Ties between Knot Theory and Quantum Mechanics

Allen Shao

May 12, 2026

Contents

1 Introduction

This paper focuses on knot theory's applications in quantum mechanics. In section 2, we provide a quick overview of the basics of knot theory, and introduce the Alexander polynomial. In section 3, we discuss a derivation of the Jones polynomial using the bracket polynomial. In section 4, we discuss how knot theory is tied to quantum mechanics by examining how knot invariants can easily be translated into the language of amplitudes.

2 Knot Theory Basics

Let's begin by giving definitions for knots and links.

Definition 2.1. A **knot** is the image of an injective homeomorphism of the circle S^1 into $\mathbb{R}^3$. A **link** is a collection of knots which do not intersect in $\mathbb{R}^3$

This can be visualized as tying a knot in a physical length of rope, and gluing its ends together. The two simplest examples of knots are the un-knot (just a circle), and the trefoil, seen below.

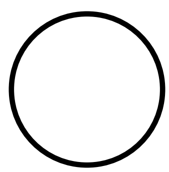


Figure 1: The Unknot

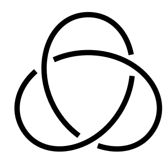


Figure 2: The Trefoil

While any knot is necessarily a link, a good example of a link containing multiple knots comes from the Borromean rings, shown below. Looking closely at the diagram, one can see that if any one of the rings are removed from the diagram, the other two become unlinked.

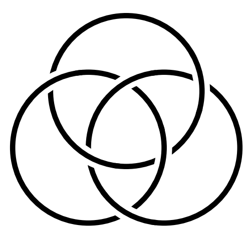


Figure 3: The Borromean Rings

In the previous figures, we've presented these links and knots in the form of a **link diagram**. A link diagram carries with it the same topological information that a knot carries by preserving the order of the crossings.

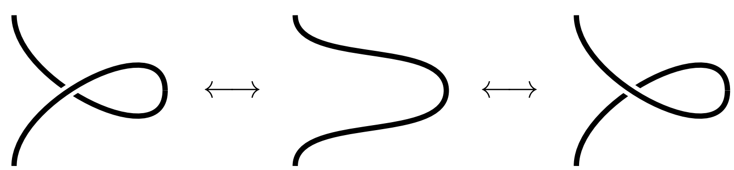
One of the central problems in knot theory is determining if two knots or links are the same in the sense that one can deform one knot into the other without breaking the knot altogether. To tackle this problem, one needs to consider the notion of an ambient isotopy.

Definition 2.2. Two links L_1 and L_2 are said to be **ambient isotopic** if there exists a continuous time-parameter family of embeddings starting with L_1 and ending with L_2.

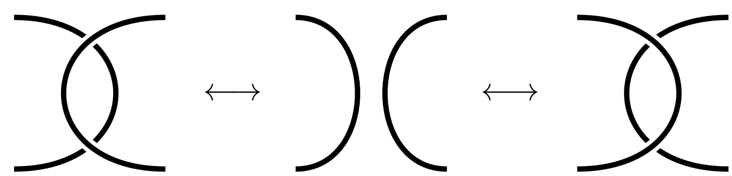
To determine if two links are ambient isotopic, it's often enough to look at their link diagrams, and perhaps find some way to manipulate them the way one might bend or unravel a physical knot. Kurt Reidemeister came up with three moves that do exactly that. These are fittingly called the Reidemeister moves.

Definition 2.3. The three Reidemeister moves are defined as follows:

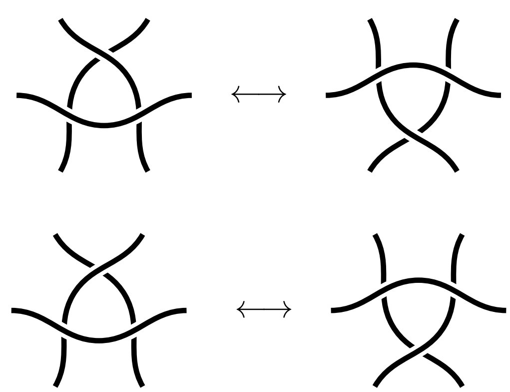
Move 1 - The Twist and Untwist:



Move 2 - The Poke and Unpoke:



Move 3 - The Slide:



Through the Reidemeister moves, we're given an excellent tool in our toolbox to help determine if two knots are ambient isotopic.

