_1$$

Since we know that the alternating harmonic series sums to $ln(2)$ we may use the linearity of the sum with respect to C to say that

$$sum_{n=1}^{\infty} a_n = ln(2) + C_0 - C_1 = ln(2) + (Z-Y) - ln(c)/2 - (Z-Y) = ln(2 * \sqrt{1/c})$$

$\square$

Consider once again our standard series $\sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,y) \rfloor} g(n)$.

Let us define the series $\{x_n\}_{1\leq n}$ as $x_n = g(n)$. Let us call a bijection $\sigma(n) : \mathbf{N} \Rightarrow \mathbf{N}$ a rearrangement. Now we will say that $\{a_n\}_{1\leq n}$ is a rearrangement of x_n if there is some rearrangement σ_n such that $a_n = x_{\sigma(n)} \forall n \in \mathbf{N}$.

We will now state the following theorem about rearrangements of $x_n = g(n)$.

Theorem 3.5. $\lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,y) \rfloor} a_n$ *converges for some y if and only if* $\lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} g(n) - \sum_{n=1}^{\lfloor x \rfloor} a_n$ *converges.*

Proof. Assume that

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x,y) \rfloor} a_n$$

converges for some y. Note that for the same y, we have the convergent series

$$\lim_{x\to\infty}\sum_{n=1}^{\lfloor x\rfloor} f(n)-\sum_{n=1}^{\lfloor p(x,y)\rfloor} g(n)$$

By the difference of limits is the limit of differences, we must have that

$$\lim_{x\to\infty}\sum_{n=1}^{\lfloor p(x,y)\rfloor} g(n)-\sum_{n=1}^{\lfloor p(x,y)\rfloor} a_n$$

converges. Note that $p(x,y)$ is continuous, strictly increasing with respect to x, and that $\lim_{x\to\infty} p(x,y)=\infty$. Then we may replace $m=p(x,y)$ and note that the following series converges, and thus we are done.

$$\lim_{x\to\infty}\sum_{n=1}^{\lfloor x\rfloor} g(n)-\sum_{n=1}^{\lfloor x\rfloor} a_n$$

We will proceed in the reverse direction by assuming that the above series converges. Choose an arbitrary y. Then note by the properties of $p(x,y)$ we may substitute $p(x,y)$ for m. Then by applying the difference of limits is the limit of the differences we can derive our goal convergent limit of

$$\lim_{x\to\infty}\sum_{n=1}^{\lfloor x\rfloor} f(n)-\sum_{n=1}^{\lfloor p(x,y)\rfloor} a_n$$

□

Corollary 3.5.1. *If $\{b_n\}_{1\le n}$ is a rearrangement of $y_n=f(n)$, then $\lim_{x\to\infty}\sum_{n=1}^{\lfloor x\rfloor} b_n-\sum_{n=1}^{\lfloor p(x,y)\rfloor} g(n)$ converges for some y if and only if $\lim_{x\to\infty}\sum_{n=1}^{\lfloor x\rfloor} f(n)-\sum_{n=1}^{\lfloor x\rfloor} b_n$ converges.*

The proof for this is the same as the proof for Theorem 2.3, and thus not included.

Remark. *The proof of Theorem 2.3 is quite straightforward, but it and Corollary 2.3.1 give us useful tools to apply our $p(x,y)$ to somewhat more exotic rearrangements.*

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