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Method of Stationary Phase and Applications

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1 Introduction

In short, the method of Stationary Phase concerns itself with obtaining bounds for oscillatory integrals of the following form

$$I(f,R) = \int_{\mathbb{R}^n} e^{iRf(x)}\ \psi(x)\ dx$$

where $\psi(x)$ is a bump function.

The method of stationary phase is of great importance in the field of Harmonic Analysis. One can find applications of this to other areas in Harmonic Analysis, Partial Differential Equations, Geometric Measure Theory, and Geometric Combinatorics to name a few. Different bounds are obtained depending on the conditions imposed on $f(x)$. In general varying conditions on the smoothness of f have yielded promising results. Throughout this paper, we will go through some of the most basic results on the subject while also providing some of the motivation for why these results are expected. Then we'll go through an application of stationary phase to getting a bound for the Fourier transform of the surface measure on a sphere.

2 Introduction to Stationary Phase

The simplest oscillatory integral one can consider is

$$I(x,R) = \int_a^b e^{iRx}\ dx$$

and a basic substitution and integral estimate tells us that

$$|I(x,R)| \leq \frac{2}{R}$$

Such an estimate is begging for a more general result. The first of these is the Vander-Corput lemma.

Lemma 2.1. *(Vander Corput) Suppose that* $f \in C^1(a,b)$, $f'(x) \geq 1$, *and* $f'(x)$ *is monotonic. Then*

$$|I(f,R)| = \left| \int_a^b e^{iRf(x)}\ dx \right| \leq \frac{2}{R}$$

Proof. We make use of the following trick

$$\begin{aligned}
|I(f,R)| &= \left| \int_a^b \frac{1}{iRf'(x)} \frac{d}{dx}\left(e^{iRf(x)}\right)\ dx \right| && (1)\\
&= \left| \left(\frac{e^{iRf(b)}}{iRf'(b)} - \frac{e^{iRf(a)}}{iRf'(a)} \right) - \int_a^b e^{iRf(x)} \frac{d}{dx}\left(\frac{1}{iRf'(x)} \right) dx \right| && (2)\\
&\leq \frac{1}{R}\left(\frac{1}{f'(b)} + \frac{1}{f'(a)} \right) + \left| \int_a^b e^{iRf(x)} \frac{d}{dx}\left(\frac{1}{iRf'(x)} \right) dx \right| && (3)
\end{aligned}$$

To deal with the last integral, we make use of the monotonicity of f and the fundamental theorem of calculus

$$\begin{aligned}
\left| \int_a^b e^{iRf(x)} \frac{d}{dx}\left(\frac{1}{iRf'(x)} \right)\ dx \right| &\leq \int_a^b \left| e^{iRf(x)} \frac{d}{dx}\left(\frac{1}{iRf'(x)} \right) \right| dx && (4)\\
&= \left| \int_a^b \frac{d}{dx}\left(\frac{1}{iRf'(x)} \right)\ dx \right| && (5)\\
&= \left| \left(\frac{1}{iRf'(b)} - \frac{1}{iRf'(a)} \right) \right| && (6)\\
&= \frac{1}{R}\left| \frac{1}{f'(b)} - \frac{1}{f'(a)} \right| && (7)
\end{aligned}$$

Putting these two estimate together, we obtain that

$$|I(f,R)| \leq \frac{1}{R}\left(\frac{1}{f'(b)} + \frac{1}{f'(a)} + \left| \frac{1}{f'(b)} - \frac{1}{f'(a)} \right| \right)$$

By considering cases $x \leq y$ and $x > y$ separately, it's easy to conclude that $x + y + |x - y| = 2\max(x, y)$. So finally

$$\frac{1}{f'(b)} + \frac{1}{f'(a)} + \left| \frac{1}{f'(b)} - \frac{1}{f'(a)} \right| = 2\max\left\{ \frac{1}{f'(a)}, \frac{1}{f'(b)} \right\} \leq 2$$

and $|I(f,R)| \leq \frac{2}{R}$

$\square$

Three observations are important in this proof. First, notice that even though we only assumed that $f \in C^1(a,b)$, we are still allowed to talk about

$$\frac{d}{dx}\left(\frac{1}{iRf'(x)}\right)$$

because the monotonicity of $f'(x)$ guarantees that this derivative exists almost everywhere. Hence, we're considering that integral as a Lebesgue integral.