Theorem 2.4. [1]
Let K_1 and K_2 be two links in $\mathbb{R}_3$, with corresponding link diagrams D_1 and D_2. Then, K_1 and K_2 are ambient isotopic if and only if D_1 and D_2 are related by a finite sequence of Reidemeister moves.

Example 2.5. Consider the following link diagram K. It can be shown that K is ambient isotopic to the trefoil.

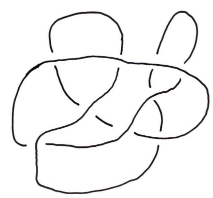


Figure 4: Our knot K

We first apply Reidemeister move 2, using the unpoke twice:

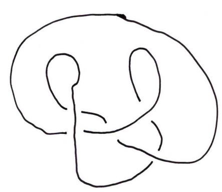


Figure 5: K after two applications of Reidemeister move 2

Using Reidemeister move 1, we can untwist the two components in the center of the diagram:

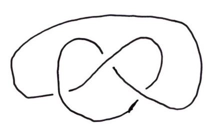


Figure 6: K after an application of two untwists

We can use Reidemeister move 3 to slide the top arc over our knot, and move it to the bottom. This final knot is ambient isotopic to the trefoil through inspection.

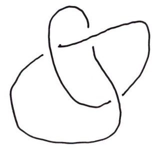


Figure 7: K after applying Reidemeister move 3

While these moves have many applications, they do have their limitations. For example, consider the following (very) unwieldy link diagram L in figure 8. While it's possible to show that L is just the unknot, getting there is an arduous task.

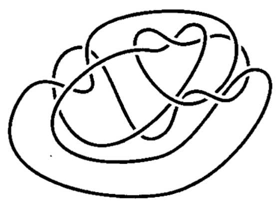


Figure 8: A terrible looking unknot from Adams' *The Knot Book* [1]

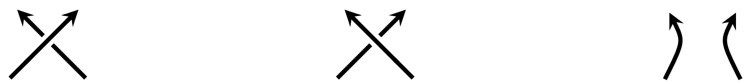
We need another way to distinguish two knots. To do this, we need to discuss the use of knot invariants, and link invariants. Simply put, an **invariant** s a well defined object associated with a given link that remains constant under ambient isotopy.

Two invariants of particular note are the Alexander polynomial, and the Jones polynomial. We start by defining the Alexander polynomial here, and save a discussion of the Jones Polynomial for the following section.

Definition 2.6. For an oriented link K, the Alexander polynomial $\Delta(K)$ is defined recursively using two rules:

1. $\Delta(U) = 1$
2. $\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0$

Where L_+, L_-, and L_0 are all knots which are identical except at one crossing. At this particular crossing, they exhibit the following differences:



L_+ L_- L_0

Example 2.7. Let K be an oriented trefoil. In figure 9, this can be seen by the green arrows attached to the link diagram. We illustrate the calculation of the Alexander polynomial by finding the Alexander polynomial of the trefoil $\Delta(K)$. In doing so, it is helpful to use a diagram called a skein tree which allows us to keep track of the local changes made to our knot. We start by considering the upper right crossing (circled in green), and identifying K with L_-. We can then make local changes to K as defined by L_+ and L_0. This is pictured below.

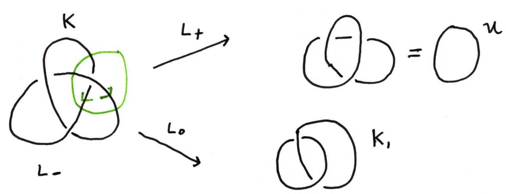


Figure 9: Skein Tree for the Trefoil K

By applying Reidemeister move 2, then move 1 on the link associated with L_+, one can easily identify this link as the unknot U. Let K_1 be the link diagram associated with L_0. By our two rules for calculating the Alexander polynomial, we have:

$$\Delta(K) = 1 + (t^{1/2} - t^{-1/2})\Delta(K_1) \tag{1}$$

However, we still don't know the Alexander Polynomial for K_1. As such, we repeat the above process on K_1, examining the crossing circled in green, and identifying K_1 with L_+.

