Minimal Surfaces in $\mathbb{R}^3$

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1 Introduction

The goal of this note is to give a brief introduction to the theory of minimal surfaces in $\mathbb{R}^3$, and to show how one would go about generalizing the theory to oriented k-dimensional hypersurfaces in $\mathbb{R}^n$. We will also touch on some interesting problems, applications, and results related to the field. We will touch on Plateau's laws for soap films, Plateau's problem, the Double Bubble theorem, and finally show some pictures.

First let's say what we mean by a minimal surface. Roughly, a minimal surface is one for which each point has neighborhood which, as itself a surface, has smallest possible area among surfaces sharing the same boundary. That is, a minimal surface is one which locally minimizes area. To have a concrete idea of how to come up with such things, let's begin with the classical derivation of the minimal surface equation as the Euler-Lagrange equation for the area functional, which is a certain PDE condition due to Lagrange circa 1762 describing precisely which functions can have graphs which are minimal surfaces. By viewing a function whose graph was a minimal surface as a minimizing function for a certain area functional (i.e. a smooth map from a function space into $\mathbb{R}$ which measures "area"), we are able to use techniques from the calculus of variations to show that for an oriented surface, locally minimizing area is equivalent with the vanishing of the divergence of the unit normal vector to that surface. It is from this perspective that we can generalize the situation to arbitrary k-dimensional submanifolds of $\mathbb{R}^n$, and in fact to any k-dimensional submanifold of a Riemannian manifold.

Recall that any surface is locally the graph of some smooth function. This is a relatively easy consequence of the inverse function theorem, based on what it means to be a regular parametrized surface, that is, where the two co-ordinate vectors for the parametrization span a two dimensional vector space. Then by the above discussion, to find more general minimal surfaces it will suffice to find minimal graphs of functions. Let $U \neq \varnothing$ be a bounded open set in $\mathbb{R}^2$, and let $\phi : U \to \phi(U) \subset \mathbb{R}^3$ be any diffeomorphism. We should think of the the set $\phi(\partial U)$ as a closed "wire" in space. In the setting of a regular surface, this role will be filled by the boundary of the neighborhood in which the surface can be viewed as the graph of a function. In any case, let's consider the set H of functions $h \in C^\infty(\overline{U})$ such that $h(\partial U) = \phi(\partial U)$ (i.e. whose boundaries are the given wire). We'll say that the graph of $f \in H$ has **minimal area** with respect to ∂U if

$$\int_{\overline{U}} d\Gamma_f \leq \int_{\overline{U}} d\Gamma_h$$

for each $h \in H$ (here Γ_h is just the graph h, with the usual parametrization, and $d\Gamma_h$ is the area element $\sqrt{\det(I)}\, dx \wedge dy$, I being the first fundamental form of Γ_h). Say that f has minimal area if its graph does. In other words, f has minimal area with respect to U if it is a minimizer of the area functional $a : H \to \mathbb{R}$ defined by

$$a(h) = \int_{\overline{U}} d\Gamma_h.$$

Here is how the minimizing functions were characterized. Suppose f minimizes a. Then chose any other smooth $\eta \in H$ so that η's value on ∂U is identically 0. We can get a parametrized family of functions $z_t = f + t\eta \in H$. The term $t\eta$ is called a variation of f. Since f was chosen to be a minimizing function, $t = 0$ must be a critical point for the function

$$\begin{aligned} A(t) = a(z_t) &= \int_{\overline{U}} d(\Gamma_{z_t}) \\ &= \int_{\overline{U}} \sqrt{1 + (z_t)_x^2 + (z_t)_y^2}\, dx \wedge dy \\ &= \iint_{\overline{U}} \sqrt{1 + |\nabla z_t|^2}\, dxdy. \end{aligned}$$

That $t = 0$ is critical means that

$$\begin{aligned} \frac{d}{dt} A(t)\bigg|_{t=0} &= \frac{d}{dt} \iint_{\overline{U}} \sqrt{1 + |\nabla z_t|^2}\, dxdy \bigg|_{t=0} \\ &= \iint_{\overline{U}} \frac{d}{dt} \sqrt{1 + |\nabla z_t|^2}\bigg|_{t=0} dxdy \\ &= 0. \end{aligned} \tag{1}$$

