Dynamics of Rotation-like Logistic Maps

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Abstract

In this paper, we study the dynamics of rotation-like logistic maps. In particular, we focus on the case of rotation-like logistic maps with Fibonacci quotient, including related parts on Hubbard tree, kneading invariant and kneading map. We shall present theorems and conjectures on their associated kneading maps.

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1 Acknowledgements

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2 Introduction

At the beginning of the twenty-first century, Ble and Douady [2] introduced a family of logistic maps inspired from ideas originated from holomophic dynamical systems. Despite the connection between the dynamics of logistic maps and holomorphic dynamics, there has been little study carried on from the perspective of both areas. Our goal is to study the kneading invariant and the kneading map of the logistic maps in Ble-Douady's family. In this paper, we shall only study one particular subset of these logistic maps: rotation-like logistic maps with Fibonacci quotient.

3 Preliminaries

3.1 Basic Concepts in Complex Dynamics

A **discrete dynamical system** (X, f) is a space/set X along with a map $f : X \to X$, where X is our "system", and f is "the law of dynamics".

Definition If $f(z) = z$, then z is called a **fixed point** of f. More generally, if $f^n(z) = z$ and $f^m(z) \neq z \forall 1 \leq m \leq n-1$, then $z = z_0 \to f(z) = z_1 \to f^2(z) = z_2 \to \cdots \to f^{n-1}(z) = z_{n-1}$ is called a **periodic orbit** of period n.

Definition Let z be a fixed point of f. If $\exists U$, a neighborhood of z such that $f(U) \subset U$ and $\forall z' \in U$, we have $\lim_{n\to\infty} f^n(z') = z$, then z is called an **attracting fixed point**. If instead, z is in an periodic orbit of period n, then $z = z_0 \to f(z) = z_1 \to f^2(z) = z_2 \to \cdots \to f^{n-1}(z) = z_{n-1}$ is called an **attracting periodic orbit** if z is an attracting fixed point of f^n.

Notice that in this way, every point in the orbit is a fixed point of f^n. In some sense, the study of an attracting periodic orbit can sometimes be "simplified" to the study of a fixed point in this way. The next step is to find a more quantitative way to identify attracting fixed points and periodic orbits. We introduce the following notion:

Definition Let z be a point in an n-periodic orbit of a differentiable map f. Then the derivative $(f^n)'(z) = \lambda$ is called the **multiplier** of this orbit.

Yes, this is the quantity we are seeking for.

In contrast to the attracting periodic orbit, we define a **repelling periodic orbit** similarly (with $|\lambda| > 1$).

Definition The **basin of attraction** B with respect to an orbit $z_0 \to f(z) = z_1 \to f^2(z) = z_2 \to \cdots \to f^{n-1}(z) = z_{n-1}$ is the collection of all points such that converges to one of the orbit points after infinite iterations of f^n. Formally,

$$B = \{z : \lim_{m\to\infty} f^{mn}(z) \in \{z_0, \cdots, z_{n-1}\}\} \tag{1}$$

The **immediate basin of attraction** is the connected components of B that contains at least one orbit point.

Now let us get to the ordinary quadratic polynomial $p_c(z) = z^2 + c$. When do p_c have attracting periodic orbits? Since this might be worth studying, we define the following set:

Definition $M_0 = \{c : p_c \text{ has a periodic orbit }\}$.

Definition Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. Then $a \in \mathbb{C}$ is called a **critical point** if $f'(a) = 0$.

Definition Let a be a critical point of a holomorphic map f. Then $f(a)$ is called a **critical value** of f. The following lemma will be helpful for understanding the motivation of introducing another special set:

Theorem (Fatou) Let p be a non-linear polynomial of order n. Then every immediate basin of attraction for a periodic orbit contains at least one critical point of p.

Corollary There can be at most $n-1$ different attracting periodic orbits.

As a corollary in our quadratic polynomial case, the immediate basin of attraction of an attracting periodic orbit, which has to be bounded (if $|z|$ is large enough, $p_c^n(z)$ will blows to infinity no matter what c we choose), must contain the orbit $\{p_c^n(0)\}$. Inspired by this observation, we now define the following more general set, called the **Mandelbrot set**:

$$M = \{c : \exists A_c, |p_c^n(0)| < A_c, \forall n \in \mathbb{N}\} \tag{2}$$

It is conjectured that M is the closure of M_0. We shall not discuss further on that.

It is proved by Douady and Hubbard that the Mandelbrot set is connected.