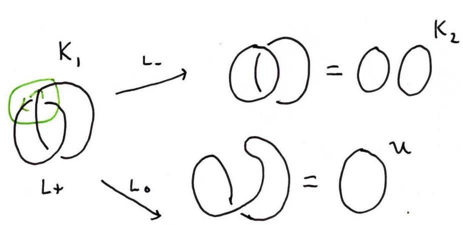


Figure 10: Skein Tree for K_1

Through an application of move 1, we can see that the link associated with L_0 in figure 10 is the unkot. This gives us:

$$\Delta(K_1) - \Delta(K_2) + (t^{1/2} - t^{-1/2}) = 0 \tag{2}$$

The skein relation to calculate $\Delta(K_2)$ is shown below. This time, we identify K_2 with L_0.

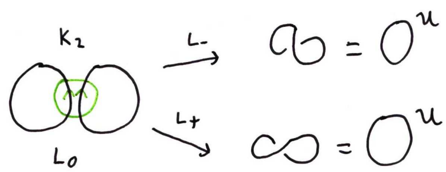


Figure 11: Skein Tree for K_2

This gives us:

$$\Delta(K_2) = 0 \tag{3}$$

Combining equations 1, 2, and 3, we can finally calculate the Alexander polynomial for our trefoil.

$$\Delta(K) = 1 - (t^{1/2} - t^{-1/2})(-t^{1/2} + t^{-1/2}) \tag{4}$$
$$= t + t^{-1} - 1 \tag{5}$$

Definition 2.8. Two link diagrams L_1 and L_2 are said to be mirror images of one another if L_2 can be obtained from L_1 by switching the order of every crossing of L_1, and vice versa.

The trefoil K and its mirror image K^* can be seen below. To avoid confusion, we will call the trefoil on the left the **left handed trefoil**, and the trefoil on the right the **right handed trefoil**.

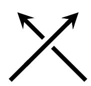
Figure 12: The left handed and right handed trefoil

Although we do not prove it here, it can be shown that the the left and right handed trefoils are not ambient isotopic. However, their Alexander polynomials are identical. This is true in general: the Alexander Polynomial cannot distinguish between a link diagram and its mirror image [1].

3 The Jones Polynomial

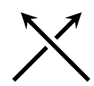
We now begin a discussion of the Jones Polynomial as a link invariant. This invariant will become pivotal in later discussions connecting quantum mechanics with knot theory. To gain a better understanding of how the polynomial works, it makes sense to follow Kauffman's construction of the bracket model of the Jones Polynomial. Before we do so, we must introduce the writhe.

Definition 3.1. Let K be a knot diagram. To this diagram, assign an (arbitrary) orientation. At each crossing, assign a crossing sign: either $+1$ or -1 as seen below.



L_+

Figure 13: Crossing Sign +1



L_-

Figure 14: Crossing Sign -1

The **writhe** $w(K)$ of K is the sum of these crossing signs for a given.

For a given link L, after orienting each of the link's constituent knots, the **writhe** $w(L)$ is again the sum of the crossing numbers in the associated link diagram.

While the writhe is clearly not invariant under the first Reidemeister move, it is invariant under the second and third.

Lemma 3.2. Let L be a link, and $w(L)$ be its writhe. $w(L)$ is invariant under Reidemeister moves two and three.

Proof. It is enough to look at the local changes made to L through Reidemeister moves 2 and 3. Refer back to Reidemeister moves 2 and 3. Regardless of the orientation attached to the two components in the figures, the sum of the crossing numbers in these local diagrams remain 0. This implies the writhe of the entire diagram L remains fixed before and after these moves. □

Although not as strong as ambient isotopy, being having two link diagrams be related by the second and third Reidemeister moves is quite useful.

Definition 3.3. Regular Isotopy is an equivalence relation where links L_1 and L_2 are said to be **regular isotopic** if and only if they are related by Reidemeister moves 2 and 3.

Let's now discuss the bracket polynomial. After performing some basic computations, we will see how the bracket polynomial naturally leads to the Jones polynomial.