It's not hard to check that

$$\begin{aligned} \frac{d}{dt} \sqrt{1 + |\nabla z_t|^2} &= \frac{-\langle \nabla z_t, \nabla \eta \rangle}{\sqrt{1 + |\nabla z_t|^2}} \\ &= -\left\langle \frac{\nabla z_t}{\sqrt{1 + |\nabla z_t|^2}}, \nabla \eta \right\rangle \end{aligned}$$

so that after evaluating at $t = 0$, equation (1) becomes

$$\begin{aligned} \frac{d}{dt} A(t)\bigg|_{t=0} &= \iint_{\overline{U}} \frac{d}{dt} \sqrt{1 + |\nabla z_t|^2}\bigg|_{t=0} dxdy \\ &= \iint_{\overline{U}} -\left\langle \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}, \nabla \eta \right\rangle dxdy \\ &= 0. \end{aligned}$$

Now, applying the formula $\operatorname{div}(\varphi\mathbf{F}) = \langle \nabla\varphi, \mathbf{F}\rangle + \varphi\big(\operatorname{div}(\mathbf{F})\big)$ (where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is smooth and $\mathbf{F}$ is a smooth vector field) we get

$$-\left\langle \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}, \nabla\eta \right\rangle = \eta\left(\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) - \operatorname{div}\left(\eta\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right)$$

so that integrating both sides and applying the divergence theorem, we can rewrite equation (1) again as

$$\begin{aligned}
\frac{d}{dt}A(t)\bigg|_{t=0} &= \iint_{\overline{U}} -\left\langle \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}, \nabla\eta \right\rangle \, dxdy \\
&= \iint_{\overline{U}} \eta\left(\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, dxdy - \iint_{\overline{U}} \operatorname{div}\left(\eta\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, dxdy \\
&= \iint_{\overline{U}} \eta\left(\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, dxdy - \int_{\partial U} \eta\left(\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, ds. \\
&= \iint_{\overline{U}} \eta\left(\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, dxdy \\
&= 0
\end{aligned}$$

Notice that we could conclude $\int_{\partial U} \eta\left(\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right) \, ds = 0$ because η was chosen to vanish on ∂U. We can now apply the fundamental lemma of the calculus of variations, which states that if $\Omega \in \mathbb{R}^n$ is an open set, and f is a continuous function on Ω, then if for all compactly supported functions $\varphi \in C^\infty(\Omega)$ one has

$$\int_\Omega \varphi(\mathbf{x})f(\mathbf{x})d\mathbf{x},$$

then f is identically 0 on Ω. In our present situation, any function φ which is compactly supported on U will extend to a function $\overline{\varphi} \in C^\infty(\overline{U})$ which will vanish on ∂U, so we can conclude, as desired, that

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right) = trace(D) = 0 \qquad (2)$$

on all of $\overline{U}$ if and only if f is a minimizer of the area functional a. Equation (2) is called the **minimal surface equation**. Here, D is the Jacobian matrix of the function $\frac{f}{\sqrt{1+|\nabla f|^2}}$. If we compute all the partials and simplify, we get the following PDE

$$f_{xx}(1+f_y^2) - 2f_xf_yf_{xy} + f_{yy}(1+f_x^2) = 0. \qquad (3)$$

It can be shown that any solution to this equation must be real-analytic. This demonstrates that functions of minimal area are not so readily abundant. As a final note, it turns out that the shape operator of any graph of minimal area parametrized via $r(x,y) = (x,y,f(x,y))$ has no trace anywhere, because the trace of this map turns out to be *exactly* the minimal surface equation (3) up to some non-zero multiple. Let's see this. It's not hard to compute the coefficients of the first and second fundamental forms of a function as

$$\begin{array}{ccc} E = 1 + f_x^2 & F = f_x f_y & G = 1 + f_y^2 \\ e = \frac{f_{xx}}{\sqrt{1+f_x^2+f_y^2}} & f = \frac{f_{xy}}{\sqrt{1+f_x^2+f_y^2}} & g = \frac{f_{yy}}{\sqrt{1+f_x^2+f_y^2}} \end{array}$$

and therefore the mean curvature is

$$\frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{f_{xx}(1+f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1+f_x^2)}{2(1+f_x^2+f_y^2)^{\frac{3}{2}}} = 0.$$

Since the trace of the shape operator of a surface is independent of choice of parametrization, one may instead say a parametrized surface is minimal whenever its shape operator has no trace anywhere. This definition provides us with some more geometric perspective. Mean curvature tells us the average value of the normal curvatures of the surface over all possible directions. It is well known that the mean curvature is determined by the average of the eigenvalues of the shape operator, which correspond respectively to the maximum and minimum of the principal curvatures of the surface at a point over all possible directions. This directions of maximal and minimal curvature are called **principal directions**. So namely, the trace of the shape operator being 0 everywhere tells us that the principal curvatures at each point cancel each other. This means that for all points on the surface, the curvature of a curve in the principal direction of the surface will be the negative of the curvature in the other principal direction.