Following from our previous discussions, there might also be of some interest to consider the collection of all points that are bounded under infinite iteration. It is called the **filled Julia set**, denoted by $K = K(f)$. The formal definition is

$$K(f) = \{z : \exists C_z > 0, f^n(z) < C_z, \forall n \in \mathbb{N}\} \tag{3}$$

Its boundary ∂K is called the **Julia set**, denoted by $J = J(f)$. We see that Julia set is indeed the boundary between being "blowing to infinity" and "forever bounded" after infinite iterations.

For a quadratic family $p_c(z) = z^2 + c$, we denote K_c to be $K(p_c)$. Actually, there is another version of the definition Mandelbrot sets, which is the primary version used in Blé's paper. Proving equivalence to our previous definition is very nontrivial and involves the use of Green function, which is somewhat a characterization of “the speed of blowing to infinity”. We shall not discuss into details here.

Definition’ The Mandelbrot set M is the collection of all c’s for which K_c is connected.

There are some very interesting properties of Julia sets. We will introduce them without detailed proofs:

Theorem If f is a rational function, then $J(f) \neq \emptyset$.

Theorem J is forward invariant. More specifically, let $z \in \hat{\mathbb{C}}$, then $f(z) \in J$ if and only if $z \in J$.

Theorem $J(f) = J(f^n) \forall n \in \mathbb{N}$.

There are two equivalent definitions of **Fatou set**. One of them is more “intuitive”, and the other is defined using the normal family of functions.

Definition The Fatou set is the complement of the Julia set. In other words, it is the union of all points that are bounded under infinite iterations of f, as well as those in the basin of attraction for ∞.

The following theorem is a great illustration of what Julia and Fatou sets really mean:

Theorem The repelling orbit points of f are completely contained in $J(f)$; on the other hand, the attracting orbit points of f are completely contained in $\hat{\mathbb{C}} \setminus J(f)$.

Let us recall how we constructed the Mandelbrot set M. The set in the first iteration, consisting of all the values of c resulting in an attracting fixed point, denoted by M_1, **main hyperbolic component**, and $\partial \bar{M}_1$ is called the **main cardioid**.

A point in M with an attracting cycle is called **hyperbolic**.

Theorem M is locally connected when restricted to the hyperbolic components.

In particular, M is locally connected when restricted to the main cardioid.

Definition The orbit of a critical point with respect to a polynomial f is called a **critical orbit**. The critical orbit is **preperiodic** if the orbit is finite. If the critical orbits are periodic or preperiodic (or both), then the polynomial f is called **PCF (postcritically finite)**. In the case that all critical orbits are periodic, f is called a **center**; if they are all preperiodic, f is called a **Misiurewicz polynomial**. Denote the set of all critical points of f by $C(f)$. Then the **postcritical set** is the union of all subsequent images of all critical points under f.

Definition Let X, Y be topological spaces and let (X, f) and (Y, g) be their corresponding dynamical systems. If h is a homeomorphism from Y to X such that $h^{-1} \circ f \circ h = g$, then (X, f) and (Y, g) are called **topologically conjugate**. If instead h is continuous and surjective (but not necessarily a homeomorphism), then (X, f) and (Y, g) are called **semi-conjugate**.

In some sense, we see that conjugation means "the same after a change of coordinates". Theorems by Koenig and Bottcher show that a polynomial map behaves "locally like a linear or a monomial map" near its attracting fixed point in the sense that the map is conjugate to a polynomial in some open neighborhood of its fixed point under a conformal map. Therefore, the dynamics (i.e. the properties of the corresponding maps under long iterations) of two conjugate dynamical systems are very similar.

It can be shown that when restricted to $J(p_c)$, p_c is semi-conjugate to the angle doubling map $\xi : \mathbb{T} \to \mathbb{T} : \theta \to 2\theta$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a representation of the circle with unit circumference. In addition, our choice of θ can ensure that the orbit of θ under the angle doubling map does not intersect the interval $[\frac{1}{4}+\frac{\theta}{4}, \frac{1}{2}+\frac{\theta}{4}]$.

3.2 Hubbard Tree

Now we construct the Hubbard tree (this follows the definition in Ble's paper [2]). A graph $\Gamma = (V, E)$ consists of a set of vertices V and a set of edges E, each connecting two vertices. If in addition, Γ is connected (intuitively, you can start from any vertex and reach any other vertex by going through the edges in E) without a loop, then it is called a **tree**. For convenience, if n edges meet at the same vertex, then the angle between any two edges is an integer multiple of $\frac{1}{n}$.