Secondly, the condition that $f'(x) \geq 1$ is far stronger than necessary. We could just suppose that $f'(x) \geq \lambda$ where $\lambda > 0$ and make the appropiate changes to have a trivially more general result.

Lastly, the computation that preceded the theorem shows that the constant 2 in the bound of $\frac{2}{R}$ is the best that we can do in terms of constants.

This result can be extended by considering derivatives of higher powers. Before stating this, result, we should consider some motivation. Previously, our motivation for Vander-Corput was the estimate

$$\left|\int_a^b e^{iRx}\ dx\right| \leq \frac{2}{R}$$

Naturally, we'd like to look at integrals of the form

$$I(x^n, R) = \int_a^b e^{iRx^n}\ dx$$

In order to gain some intuition on what results we should expect, we recall the classic Fresnel Integral. I won't prove this result as I will prove a more general result in a moment; however, the Fresnel integral tell us that

$$\int_0^\infty e^{iRx^2}\ dx = \frac{1}{R^{1/2}}\int_0^\infty e^{ix^2}\ dx = \frac{1+i}{R^{1/2}}\sqrt{\frac{\pi}{8}}$$

By similar methods, we can also deduce that

$$\int_0^\infty e^{iRx^n}\ dx = \frac{C_n}{R^{1/n}}$$

where $C_n = \int_0^\infty e^{ix^n}\ dx$

This suggests conditions on $f(x)$ similar to that of x^n and bounds of the form

$$|I(f,R)| \leq \frac{C}{R^{1/n}}$$

Lemma 2.2. *(Vander Courput) Let $f(x) \in C^m(a,b)$. Suppose that $f^{(m)}(x) \geq 1$. Then*

$$|I(f,R)| \leq C_m R^{-1/m}$$

where C_m will depend solely on m

Proof. We proceed with induction. Suppose that the result holds for $m-1$. If $f^{(m-1)}(x) \neq 0$ on (a,b) then we follow the proof of Lemma 2.1 where we again use the fact that

$$\left| \int_a^b \frac{1}{iRf'(x)} \frac{d}{dx} \left(e^{iRf(x)} \right) \right|$$

Since $f^{(m)}(x) \geq 1$ and $f^{(m-1)}(x)$ is monotonic, there is at most one point, x_0 such that $f^{(m-1)}(x_0) = 0$. Then there exist at most one x_0

$$I(f,R) = \int_a^{x_0-\delta} e^{iRf(x)}\ dx + \int_{x_0-\delta}^{x_0+\delta} e^{iRf(x)}\ dx + \int_{x_0+\delta}^{b} e^{iRf(x)}\ dx$$

By the fact that $|e^{iRf(x)}| = 1$

$$\left| \int_{x_0-\delta}^{x_0+\delta} e^{iRf(x)}\ dx \right| \leq 2\delta$$

Moreover, we can observe that if $x \in (x_0+\delta, b)$, then by using the mean value theorem, we deduce that

$$\frac{f^{(m-1)}(x) - f^{(m-1)}(x_0)}{x - x_0} = f^{(m)}(c)$$

for $c \in (x_0, x)$. Since $f^{(m-1)(x_0)} = 0$ and $f^{(m)}(c) \geq 1$ we have

$$f^{(m-1)}(x) \geq \delta$$

on $(x_0+\delta, b)$ which further implies that $\frac{f^{(m-1)}(x)}{\delta} \geq 1$.

Applying the induction hypothesis, we obtain

$$\left|\int_{x_0+\delta}^{b} e^{iRf(x)}\ dx\right| = \left|\int_{x_0+\delta}^{b} e^{iR\delta\frac{f(x)}{\delta}}\ dx\right| \leq C_{m-1}(R\delta)^{-1/(m-1)}$$

We can apply the exact same argument to $-f(x)$ in the first integral and obtain

$$\left|\int_{a}^{x_0-\delta} e^{iRf(x)}\ dx\right| \leq C_{m-1}(R\delta)^{-1/(m-1)}$$