Let us now try to generalize the situation. Basically the same procedure we outlined above will make sense for an arbitrary k-submanifold of $\mathbb{R}^n$, but first we have to say clearly what the "area" (volume more generally) element is for such a thing. For a regular surface, the most natural definition is that the area element would be the size of the area of the parallelogram spanned by the co-ordinate tangent vectors to the surface at a point times the usual area element $dx \wedge dy$. In general, any k-dimensional linear subspace of some n-dimensional inner product space is itself an inner product space via the induced inner product on the bigger space. We already know from linear algebra that the area of the parallelogram spanned by the linearly independent set $\{v_1, \ldots, v_k\}$ of vectors is gotten by taking the size of the determinant of the symmetric matrix which encodes how the inner product operates on the subspace generated by the vectors. The matrix in other words encodes all of the information about lengths and angles in the span of the set. This matrix is called the **Gram matrix** of the linearly independent set. It is given by

$$G(v_1, \ldots, v_k) = (v_1 \ \ldots \ v_k)^T (v_1 \ \ldots \ v_k),$$

or perhaps more clearly, $[G(v_1, \ldots, v_k)]_{ij} = \langle v_i, v_j \rangle$. All local geometric properties (i.e. those relating to concepts of size (length, area, volume etc) and angle) of a submanifold of $\mathbb{R}^n$ are described by the way the canonical basis for the tangent space at a point inherits an inner product structure from $\mathbb{R}^n$. In the very particular situation of a regular surface

S parametrized via $r : \mathbb{R}^2 \to \mathbb{R}^3$, we have that the vectors $r_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ and $r_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ form a basis for the tangent space at the point $p = r(u, v)$. The corresponding Gram matrix is

$$
\begin{aligned}
G(r_u, r_v) = (r_u\ r_v)^T (r_u\ r_v) &= \begin{pmatrix} r_u^T r_u & r_u^T r_v \\ r_v^T r_u & r_v^T r_v \end{pmatrix} \\
&= \begin{pmatrix} \langle r_u, r_u \rangle & \langle r_u, r_v \rangle \\ \langle r_v, r_u \rangle & \langle r_v, r_v \rangle \end{pmatrix} \\
&= I_p(S).
\end{aligned}
$$

So that the Gram matrix is exactly the first fundamental form of the surface at a point. This is clearly the right generalization. We can now define the area element for an arbitrary immersed k-dimensional submanifold $\mathcal{S}^k$ of $\mathbb{R}^n$ as simply $d\mathcal{S} = \sqrt{\det\left(G(r_{u_1}, \ldots, r_{u_k})\right)}\ dV$ where dV is the usual volume form on $\mathbb{R}^n$ and r is the immersion, r_{u_i} are the basis vectors for the tangent space. When our submanifold is an oriented hypersurface, we are able to basically repackage the above argument to get that a function minimizing the “volume functional” $v(h) = \int_{\overline{U}} d\Gamma_h$ (where now h is a smooth $\mathbb{R}^{n-1} \to \mathbb{R}$ function) is equivalent with the divergence of the unit normal vector field (which we can define because we're working with an oriented hypersurface) of the graph of that function vanishing. So we can study minimal hypersurfaces too.