Definition 3.4. Let K be a link. The bracket polynomial $\langle K\rangle(A, B, C) \in \mathbb{Z}[A, B, C]$ is a polynomial in 3 variables, defined recursively by the following three rules:

1. The bracket polynomial for the unknot is the identity.

$$\left\langle \bigcirc \right\rangle = 1$$

2. For any crossing in K, let $\left\langle \asymp \right\rangle$ and $\left\langle)\,(\right\rangle$ be the bracket polynomials by smoothing this crossing in the ways prescribed by the respective diagrams. Letting $\left\langle \times \right\rangle$ represent the bracket polynomial of K, we have:

$$\langle K\rangle = \left\langle \times \right\rangle = A\left\langle \asymp \right\rangle + B\left\langle)\,(\right\rangle$$

   For a crossing $\left\langle \times \right\rangle$ in K with the opposite ordering, we also have:

$$\langle K\rangle = \left\langle \times \right\rangle = A\left\langle)\,(\right\rangle + B\left\langle \asymp \right\rangle$$

3. The union of the unknot to any link diagram K is equivalent to multiplying $\langle K\rangle$ by C:

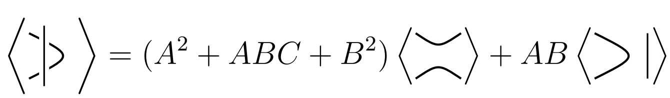
$$\left\langle \bigcirc \cup K \right\rangle = C\langle K\rangle$$

Often, we will use the bracket to denote local changes between two link diagrams.

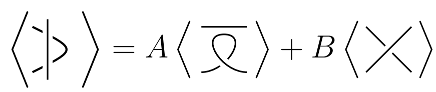
As it currently stands, the bracket polynomial is not invariant under any of the Reidemeister moves. However, we claim that there exists a certain relationship between A, B, and C such that the bracket will becomes invariant under moves 2 and 3. The following lemma will help us determine what that relation is.

Lemma 3.5. Let K and K' be two link diagrams which are related by Reidemeister move 2. (That is to say, K and K' are identical except at one region, where Reidemeister move 2 has been applied). Using the bracket Polynomial to represent local changes, define:

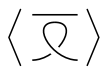
Then:

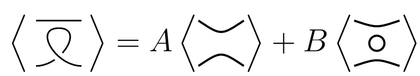


Proof. Let's start with the left hand side, and apply the second bracket polynomial rule.

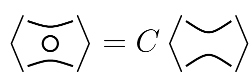


(6)

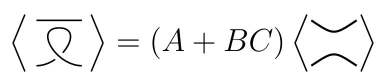
Note that we've applied the smoothings in $\langle K \rangle$ to its upper crossing. Applying rule 2 to , we have:



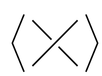
By rule 3, we should note that:

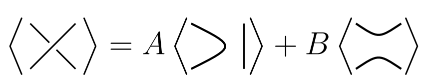


So that:



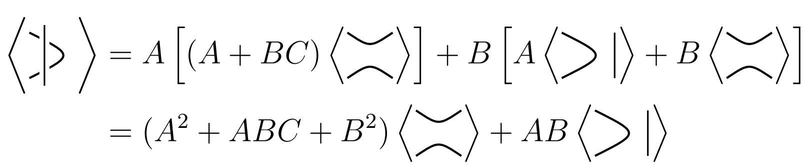
(7)

Similarly, applying the rule 2 to  gives us:



(8)

Substituting equations (7), and (8) into equation 6, the desired result follows almost immediately.



□

Through lemma 3.5, we can see that the bracket polynomial is invariant under the second Reidemeister move if and only if both of the following equations hold:

$$AB = 1 \tag{9}$$

$$A^2 + ABC + B^2 = 0 \tag{10}$$

By 9, we clearly have:

$$B = A^{-1} \tag{11}$$

(Where A^{-1} denotes the multiplicative inverse of A). Multiplying (10) by A, we have:

$$A^2 + C + A^{-2} = 0 \implies C = -A^2 - A^{-2} \tag{12}$$

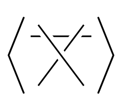
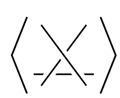
We repeat our findings in the following lemma.