Putting this together, we obtain that

$$|I(f,R)| \leq 2C_{m-1}(R\delta)^{\frac{-1}{m-1}} + 2\delta$$

If we choose $\delta = R^{-1/m}$, we have

$$|I(f,R)| \leq 2C_{m-1}(RR^{-1/m})^{\frac{-1}{m-1}} + 2R^{-1/m}$$

and since

$$\begin{aligned} \frac{-1}{m-1} + \left(\frac{-1}{m}\right)\frac{-1}{m-1} &= \frac{-m}{m(m-1)} + \frac{1}{m(m-1)} && (8)\\ &= \frac{1-m}{m(m-1)} && (9)\\ &= \frac{-1}{m} && (10) \end{aligned}$$

we finally have that

$$|I(f,R)| \leq (2C_{m-1}+1)R^{-\frac{1}{m}}$$

where if $C_m = 2C_{m-1} + 1$, then our proof is complete. □

It may be worth nothing that we can play around with the constants by choosing δ in the above proof to be different.

It is natural to wonder whether we can keep extending this result. The motivation we used suggests the consideration of fractional powers of x.

$$I(x^\alpha, R) = \int_a^b e^{iRx^\alpha}\ dx$$

where $\alpha > 1$.

To show that

$$\lim_{A\to+\infty}\int_0^A e^{iRx^\alpha}\ dx$$

exists, we first use the fact that

$$\left|\int_0^1 e^{iRx^\alpha}\ dx\right|\le 1$$

which tells us that

$$\lim_{A\to+\infty}\left|\int_0^A e^{iRx^\alpha}\ dx\right|\le 1+\lim_{A\to+\infty}\left|\int_1^A e^{iRx^\alpha}\ dx\right|$$

If we make the substitution

$$t=Rx^\alpha,\ \text{so that}\ x=\left(\frac{t}{R}\right)^{\frac{1}{\alpha}}\ \text{and}\ \ dx=\frac{t^{1/\alpha-1}}{\alpha R^{1/\alpha}}\ dt$$

.

We have

$$\int_1^A e^{iRx^\alpha}\ dx=\int_1^{RA^\alpha} e^{it}\ \frac{t^{1/\alpha-1}}{\alpha R^{1/\alpha}}\ dt \tag{11}$$

$$=\frac{1}{\alpha R^{1/\alpha}}\left[ie^{it}t^{1/\alpha-1}\Big|_1^{RA^\alpha}-\int_1^{RA^\alpha}\frac{ie^{it}t^{1/\alpha-2}}{\frac{1}{\alpha}-1}\right] \tag{12}$$

$$=\frac{1}{\alpha R^{1/\alpha}}\left[ie^{iRA^\alpha}(RA^\alpha)^{1/\alpha-1}-ie^{i}-\int_1^{RA^\alpha}\frac{ie^{it}t^{1/\alpha-2}}{\frac{1}{\alpha}-1}\right] \tag{13}$$

Since $\alpha>1$ we have that $\frac{1}{\alpha}-2<-1$. Therefore; the leftmost integral converges as $A\to\infty$. So taking the limit as A $\to\infty$ and letting the value of the integral be denoted by C, we finally have

$$\lim_{A\to+\infty}\left|\int_0^A e^{iRx^\alpha}\ dx\right|\le 1+\frac{1+C}{\alpha R^{1/\alpha}}$$

For our purposes, what's important about this result is the factor $R^{1/\alpha}$ which appears in the proof of convergence. More motivation is given by the following calculation below.

Theorem 2.3. *Let* $\alpha>1$.

$$\int_0^\infty e^{iRx^\alpha}\ dx=\frac{1}{\alpha R^{1/\alpha}}\ \Gamma\left(\frac{1}{\alpha}\right)\left(cos\left(\frac{\pi}{2\alpha}\right)+isin\left(\frac{\pi}{2\alpha}\right)\right)$$

Proof. Consider the quarter annulus with inner radius ϵ and outer radius r, denoted, $\lambda_{\epsilon,r}$. Let $\alpha \in (0,1)$. By Cauchy's Theorem we have that for all $r, \epsilon > 0$

$$\int_{\lambda_{\epsilon,r}} z^{\alpha-1}e^{iz} = 0$$

We can break up the contour we're integrating over into four parts yielding the following

$$0 = \int_{\epsilon}^{r} x^{a-1}e^{ix}\ dx\ +\ \int_0^{\pi/2} (re^{it})^{\alpha-1}e^{i(re^{it})}ire^{it}$$

$$- \int_{\epsilon}^{r} (ix)^{a-1}ie^{-x}\ dx - \int_0^{\pi/2} (\epsilon e^{it})^{\alpha-1}e^{i(\epsilon e^{it})}i\epsilon e^{it}$$