2 The Helicoid and the Catenoid

Let's look closely at an example of a minimal surface. Suppose f is minimal, and consider a level curve of f given implicitly by $f(x, y) = c$ for some constant c. It's not hard to compute the curvature k of this curve as

$$
k = \frac{-f_{xx} f_y^2 + 2 f_x f_y f_{xy} - f_{yy} f_x^2}{|\nabla f|^3}.
$$

Should f have level curves which are straight lines, it must be that $k \equiv 0$, implying that f is harmonic. To see this, simply notice that we can simplify equation (1) to

$$
\begin{aligned}
f_{xx}(1 + f_y^2) - 2 f_x f_y f_{xy} + f_{yy}(1 + f_x^2) &= (f_{xx} + f_{yy}) + (f_{xx} f_y^2 - 2 f_x f_y f_x + f_x^2) \\
&= f_{xx} + f_{yy} \\
&= 0.
\end{aligned}
$$

It turns out that the only harmonic functions with linear level curves are of the form

$$
f(x, y) = \lambda \arctan\left(\frac{y - y_0}{x - x_0}\right) + \mu
$$

for constants $\lambda, \mu, x_0, y_0 \in \mathbb{R}$.

If $\lambda \neq 0$, the graph of such a function is called a **helicoid** (otherwise we get a plane, a somewhat trivial minimal surface). To see what's happening, let's work with a more illuminating parametrization. It can be shown that the following parametrization gives the same surface as f.

Remark. *We are taking the whole plane as our domain here, but there's no need for that. We could instead consider compact helicoids and get the same results.*

$$
\begin{aligned}
x &= u\cos v + x_0 \\
y &= u\sin v + y_0 \\
z &= \lambda v + \mu.
\end{aligned}
$$