Lemma 3.6. Let K and K' be two link diagrams, and let $B = A^{-1}$, and $C = -A^2 - A^{-2}$ in the bracket polynomial. If K and K' are related by the Reidemeister move 2, then $\langle K \rangle = \langle K' \rangle$.

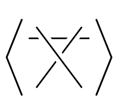
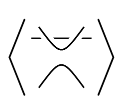
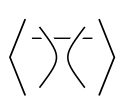
Under relations (11) and (12), we can also show that the bracket polynomial is invariant under the third Reidemeister move.

Lemma 3.7. Let K and K' be two link diagrams, and let $B = A^{-1}$, and $C = -A^2 - A^{-2}$. If K and K' are related by Reidemeister move 3, then their bracket polynomials are identical.

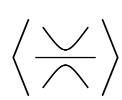
Proof. Using the bracket polynomials to define represent local changes define:

$\langle K \rangle :=$  $\langle K' \rangle :=$ 

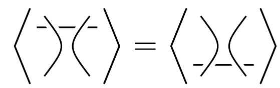
It is enough to show that $\langle K \rangle = \langle K' \rangle$. Showing that the bracket polynomial is invariant when the central crossing has its order reversed follows identically. By rule 2, we have:

 $= A$  $+ A^{-1}$    (13)

By Lemma 3.6, we can apply the unpoke and poke to see that:

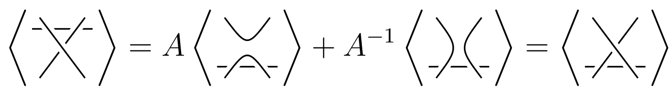
 $=$  $=$    (14)

Additionally, we have:



(15)

So, by a simple substitution, we have the desired result.



□

We've managed to modify the bracket polynomial so that it's invariant under Reidemeister moves 2 and 3. One more modification using the writhe gives us the Jones Polynomial, which is invariant under every move. The following definition follows as a natural consequence of the modifications we've made to the the bracket Polynomial.

Definition 3.8. Let K be a link diagram. The Jones polynomial $V_K(t) \in \mathbb{Z}[t, t^{-1}]$ for K is defined by:

$$V_K(t) = (-t^{-3/4})^{-w(K)} \langle K \rangle (t^{-1/4})$$

Where $w(K)$ is the writhe of K, and $\langle K \rangle (t^{-1/4})$ is the bracket polynomial with $A = t^{-1/4}$, $B = t^{1/4}$, and $C = -t^{1/2} - t^{-1/2}$.

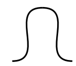
For the rest of this paper, we will be using the bracket polynomial in the variable t instead of A, B and C. Using our previous two lemmas, and our earlier work with the writhe, it's relatively easy to show that the Jones polynomial is invariant under ambient isotopy.

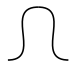
Theorem 3.9. The Jones polynomial is invariant under all three Reidemeister moves, and is thus a link invariant.

Proof. By Lemmas 3.6, and 3.7, we know that the bracket polynomial is invariant under the Reidemeister moves 2 and 3. Furthermore, lemma 3.2, tells us that the writhe is invariant under Reidemeister moves 2 and 3 as well. It then follows immediately that the Jones polynomial is invariant under these moves. We need only show that it's also invariant under move 1.

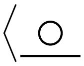
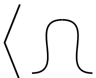
Let K and K' be two link diagrams which are related by move 1. Without loss of generality, suppose that K represents the link with the "twist", and

K' represents the link with the untwist. Using the bracket to represent local changes, we have:

$\langle K \rangle := \Big\langle$  $\Big\rangle$ $\qquad\qquad$ $\langle K' \rangle := \Big\langle$  $\Big\rangle$

Noting that —— and  are isotopic, calculating $\langle K \rangle$, gives us:

$\langle K \rangle = \Big\langle$  $\Big\rangle$

$= t^{-1/4}\Big\langle$  $\Big\rangle + t^{1/4}\Big\langle$  $\Big\rangle$ (by rule 2)

$= t^{-1/4}\left(-t^{1/2} - t^{-1/2}\right)\langle \text{——} \rangle + t^{1/4}\langle \text{——} \rangle$ (by rule 3)