We will get the second integral and the fourth integral out of the way by using some basic integral estimates and basic results in measure theory. We proceed as follows

$$\left| \int_0^{\pi/2} i(\epsilon e^{it})^{\alpha}e^{i\epsilon e^{it}}\ dt \right| \leq \int_0^{\pi/2} \left| i(\epsilon e^{it})^{\alpha}e^{i\epsilon e^{it}} \right| dt \tag{14}$$

$$= \int_0^{\pi/2} \epsilon^{\alpha} \left| e^{i\epsilon(cos(t)+isin(t)\ )} \right| dt \tag{15}$$

$$= \int_0^{\pi/2} \epsilon^{\alpha}e^{-\epsilon sin(t)}\ dt \tag{16}$$

On this domain, our integrand is a family of functions indexed by ϵ which is uniformly bounded by $\epsilon^{\alpha}e^{\epsilon}$ on a set of finite measure and approaches 0 point-wise as $\epsilon \to 0$. Therefore, we may apply the dominated convergence theorem and obtain

$$\lim_{\epsilon\to 0} \int_0^{\pi/2} \epsilon^{\alpha}e^{-\epsilon sin(t)}\ dt = \int_0^{\pi/2} \lim_{\epsilon\to 0} \epsilon^{\alpha}e^{-\epsilon sin(t)}\ dt = 0$$

Similarly we have that

$$\left| \int_0^{\pi/2} (re^{it})^{\alpha-1}e^{i(re^{it})}ire^{it}\ dt \right| \leq \int_0^{\pi/2} re^{-rsin(t)}\ dt$$

Recall that removing a point from the domain of integration, $[0, \pi/2]$ does not change the value of the integral. Therefore we may view the last integral as being over $(0, \pi/2]$. The purpose of doing this is because on this domain, our integrand is a family of functions indexed by ϵ which is uniformly bounded by the integrable function $\frac{\pi}{2}\, r$. The domain is a set of set of finite measure and the family of functions inside the integral approaches 0 point-wise as $r \to \infty$. We have satisfied the conditions so that the dominated convergence theorem can be applied. We have

$$\lim_{r\to\infty} \int_0^{\pi/2} re^{-rsin(t)}\ dt = 0$$

So we are now left with the following

$$\int_0^{\infty} x^{\alpha-1}e^{ix} = \int_0^{\infty} (ix)^{\alpha-1}e^{-x}\ dx = \Gamma(\alpha)e^{i\pi\alpha/2}$$

The result follows from the fact that a change of variables and the fact that if $\alpha > 1$, $\frac{1}{\alpha} \in (0,1)$. So we have

$$\int_0^{\infty} e^{iRx^{\alpha}}\ dx = R^{-1/\alpha} \int_0^{\infty} e^{ix^{\alpha}}\ dx \tag{17}$$

$$= \frac{1}{\alpha R^{1/\alpha}} \int_0^{\infty} x^{\frac{1}{\alpha}-1}e^{ix} \tag{18}$$

$$= \frac{1}{\alpha R^{1/\alpha}} \Gamma\left(\frac{1}{\alpha}\right) e^{i\pi/2\alpha} \tag{19}$$

□

Note that if $R = 1$ and $\alpha = 2$, we obtain the result of Fresnel integral.

It would seem that an extension of the Vander-Corput lemma would require a treatment of fractional derivatives. I'm going to finish our discussion of the method of stationary phase here however, my hope is to convince the reader that such an extension of the Vander-Corput lemma is quite likely to exist.

3 Applications

We will use what we developed so far to give estimates for the Fourier transform of the surface measure σ on a sphere $S^{n-1} \subset \mathbb{R}^n$. First we start off by diving a bit more into our oscillatory integrals. We assume an introduction of the Fourier transform and in particular, we will assume the following theorem.