Definition A **Hubbard tree** is a tree that further satisfies the following properties:

(1) There exists a skew-symmetric function $\xi : (l, l') \to \mathbb{T}$, assigning each pair of edges meeting at the same vertex an angle, that satisfies ($\xi(l, l') = 0$ if and only if $l = l'$) and ($\xi(l, l') + \xi(l', l'') = \xi(l, l'')$).

(2) There exists a map $\delta : V \to \mathbb{N}$, called a **local degree function**, that assigns a *degree* to each vertex, with the property $1+\sum(\delta(v)-1) > 1$. (Here, $\mathbb{N}$ does not include 0.) If $\delta(v) > 1$, then v is **critical**. Note that if there is no critical vertex, then $1 + \sum(\delta(v) - 1) = 1 + 0 = 1$, so there must be at least one critical point.

(3) There exists a homeomorphism $\tau : H \to H$ mapping vertices to vertices and edges to edges, with the property that $\xi(\tau(l), \tau(l')) = \delta(v)\xi(l, l')$ where v is the vertex on which the two edges meet. If $\tau^n(v) = v$ for a positive integer n, then v is called a **periodic**. If v is a critical point in addition, then its orbit is called a **critical cycle**. If v is not in the mage of a critical cycle under $\tau^{-n} \forall n \in \mathbb{N}$, the v is a **Julia vertex**; otherwise, v is called a **Fatou**

vertex.

(4) There exists a metric $d : V \times V \to \mathbb{N} \cup \{0\}$, counting the number of edges in the shortest path between two vertices.

(5) H is **expanding**. This means that for all Julia vertices v, v' that are connected by an edge l, $d(\tau^n(v), \tau^n(v')) > 1$ for some n_l.

Here we provide three concrete examples for the Hubbard tree. Let us consider a **rotation-like logistic map**. The general form of the map is $f(z) = e^{2\pi i\gamma}z, z \in \mathbb{C}$, where γ is a characterizing quantity of the map. If γ is a rational number, then all points on the unit circle are periodic points with period being the denominator of γ (written in lowest terms). In this paper, we are particularly interested in the case when γ is a **Fibonacci quotient**. This means the numerator and denominator in each gamma are neighboring terms in the Fibonacci sequence, or $\gamma = \frac{F_{n-1}}{F_n}$, where F_n is the $n-th$ Fibonacci number. The version of Fibonacci number we are using is that $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For the purpose of Hubbard trees, we shall only consider those starting from $\frac{F_3}{F_4}$ The first few Fibonacci quotients are $\gamma = \frac{2}{3}$, $\gamma = \frac{3}{5}$, and $\gamma = \frac{5}{8}$. For the purpose of kneading invariants and kneading maps, we shall only consider those starting from $\frac{F_4}{F_5} = \frac{3}{5}$.

The steps to construct the Hubbard tree in the case of Fibonacci quotient is as follows (this is also from [2] by G. Blé):

1. Draw a unit circle. Set $(1, 0)$ as x_0, and label all iterations of x_0. There are finitely many of them, since $\gamma \in \mathbb{Q}$.

2. Split the circle into two half-circles at x_1 and x_2. There will be a longer half-circle and a

shorter half-circle (unless $\gamma = 1/2$, which we shall not consider here). Just for mathematical rigor, take the closure of the two half-circles so they are both compact segments. Relabel x_1 and x_2 on the longer half-circle as x_1' and x_2'.

3. "Paste" x_2 and x_1' together (more rigorously, we can consider performing a quotient map by identifying these two points on the two half-circles) and label it as α. (We do not have to use α in our discussion below. This is just for construction.)

4. Now we have an interval. The interval goes from x_1 to x_2' in ascending order. We can map this interval to the real line homeomorphically and set $x_0 = 0$ for simplicity.

We can see the first three examples in the picture below:

Now that we have a Hubbard tree, we are looking for a map such that $f(x_0) = x_1$, $f(x_1) = x_2$, etc. It is not hard to imagine that there are many maps that satisfy this condition, even if we stipulate that the map must be holomorphic.

Theorem A Hubbard tree corresponds to a unique PCF (a polynomial map that have finitely many critical points along with their iterations under f) up to an affine conjugation.

This theorem is critical in that we can construct our Hubbard trees with more freedom in choosing a corresponding map.

We now introduce the notion of kneading invariant.