$= (-t)^{-3/4}\langle \text{——} \rangle$

In particular, we have:

$$\langle K \rangle = t^{-3/4}\langle K' \rangle \tag{16}$$

Multiplying the above by $(-t^{-3/4})^{w(K)}$, we have:

$$V_K(t) = (-t^{-3/4})^{w(K)}\langle K \rangle = \langle K' \rangle = (-t^{-3/4})^{w(K')}\langle K' \rangle = V_{K'}(t) \tag{17}$$

□

At the end of section 2, we noted that the Alexander polynomial was unable to distinguish between a knot K and its mirror image. If K is a link diagram with mirror image K^*, we can see that $w(K) = -w(K^*)$. This implies that the Jones polynomial can in fact distinguish a link diagram from its mirror image, since we would have that $V_K(t) = V_{K^*}(t^{-1})$. This is one of many advantages that the the Jones polynomial provides over the Alexander polynomial.

Example 3.10. Let K be the left handed trefoil. We compute the Jones polynomial for K. We start by calculating the bracket polynomial for K. We examine the upper right crossing on the link, and obtain links K_1 and K_2 using a horizontal, and vertical smoothing respectively.

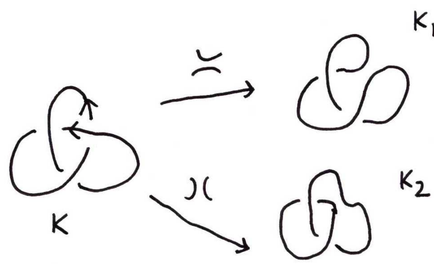


Figure 15: Application of smoothings to K

Applying rule 2 of the bracket polynomial gives:

$$\langle K\rangle = t^{-1/4}\langle K_1\rangle + t^{1/4}\langle K_2\rangle \tag{18}$$

As seen in the above figure, we see that $w(K) = -3$. This implies:

$$V_K(t) = (-t^{-3/4})^3\left[t^{-1/4}\langle K_1\rangle + t^{1/4}\langle K_2\rangle\right] \tag{19}$$

Referring back to figure 15, we can also see that $w(k_1) = 2$. In particular, we should notice that:

$$V_{K_1}(t) = (-t^{-3/4})^{-2}\langle K_1\rangle \tag{20}$$

Substituting 20 into 19 gives:

$$V_K(t) = t^{-1/4}(-t^{-3/4})^5 V_{K_1}(t) + t^{1/4}(-t^{-3/4})^3\langle K_2\rangle \tag{21}$$

However, we should recognize that K_1 is ambient isotopic to the unknot U. This can be seen by applying the untwist to the two crossings present in K_1. By by applying theorem 3.9, and noticing that the writhe of the unknot is 0, we have:

$$V_{K_1}(t) = V_U(t) = 1 \tag{22}$$

Substituting back into (21) gives:

$$V_K(t) = -t^{-4} + t^{1/4}(-t^{-3/4})^3\langle K_2\rangle \tag{23}$$

We can then use the bracket polynomial rules to determine $\langle K_2 \rangle$. We apply vertical and horizontal smoothings to the lower crossing of K_2, obtaining K_3 and K_4 respectively.

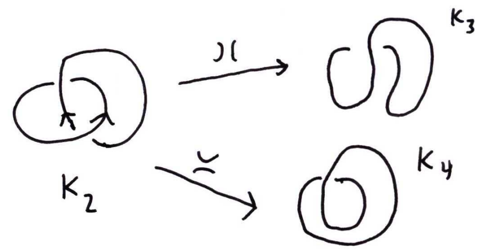


Figure 16: Application of smoothings to K_2

By rule 2, figure 16 gives:

$$V_K(t) = -t^{-4} + t^{1/4}(-t^{-3/4})^3 \Big[t^{-1/4}\langle K_3 \rangle + t^{1/4}\langle K_4 \rangle \Big] \tag{24}$$

Noting that $w(K_3) = +1$ and $w(K_4) = -1$, we can also see that:

$$V_{K_3}(t) = (-t^{-3/4})^{-1}\langle K_3 \rangle \quad \text{and} \quad V_{K_4}(t) = (-t^{-3/4})^{1}\langle K_4 \rangle$$