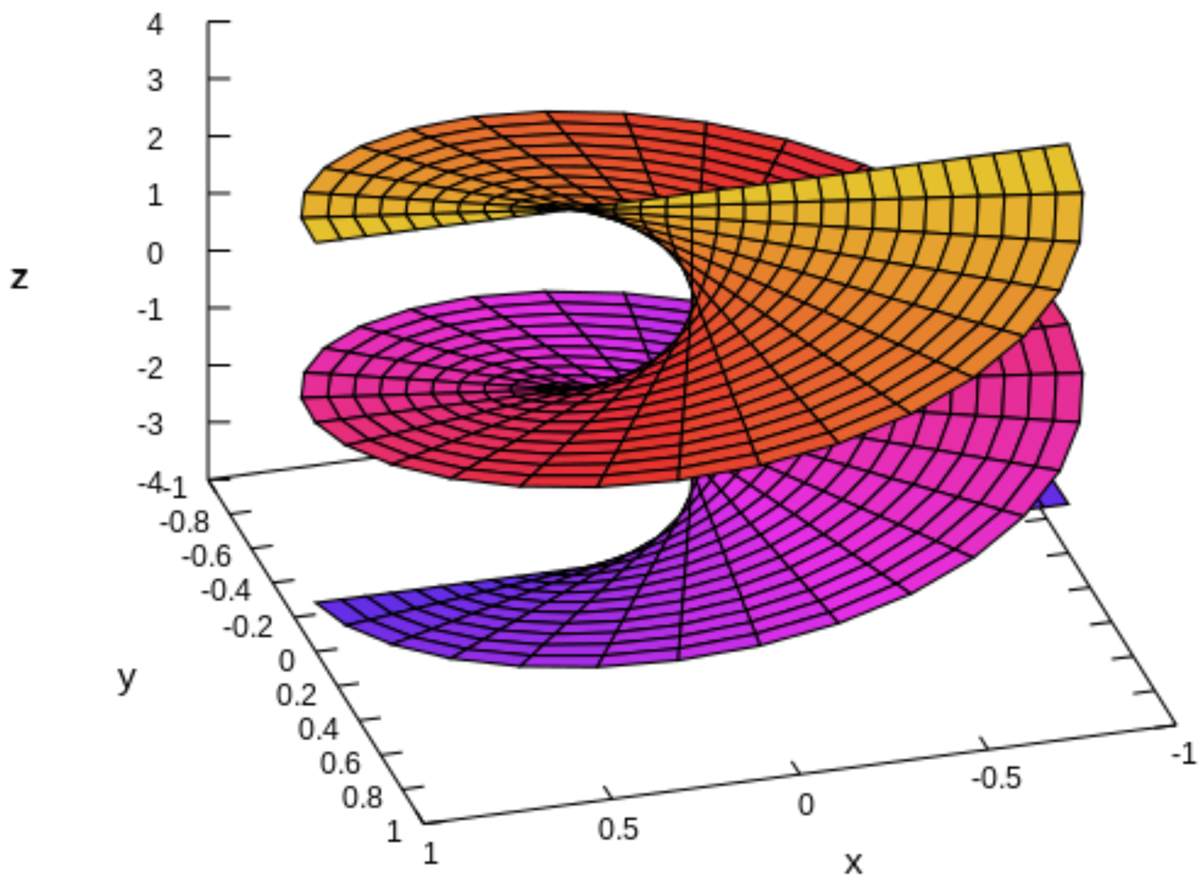


Figure 1: A Helicoid

This surface is special in that it is **ruled**, meaning that it is gotten by sweeping a line along some base curve, admitting a parametrization of the form $r(u,v) = b(v) + u\delta(v)$, where b is an analytic curve (called the directrix or base curve) perpendicular to the rulings (i.e. the individual lines) of the surface, and $\delta(v)$ is a unit vector in the direction of the ruling through $r(u,v)$.

Indeed, notice that we can rewrite the given parametrization r for the helicoid as $r(u,v) = (0,0,v) + u\big(\cos v, \sin v, 0\big)$, so it is ruled, and indeed it can be easily seen as a partition into lines along helixes. Of particular interest is that any ruled minimal surface (except the plane) can be isometrically embedded into some helicoid. Thus the helicoid actually characterizes all ruled minimal surfaces.

As an exercise in computing surface curvatures, let's verify that the helicoid has no mean curvature. In what follows, put $X = r(\mathbb{R}^2) \subset \mathbb{R}^3$. Since we'll be working with a single smooth regular parametrization $r : \mathbb{R}^2 \to \mathbb{R}^3$, we get a natural choice of smooth non-vanishing unit normal vector field $\tilde{N}$ gotten by taking the cross product of the basis vectors (r_u, r_v) in T_pX. Namely,

$$\tilde{N}(u,v) = \frac{r_u \wedge r_v}{\|r_u \wedge r_v\|}.$$

This gives us the "outward facing" normals. Note $\tilde{N}$ only has the properties we claim because r is regular (i.e. r_u and r_v are linearly independent). Then the Gauss map (with respect to the orientation $\tilde{N}$) is the smooth map $N : X \to \mathbb{S}^2$ gotten by $N(u,v) = \tilde{N}(u,v)$. We have $r_u = \big(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\big) = \big(\cos v, \sin v, 0\big)$ and $r_v = \big(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\big) = \big(-u\sin v, u\cos v, \lambda\big)$. Thus,

$$r_u \wedge r_v = \big(\lambda \sin v, -\lambda \cos v, u\big),$$

so that

$$N(r(u,v)) = \frac{r_u \wedge r_v}{\|r_u \wedge r_v\|} = \big(\frac{\lambda \sin v}{\sqrt{\lambda^2 + u^2}}, \frac{-\lambda \cos v}{\sqrt{\lambda^2 + u^2}}, \frac{u}{\sqrt{\lambda^2 + u^2}}\big) = (N_1, N_2, N_3).$$

The idea here is to generalize Euler's characterization of curvature as the infinitesimal change of angle between tangent vectors with respect to position along the curve in order to get a curvature measure. Then the natural generalization for the "curvature" of a surface at a point $p = r(u,v)$ would be the Jacobian determinant of N (the Jacobian matrix dNp of N is called the **shape operator**). This number is called the **Gaussian curvature** of X at p. We might as well start by computing the Gaussian curvature, as we need to coefficients of the fundamental forms in order to compute mean curvature anyway. Recalling that by definition $dN_p(r_u) = N_u$ and $dN_p(r_v) = N_v$ where

$$N_u = \big(\frac{\partial N_1}{\partial u}, \frac{\partial N_2}{\partial u}, \frac{\partial N_3}{\partial u}\big) = \big(\frac{-\lambda u \sin v}{(\sqrt{\lambda^2 + u^2})^3}, \frac{\lambda u \cos v}{(\sqrt{\lambda^2 + u^2})^3}, \frac{1}{\sqrt{\lambda^2 + u^2}} \cdot (1 - \frac{u}{\sqrt{\lambda^2 + u^2}}) \cdot (1 + \frac{u}{\sqrt{\lambda^2 + u^2}})\big)$$

and

$$N_v = \big(\frac{\partial N_1}{\partial v}, \frac{\partial N_2}{\partial v}, \frac{\partial N_3}{\partial v}\big) = \big(\frac{\lambda \cos v}{\sqrt{\lambda^2 + u^2}}, \frac{\lambda u \sin v}{\sqrt{\lambda^2 + u^2}}, 0\big).$$