3.3 Kneading Invariant

Suppose we have a **unimodal map** $f : [0, 1] \to [0, 1]$, which means that f is a continuous map with one and only one maximum x_0 and that $f(0) = 0 = f(1)$. Let x_0^+ be the right limit of x_0.

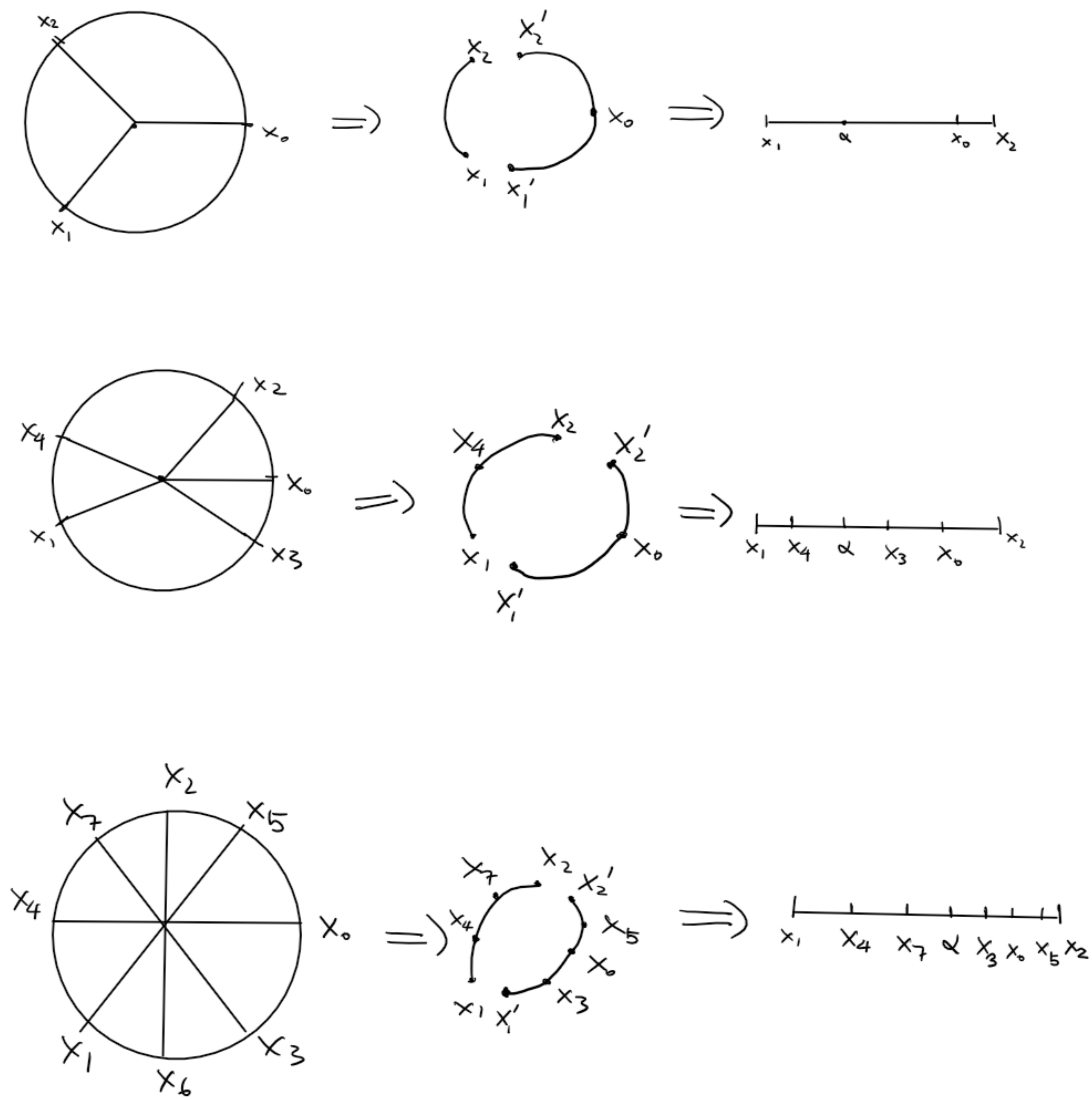

Figure 1: Examples on construction of the Hubbard Tree

Definition The **kneading invariant** of f is defined as a sequence $v(f) = \overline{v_1 v_2 v_3 \cdots}$, where each v_i denote the i-th digit in the sequence, and is defined to be:

$$v_i = \begin{cases} 0, & \text{if } f^i(x_0^+) < x_0 \\ \\ 1, & \text{if } f^i(x_0^+) > x_0 \end{cases}$$

Here, we use x_0^+ instead of x_0 in order to avoid the case when $f^{(i)}(x_0) = x_0$.