Theorem 3.1. *Let T be an invertible $n \times n$ real symmetric matrix with signature σ. Define*

$$G_T(x) = e^{-i\pi\langle Tx,x\rangle}$$

G_T has a distributional Fourier transform which is equal to

$$e^{-\pi i\frac{\sigma}{4}}\ |detT|^{-1/2}\ G_{-T^{-1}}(\xi)$$

Put another way, we have that for any $\phi(x) \in S(\mathbb{R}^n)$

$$\int e^{-i\pi\langle Tx,x\rangle}\hat{\phi}(x) = e^{-\pi i\frac{\sigma}{4}}\ |detT|^{-1/2}\int e^{i\pi\langle T^{-1}x,\ x\rangle}\phi(x)\ dx$$

Using this theorem, we're able to prove the following:

Theorem 3.2. *Let T be a real symmetric invertible matrix with signature σ. Let $\psi(x) \in C_0^\infty$ and define*

$$I(R) = \int_{\mathbb{R}} e^{-i\pi R\langle Tx,x\rangle}\ \psi(x)\ dx$$

Then for any N, we have the following equality

$$I(R) = e^{-\pi i\frac{\sigma}{4}}|detT|R^{-n/2}\left(\psi(0) + \sum_{j=1}^{N} R^{-j}D_j\psi(0) + O(R^{-(N+1)})\right)$$

For our purposes the constants D_j will not be of importance here.

Proof. If we invoke Lemma 3.1, and the Fourier inversion formula, we have that

$$I(R) = e^{-\pi i\frac{\sigma}{4}}\ |\det T|\ R^{\frac{-n}{2}}\int\hat{\psi}(-\xi)\ e^{R^{-1}\pi i\langle T^{-1}\xi,\xi\rangle}\ d\xi$$

Using Taylor's theorem on e^{ix}, we have that

$$e^{\pi i\langle T^{-1}\xi,\xi\rangle} = \sum_{j=0}^{N}\frac{(\pi i\langle T^{-1}\xi,\xi\rangle)^j}{j!} + O\left(\frac{|\xi|^{2N+2}}{\mathbb{R}^{N+1}}\right)$$

where the convergence in uniform in ξ and R. Putting this together with our equation for $I(R)$,

$$\int \hat{\psi}(\xi)\ e^{\pi i R^{-1}\langle\ T^{-1}\xi,\ \xi\rangle}\ d\xi = \int \hat{\psi}(\xi)\Big(1 + \sum_{j=1}^{N} \frac{(\pi i\langle T^{-1}\xi, \xi\rangle)^j}{j!}\Big)\ d\xi$$

$$+\ O\Big(\int |\psi\hat{(}\xi)|\ \frac{|\xi|^{2N+2}}{R^{N+1}}\ d\xi\Big)$$

The proof follows from the fact that $\int \hat{\psi}(\xi)\ d\xi = \psi(0)$ and that from Distribution theory,

$$\int \hat{\psi}(\xi)\frac{(\pi i\langle T^{-1}\xi, \xi\rangle)^j}{j!}\ d\xi = D_j\psi$$

for appropriate differential operators D_j, and lastly by the fact that

$$\int |\hat{\psi}(\xi)||\xi|^{2N+2}\ d\xi < C$$

since $\hat{\psi}(\xi)$ is also a Schwartz function.

$\square$

The following theorem, we will take for granted; however, it is a corollary of the theorem just proven.

Theorem 3.3. *Let ϕ be C^∞ and assume that $\nabla\phi(p) = 0$ and $H_{\phi(p)}$ is invertible. Let σ be the signature of $H_{\phi(p)}$, and let $\Delta = 2^{-n}|det(H_{\phi(p)}|$. Let ψ be a bump function supported in a sufficiently small neighborhood of p. Defining*

$$I(R) = \int e^{-i\pi Rf(x)}\ \phi(x)\ dx$$

we have the following

$$I(R) = e^{-\pi i R\phi(p)}e^{-i\pi\frac{\sigma}{4}}\Delta^{\frac{-1}{2}}R^{-n/2}\Big(\psi(p) + \sum_{j=1}^{N} R^{-j}D_j\psi(p) + O(R^{-(N+1)})\Big)$$

Theorem 3.4. *Assume that $\nabla\phi(p) = 0$ and $H_{\phi(p)}$ is invertible. Then, for ψ supported in a small neighborhood of p, we have the following estimate*

$$\left|\frac{d^k}{dR^k}(e^{i\pi R\phi(p)})I(R)\right| \le C_k R^{-(\frac{n}{2}+k)}$$

We need to demonstrate the following claim: Let $\phi_{i=1}^M$ be real valued smooth functions and assume that $\phi(p) = 0$ and that $\nabla\phi(p) = 0$. Lastly, let

$$\Phi = \prod \phi_i$$

Then all partial derivatives of Φ of order less than $2M$ also vanish at p.