However, by applying an untwist to both K_3 and K_4, we can see that both knots are ambient isotopic to the unknot. By theorem 3.9, this simplifies our work quite a bit.

$$V_K(t) = -t^{-4} + t^{1/4}\Big[t^{-1/4}(-t^{-3/4})^4 V_{K_3}(t) + t^{1/4}(-t^{-3/4})^1 V_{K_4} \Big] \tag{25}$$

$$= -t^{-4} + t^{1/4}\Big[t^{-13/4} + t^{-5/4} \Big] \tag{26}$$

$$\implies V_K(t) = -t^{-4} + t^{-3} + t^{-1} \tag{27}$$

In [2], Kauffman works through a calculation of the Jones polynomial for the right-handed trefoil K^*, where he shows that:

$$V_{K^*}(t) = -t^4 + t^3 + t$$

While the Alexander polynomial was unable to distinguish between the left and right handed trefoil, the Jones polynomial can.

4 Vacuum-Vacuum Amplitudes

In [2], Kauffman states: "Our strategy for bringing forth relations between quantum theory and topology is to pivot on the Dirac bracket". In this section, we will do exactly that by discussing the relevance of the Jones Polynomial in creation-annihilation amplitudes. Much of our work will also stem from [3], [5], and [4]. We begin by placing a link diagram in a spacetime diagram, with time and space running horizontally and vertically on the page. First consider the humble unknot yet again.

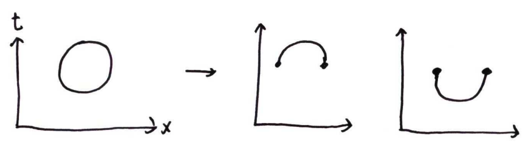


Figure 17: The unknot decomposed into a "cap" and "cup"

This circle represents a vacuum to vacuum process with two parts: the creation of two particles, and their annihilation. We call the bottom half of the circle the cup, and associate it with creation, and the top half the cap, and associate it with annihilation. It's then natural to look at the probability amplitude associated with this process by taking ket associated with the cup, and the bra associated with the cap. We call these $|a\rangle$ and $\langle b|$ respectively. Note that these amplitudes are a generalization of the amplitudes in quantum mechanics, but still they still obey two important rules [3].

1. If an event occurs such that it can be decomposed into a set of individual steps, then the amplitude of the given event is the product of the amplitudes of the individual steps.

2. If an event may occur in several disjoint alternative processes, then the amplitude of this event is the sum of the amplitudes of these processes.

In addition to looking at "caps" and "cups", we also want to examine the crossings that occur in a knot. These crossings can be thought of as an "interaction" between two particles. To each of these caps, cups, and crossings, we assign a matrix, where the indicies of these matrices vary over the possible states of the particle (e.g. spin), and the entries of these matrices are the probability amplitudes associated with these events. This is seen below.

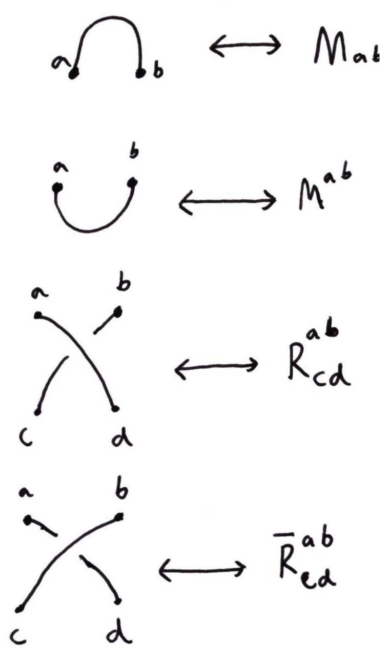
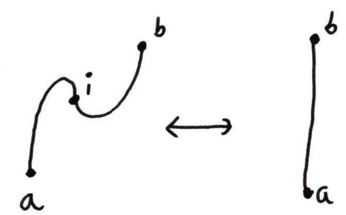


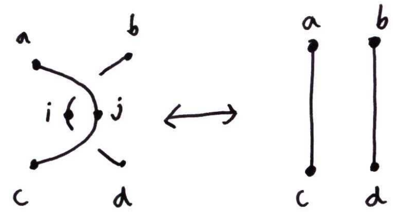
Figure 18: Matrix assignments for crossings, and caps and cups.