Also, we are only considering the case where we are in the "dynamical core". In other words, $f^{(2)}(x_0) < x_0 < f(x_0)$ or $f(x_0) < x_0 < f^{(2)}(x_0)$. This is to ensure that the dynamics of this one-dimensional map is "interesting", in the sense that if not, we will see that the map is "too predictable". For example, if $x_0 < f^{(2)}(x_0) < f(x_0)$, then $f((x_0, f(x_0))) = (f^{(2)}(x_0), f(x_0))$, which is a proper subset of the interval $(x_0, f(x_0))$. If we consider $f((x_0, f(x_0)))$, we will obtain an even smaller interval. In other words, the map is "strictly shrinking".

Now, in the case of a rotation-like logistic map f with rational quotient $\gamma = \frac{m}{n}$ (including those with Fibonacci quotient), the kneading invariant is periodic, in the sense that there will be some string $d = \overline{d_1 d_2 d_3 \cdots d_n}$ such that $v(f) = \overline{ddd \cdots}$. This can be easily seen from the fact that there are n images of x_0. In some sense, the kneading invariant preserves the information on the position of images of x_0 under f relative to x_0. Let $r(n)$ denote the length of the $n-th$ block. Let $R(n)$ denote the total length of the first n blocks and define $R(0) = 0$. There are some more results on the properties of the kneading invariant and the blocks in Hofbauer's paper [8]. In general, these can be proved easily by considering and visualizing the positions of the images of x_0 under iterations of f.

A theorem by Hofbauer shows that a kneading invariant $v(f) = \overline{v_1 v_2 v_3 \cdots}$ can be decomposed into an infinite number of blocks in a unique way through an algorithm.

The algorithm for the decomposition is as follows [8]:

1. Start with either 0 or 1, depending on whether $v(f)$ starts with 0 or 1, and consider $0v(f)$ or $1v(f)$ accordingly.

2. Now, compare the portion of $v(f)$ starting from the digit of interest (in this case, the first digit) with $0v(f)$ or $1v(f)$. For rotation-like logistic maps with rational γ, the x_1 is always negative, which means the first digit of the kneading invariant is always 0. Thus, we will always start with $0v(f)$.

3. The comparing process is as follows: check whether the next digit, v_2, is the same as v_1; if $v_2 = v_1$, continue to v_3; if not, end the block at v_2.

4. Now, in the case when $v_2 = v_1$, if $v_3 = v_2$, continue; if $v_3 \neq v_2$, end the block at v_3.

5. Continue the process until the block ends.

6. Next, comparing the portion of $v(f)$ starting from the $r(1)+1-th$ digit with either $0v(f)$ or $1v(f)$, depending on whether the $r(1)+1-th$ digit is 0 or 1. If the $r(1)+2-th$ digit is the same as the first digit of $v(f)$, continue to compare the $r(1)+3-th$ digit and the second digit of $v(f)$; if not, stop the block.

7. Continue the comparing process indefinitely.

3.4 Kneading Map

The kneading map, compared to kneading invariant, is defined in a much more implicit way.

Definition The **kneading map** $Q : \mathbb{N} \to \mathbb{N}$ is the map that satisfies:

$$R_{Q(n)} + 1 = r(n) \tag{4}$$

The image of the kneading map of a rotation-like logistic map with Fibonacci quotient is a repetitive sequence. In other words, there is some period p such that $Q(N) = Q(N \bmod p)$. This directly follows from the fact that the kneading invariant of f is periodic, since this means the block decomposition of $v(f)$ is also periodic, so it follows from the definition of the kneading map that Q_f must be periodic also.

To have a better understanding or more intuition on the kneading map, it might be helpful to introduce Hofbauer's tower. A more detailed discussion can be found in [4].

Definition Let I be a unit interval, $f : I \to I$ be a unimodal map, and c. The **Hofbauer's tower** $H = H_1 \times H_2 \times H_3 \times \cdots$ is a subspace of $I^{\mathbb{N}}$ such that $H_1 = (x_0, f(x_0))$, and

$$H_{i+1} = \begin{cases} (f^{(i+1)}(x_0), f(x_0)), & \text{if } x_0 \in \overline{H_i} \\ f(H_i), & \textit{otherwise} \end{cases}$$

Definition If $x_0 \in H_i$ for some $i \in \mathbb{N}$, then i is a **cutting time**. We denote the $n-th$ cutting time by S_n and set $S_0 = 1$. In the case where n is the smallest natural number such that there is no $n-th$ cutting time, we set $S_n = \infty$.