Proof. By the product rule any partial $D^\alpha\Phi$ is a linear combination of terms of the form

$$\prod_{i=1}^{M} D^{\beta_i}\phi_i$$

with $\sum \beta_i = \alpha$. If $|\alpha| < 2M$, then some β_i must be less than 2, so by hypothesis all such terms vanish at p.

To prove our theorem, we need only differentiate $I(R)$ under the integral sign in order to obtain

$$\frac{d^k}{dR^k}(e^{-\pi i R(\phi(x)-\phi(p))}) = (-\pi i)^k \int (\phi(x) - \phi(p)))^k \psi(x) e^{-\pi i R(\phi(x)-\phi(p))}\ dx$$

Let $b(x) = (\phi(x) - \phi(p))^k\psi(x)$. By the above claim all partials of b of order less than $2k$ vanish at p. Now look at the expansion in previous theorem replacing ψ with b and setting $N = k-1$. By the claim the terms $D_j b(p)$ must vanish when $j < k$, as well as $b(p)$ itself. Hence, the above theorem shows the desired result.

□

We now have all the tools we need to give our estimate. Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}$ is smooth and M is a k-dimensional submanifold. Recall that if we are given $p \in M$, $U \subset \mathbb{R}^n$ and a local coordinate map $F : U \to M$ $\phi \circ F$ will have a critical point at $F^{-1}(p)$ iff $\nabla\phi(p)$ is orthogonal to the tangent space to M at p. Henceforth this is independent of the choice of F.

In order to estimate the Fourier transform of σ, it suffices to consider $\hat{\sigma}(\lambda e_n)$ where $e_n = (0,\ \ldots\ 0,\ 1)$ and $\lambda > 0$. We can put local coordinates on the sphere the standard way:

$$F_1(x) = \sqrt{1 - |x|^2} \tag{20}$$

$$F_2(x) = -\sqrt{1 - |x|^2} \tag{21}$$

and all that is required of the remaining maps is that they map onto sets whose closures do not contain $\{\pm e_n\}$

Let ϕ_k be a suitable partition of unity subordinate to this covering by charts. Let P_n be the projection map defined by

$$P_n(x) = e_n \cdot x$$

The gradient of P_n is normal to the sphere at $\{\pm e_n\}$ only.

Putting all this together, we have

$$\begin{aligned}
\hat{\sigma}(\lambda e_n) &= \int e^{2i\pi\lambda e_n \cdot x}\ d\sigma(x) && (22)\\
&= \sum_{j=1}^{k} \int e^{2i\pi\lambda e_n \cdot x}\phi_j(x)\ d\sigma(x) && (23)\\
&= \int e^{2i\pi\lambda F_1(x)} \frac{\phi_1(x)}{F_1(x)}\ dx + \int e^{2i\pi\lambda F_2(x)} \frac{\phi_2(x)}{F_2(x)}\ dx && (24)\\
&+ \sum_{j=3}^{k} \int e^{2i\pi\lambda F_j(x)}\phi_j(x)\ dx && (25)
\end{aligned}$$

The integrals are all in $\mathbb{R}^{n-1}$, and the phase functions $\phi_k(x)$ have no critical points if $k \geq 3$ in the support of $\phi_k(x)$. The Heissian of $2\sqrt{1-|x|^2}$ is invertible. Moreover the first and second terms are complex conjugates. So we have that

$$\hat{\sigma}(\lambda e_n) = \mathrm{Re}(a(\lambda)e^{2\pi i\lambda}) + y(\lambda)$$

And by invoking, we have

$$\begin{aligned}
\frac{d^j a(\lambda)}{d\lambda^j} &\lesssim \lambda^{-\frac{n-1}{2}-j} && (26)\\
\frac{d^j y(\lambda)}{d\lambda^j} &\lesssim \lambda^{-N} && (27)
\end{aligned}$$

for all N.

4 References

1. Thomas M Wolff. Lectures in Harmonic Analysis(Revised). March 2002
2. Pertti Matilla. Fourier Analysis and Hausdorff Dimension. Cambridge University Press 2015