We must impose a few restrictions on our matrices if we want them to be invariant under regular isotopy. These restrictions can be seen below.

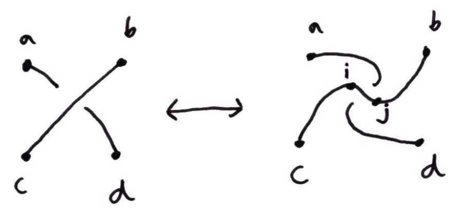
1. $M_{ai}M^{ib} = I^b_a = I$



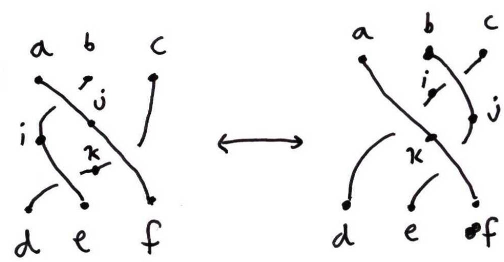
2. $R^{ab}_{ij} = \overline{R}^{ab}_{cd} = I^a_c I^b_d = I$



3. $\overline{R}^{ab}_{cd} = M_{ci} R^{ia}_{dj} M^{jb}$



4. $R^{ab}_{ij} R^{ie}_{kf} R^{ik}_{de} = R^{be}_{ij} R^{ai}_{dk} R^{kj}_{ef}$



Note that restriction 4 can be viewed as the Yang-Baxter equation, which is incredibly important to quantum mechanics. It's possible to choose our matrices M_{ab} and M^{ab} such that they obey relations similar to the bracket polynomial. The easiest such choice is as follows:

$$M_{ab} = M^{ba} = \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix}$$

As soon as we identify M_{ab} and M^{ba}, our R matrices for crossings begin to obey a very familiar relationship:

$$R^a_{cd}b = AM^{ab}M_{cd} + A^{-1}I^a_c I^b_d$$

With this, we're now able to identify the Jones polynomial with amplitudes associated with vacuum-vacuum processes.

5 Conclusion

Throughout this paper, we've managed to explore some of the basics of knot theory by working with the Alexander polynomial, calculating the Alexander polynomial for the left-handed trefoil. We then derived the Jones polynomial by using the bracket polynomial. We were then able to connect the Jones polynomial to vacuum-vacuum amplitudes only by specifying two matrices: M_{ab} and M^{ab}. The connections between knot theory and quantum mechanics are much deeper than just the Jones polynomial. In particular, the presence of the Yang-Baxter equation in restriction (4) indicates this! The Jones polynomial is only one example of what is called a quantum link invariant, and the study of these objects reveals a great deal about quantum mechanics.

Acknowledgments

- I would like to thank my advisors Jonathan Pakianathan and Elizabeth Tatum for their continued support and patience throughout the semester.
- Mom and Dad: Thank you for encouraging me to study hard, and helping me find my path forward.
- Professor Naomi Jochnowitz: Thank you for putting up with me in your algebra classes, and encouraging me to not give up on math.
- I would also like to thank the "POA kids" for their endless math and physics help throughout all four years of college.

References

[1] C.C. Adams. *The Knot Book*. W.H. Freeman, 1994. ISBN: 9780821886137.

[2] Louis Kauffman. "Knot Theory". URL: `https://homepages.math.uic.edu/~kauffman/Knots.pdf`.

[3] Louis Kauffman. "Knots and Physics". In: *Note di Matematica* (1989). ISSN: 1123-2536. URL: `http://siba-ese.unisalento.it/index.php/notemat/article/viewFile/1559/1339`.

[4] Louis H Kauffman. *Knots and Physics*. 4ed. Knots and everything 53. World Scientific Publishing Company, 2013. ISBN: 9814383007.

[5] J. Mostovoy S. Chmutov S. Duzhin. “Introduction to Vassiliev Knot Invariants”. In: (). URL: https://doi.org/10.48550/arXiv.1103.5628.