Now, it can be shown from the definition of H that $H_{S_n} \subset H_{S_n - S_{n-1}}$, so that $S_n - S_{n-1}$ must also be a cutting time. This motivates the following definition:

Definition' The **kneading map** $Q : \mathbb{N} \to \mathbb{N}$ is the map that satisfies:

$$S_{Q(n)} = S_n - S_{n-1} \tag{5}$$

4 Examples

We now compute two examples of the kneading invariant of the logistic maps with Fibonacci quotient. Before that, we need to have a map which takes x_0 to x_1, x_1 to x_2, $\cdots$. This map does not have to be unique; rather, as we have previously discussed, they only have to be equivalent up to an affine conjugation. In this case, just for simplicity, we use the quadratic map $f(x) = x^2 + c$. Now, a requirement is that when iterated enough times (in particular, multiples of the period of the number of points in the Hubbard tree), f takes x_0 back to itself. Here, we assume that the points in the Hubbard tree are mapped to the real line and let $x_0 = 0$ be the turning point of the unimodal map f. Actually, f is not exactly a unimodal map here, since it is not mapping from $[0, 1]$ to $[0, 1]$. However, it is equivalent to a unimodal map up to an affine conjugation, so they have the same dynamics and therefore can be treated equally. We are mapping the Hubbard tree to a real interval, and it does not matter much where the interval is on the real line regarding the dynamics on the Hubbard tree.

4.1 Case $\gamma = \frac{3}{5}$

We can actually solve for c by noting that $f(x_0) = f(0) = c = x_1$, $f(x_1) = c^2 + c = x_2$, $f(x_2) = (c^2+c)^2+c = x_3$, $f(x_3) = ((c^2+c)^2+c)^2+c = x_4$, $f(x_4) = (((c^2+c)^2+c)^2+c)^2+c =$

$x_0 = 0$. Although the value of c does not matter much in our discussion later, we will just compute c as a reference. Note that since this is a polynomial, we will have multiple solutions of c, which may or may not be real, and not all of them will satisfy the requirement that the relative position of the images of x_0 must be the same as we desired. For example, the trivial solution $c = 0$ is not valid for our purposes, since if we take $c = 0$, we will have $x_0 = x_1 = x_2$ instead of $x_1 < x_0 < x_2$. An unproved conjecture is that only one c is valid. The only valid c here is $c \approx -1.6254$.

The kneading invariant is $01000\cdots$. The repeating part is 01000. Let us practice the block decomposition for this map as an example:

1.Starting with the first digit 0 and comparing the sequence of $v(f)$ starting from the first digit with $0v(f)$ (recall that if the digit we are considering is 0, we compare the kneading invariant starting with this digit with $0v(f)$; otherwise, we compare it with $1v(f)$), since 1 is the second digit in $v(f)$, and 0 is the first digit of $v(f)$, and $1 \neq 0$, we stop, so the first block is (0).

2.Next, starting from the second digit 1 and comparing the portion of $v(f)$ starting from the second digit with $1v(f)$, since the third digit in $v(f)$ is 0, which is equal to the first digit in $v(f)$, we continue; comparing the fourth digit of $v(f)$, which is 0, with the second digit of $v(f)$, which is 1, we find $0 \neq 1$, so we stop on this block, and the the second block is (10).

3.Next, starting from the fourth digit of $v(f)$, which is 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the fifth digit of $v(f)$ is 0, and the first digit of $v(f)$ is 0, we continue; now, the sixth digit of $v(f)$ is 0, and the second digit of $v(f)$ is 1, $1 \neq 0$, so we stop with this block, so the third block is (00).

4.Since the kneading invariant forward is just repeating the first five digits, the block decomposition will be exactly the same as the three we did above, which means we will have blocks (0)(10)(00) repeating forever.

Now we calculate the sequence of kneading map of f:

1.First, $R(Q(1)) + 1 = r(1) = 1$, so $R(Q(1)) = 0$. Since R is a strictly increasing function (a block must have length at least 1), and $R(0) = 0$, we must have $Q(1) = 0$.

2.Next, $R(Q(2)) + 1 = r(2) = 2$, or $R(Q(2)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(2) = 1$.

3.Next, $R(Q(3)) + 1 = r(3) = 2$, or $R(Q(3)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(3) = 1$.

4.From this point on, for any $j \geq 4$, $R(Q(j \bmod 3)) + 1 = r(j \bmod 3)$, so we must have $Q(j) = Q(j \bmod 3)$.

4.2 Case $\gamma = \frac{5}{8}$

Similarly as the procedures in the last example, we have $f(x_0) = f(0) = c = x_1$, $f(x_1) = c^2 + c = x_2$, $f(x_2) = (c^2 + c)^2 + c = x_3$, $f(x_3) = ((c^2 + c)^2 + c)^2 + c = x_4$, $f(x_4) = (((c^2+c)^2+c)^2+c)^2+c = x_5$, $f(x_5) = ((((c^2+c)^2+c)^2+c)^2+c)^2+c = x_6$, $f(x_6) = (((((c^2+c)^2+c)^2+c)^2+c)^2+c)^2+c = x_7$, $f(x_7) = ((((((c^2+c)^2+c)^2+c)^2+c)^2+c)^2+c)^2+c = x_0 = 0$. By checking the relative positions of the images of x_0 under f for different c, we see that the only valid c here is $c \approx -1.7111$.

The kneading invariant is $01001000\cdots$. The repeating part is 01001000. Let us practice the block decomposition for this map:

1.Starting with the first digit 0 and comparing the sequence of $v(f)$ starting from the first digit with $0v(f)$ (recall that if the digit we are considering is 0, we compare the kneading invariant starting with this digit with $0v(f)$; otherwise, we compare it with $1v(f)$), since 1 is the second digit in $v(f)$, and 0 is the first digit of $v(f)$, and $1 \neq 0$, we stop, so the first block is (0).

2.Next, starting from the second digit 1 and comparing the portion of $v(f)$ starting from the second digit with $1v(f)$, since the third digit in $v(f)$ is 0, which is equal to the first digit in $v(f)$, we continue; comparing the fourth digit of $v(f)$, which is 0, with the second digit of $v(f)$, which is 1, we find $0 \neq 1$, so we stop on this block, and the the second block is (10).

3.Next, start from the fourth digit 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the fifth digit of $v(f)$ is 1, and the first digit of $v(f)$ is 0, we stop here, so the third block is (0).

4.Next, start from the fifth digit 1, and compare the sequence of $v(f)$ starting from it with $1v(f)$: since the sixth digit of $v(f)$ is 0, which is equal to the first digit 0, we continue; since the seventh digit is 0, which is not equal to the second digit 1, we stop here, so the fourth block is (10).

5.Next, start from the seventh digit 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the eighth digit 0 is equal to the first digit 0, we continue; since the ninth digit 0 is different from the second digit 1, we stop here, so the fifth block is (00).

6.Since the kneading invariant forward is just repeating the first eight digits, the block decomposition will be exactly the same as the three we did above, which means we will have blocks (0)(10)(0)(10)(00) repeating forever.

Now we calculate the sequence of kneading map of f:

1.First, $R(Q(1)) + 1 = r(1) = 1$, so $R(Q(1)) = 0$. Since R is a strictly increasing function (a block must have length at least 1), and $R(0) = 0$, we must have $Q(1) = 0$.

2.Next, $R(Q(2)) + 1 = r(2) = 2$, or $R(Q(2)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(2) = 1$.

3.Next, $R(Q(3)) + 1 = r(3) = 1$, or $R(Q(3)) = 0$. By the same reason, since $R(0) = 0$, we must have $Q(3) = 0$.

4.Next, $R(Q(4)) + 1 = r(4) = 2$, or $R(Q(4)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(4) = 1$.

5.Next, $R(Q(5)) + 1 = r(5) = 2$, or $R(Q(5)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(5) = 1$.

6.From this point on, for any $j \geq 6$, $R(Q(j \bmod 6)) + 1 = r(j \bmod 5)$, so we must have $Q(j) = Q(j \bmod 5)$.

5 Main Conjectures

For a rotation-like logistic map $f_n = e^{2\pi i \gamma_n}$ with Fibonacci quotient $\gamma_n = \frac{F_n}{F_{n+1}}$ and its associated kneading map Q_n:

(a) Let B_n denote the smallest repeating part of the image of Q_n. Then for $n \geq 6$:

(i) If n is even, then $B_n = B_{n-1}B_{n-2}$;

(ii) If n is odd, then $B_n = B_{n-2}B_{n-1}$;

(b) Let $v(f_n) = \overline{v_1 v_2 v_3 \cdots}$ be the kneading invariant of f_n. Then:

(i) $v_{F_k} = 1$ if $k \equiv 1 \bmod 2$ and $k \neq n + 1$;

(ii) $v_{F_k} = 0$ if $k \equiv 0 \mod 2$;

(iii) $v_{F_{n+1}} = 0$;

To get an idea about what these conjectures really mean, we list the first few values of B:

$\gamma = \frac{3}{5}$: $B_4 = 011$

$\gamma = \frac{5}{8}$: $B_5 = 01011$

$\gamma = \frac{8}{13}$: $B_6 = (01011)(011)$

$\gamma = \frac{13}{21}$: $B_7 = [(01011)][(01011)(011)]$

$\gamma = \frac{21}{34}$: $B_8 = [(01011)(01011)(011)][(01011)(011)]$

$\gamma = \frac{34}{55}$: $B_9 = [(01011)(01011)(011)][(01011)(01011)(011)(01011)(011)]$

Conjecture (a) just states that $B_6 = B_5 B_4$, and in order to obtain B_7, we take B_5 and paste to the left of B_6; then, in order to obtain B_8, we take B_6 and paste it to the right of B_7; then, in order to obtain B_9, we take B_7 and paste it to the left of B_8, $\cdots$.

For conjecture (b), let us look at the case $\gamma = \frac{F_4}{F_5} = \frac{3}{5}$ as an example. The kneading invariant for f_4 is $v(f_4) = (01000)(01000)\cdots$. Now, the $F_2 = 1st$ digit is 0 by (ii), the $F_3 = 2nd$ digit is 1 by (i), the $F_4 = 3rd$ digit is 0 by (ii), the $F_5 = 5th$ digit is 0 by (i) and (iii).

These conjectures above have not been proved yet. However, we have conducted a decent amount of computation and verification so far. For (a), we have verified the conjecture up until $\gamma = \frac{F_{17}}{F_{18}}$. For (b), we have verified the conjecture up until $\gamma = \frac{F_9}{F_{10}}$.

Shenxiong Li [10] has recently shown that Conjecture (a) is true if Conjecture (b) is true. Unfortunately, neither of us could prove Conjecture (b) up until this point.

Remark These conjectures are greatly inspired by examples of computations using Mathematica, provided by Shenxiong Li, which have been up until case $\frac{F_{17}}{F_{18}}$.

References

[1] Lluís Alsedà and Núria Fagella, *Dynamics on Hubbard trees*, Fundamenta Mathematicae **164**(2) (2000), 115–141.

[2] Gamaliel Blé, *External arguments and invariant measures for the quadratic family*, Discrete and Continuous Dynamical Systems. **11** (2004), 241–260.

[3] Michael Brin and Garrett Stuck, *Introduction to dynamical systems.* Cambridge University Press, Cambridge, 2002.

[4] Henk Bruin, *Combinatorics of the Kneading Map*, International Journal of Bifurcation and Chaos, **5** (1995), 1339–1349.

[5] Henk Bruin, *Topological conditions for the existence of invariant measures for unimodal maps*, Ergodic Theory and Dynamical Systems, **14**(3) (1994), 433-451.

[6] Henk Bruin, Alexandra Kaffl and Dierk Schleicher, *Existence of quadratic Hubbard trees*, Fundamenta Mathematicae **202** (2009), 251–279.

[7] Sylvain Crovisier, Pablo Guarino, and Liviana Palmisano, *Ergodic properties of bimodal circle maps*, Ergodic Theory and Dynamical Systems, **39**(6) (2019), 1462–1500.

[8] Franz Hofbauer, *The topological entropy of the transformation* $x \mapsto ax(1-x)$, Monatshefte für Mathematik, **90** (1980), 117–141.

[9] Mikhail Lyubich and John Milnor, *The Fibonacci unimodal map*, Journal of the American Mathematical Society, **6** (1993), 425-457.

[10] Shenxiong Li, private communication.

[11] Juan Rivera-Letelier, private communication.

[12] Roland Roeder, *Around the Boundary of Complex Dynamics*, `https://math.iupui.edu/$\sim$roederr/complex_dynamics.pdf`. Accessed on July 22, 2020.

[13] Danny Stoll, *A Brief Introduction to Complex Dynamics*, `https://math.uchicago.edu/$\sim$may/REUDOCS/Stoll.pdf`. Accessed on July 2, 2020.

[14] Chuanyi Wang, unpublished notes.