We seek to write $N_u = a_{11}r_u + a_{21}r_v$ and $N_v = a_{12}r_u + a_{22}r_v$, then the Gaussian curvature K will simply be the number $a_{11}a_{22} - a_{12}a_{21}$. Thankfully we can appeal to the Weingarten equations to compute these a_{ij}. That is,

$$a_{11} = \tfrac{eG-fF}{EG-F^2} \quad a_{12} = \tfrac{fG-gF}{EG-F^2}$$

$$a_{21} = \tfrac{fE-eF}{EG-F^2} \quad a_{22} = \tfrac{gE-fF}{EG-F^2},$$

where (E, F, G) and (e, f, g) are the coefficients of the first and second fundamental forms of X, respectively. Recalling that

$$E = r_u \cdot r_u \quad F = r_u \cdot r_v \quad G = r_v \cdot r_v$$

$$e = N_u \cdot r_u \quad f = N_u \cdot r_v \quad g = N_v \cdot r_v,$$

we can easily conclude that $E = 1$, $F = 0$ and $G = \lambda^2 + u^2$, while $e = 0$, $f = -\frac{\lambda}{\sqrt{\lambda^2+u^2}}$ and $g = 0$. Then also $EG - F^2 = G$, and $eG - fF = 0, fG - gF = fG, fE - eF = f, gE - fF = 0$, so that finally we can conclude

$$a_{11} = 0 \quad a_{12} = f$$

$$a_{21} = \tfrac{f}{G} \quad a_{22} = 0,$$

thus the Gaussian curvature is $-\frac{f^2}{G} = -\frac{\lambda^2}{(\lambda^2+u^2)^2}$. Finally, the eigenvalues of dNp (i.e. the principal curvatures) are easily seen to be $\pm\frac{\lambda}{\lambda^2+u^2}$, and thus their sum is 0, meaning that the helicoid X has no mean curvature anywhere, and hence is a minimal (parametrized) surface as desired. (Notice here that it is crucially important that dN_p is a self-adjoint linear operator, because that allows us to orthonormally diagonalize dN_p thanks to the spectral theorem. This allows us to conclude the principal curvatures agree with the eigenvalues of dN_p.)

As a final note, since once again dN_p is self-adjoint, we can express the Gaussian curvature as simply the product of the eigenvalues. This tells us immediately that the Gaussian curvature of any minimal surface is going to be negative, meaning that in particular each point of a minimal surfaces is a saddle point.

Now, we've seen that angles and lengths of tangent vectors on a surface are completely determined by the first fundamental form. It stands to reason that any regular parametrized surface $s(u, v)$ which shares the same fundamental form with r at a point must be **locally isometric** to r. That is, they measure distances and angles exactly the same way. As an ant walked around from point to point, it would not be able to distinguish between these two surfaces geometrically.

It turns out that the helicoid possess a certain parametrization, called a **Weierstrass parametrization**, which generated a parametrized family of minimal surfaces, each locally isometric to the helicoid. One might imagine the family as a system of local isometric deformations of the helicoid into a "dual" minimal surface, in this case called a **catenoid**. Specifically, a Weierstrass parametrization has the form

$$x_k(\zeta) = Re\bigg(\int_0^\zeta \phi_k(z)dz\bigg) + c_k$$

where $\phi_1 = \frac{f(1-g^2)}{2}$, $\phi_2 = \frac{if(1+g^2)}{2}$, and $\phi_3 = fg$, where f is analytic, g is meromorphic (analytic except at a set of isolated points), and fg^2 is analytic. This parametrizes an associate family of pairwise locally isometric minimal surfaces by

$$x_k(\zeta, \theta) = Re\bigg(e^{i\theta}\int_0^\zeta \phi_k(z)dz\bigg) + c_k$$

for $\theta \in [0, 2\pi]$ and $c_k \in \mathbb{R}$.

It can be shown that the parametrization

$$\begin{aligned} x &= \cos\theta \sinh v \sin u + \sin\theta \cosh v \cos u \\ y &= -\cos\theta \sinh v \cos u + \sin\theta \cosh v \sin u \\ z &= u\cos\theta + v\sin\theta. \end{aligned}$$

has this form, with $(u, v) \in (-\pi, \pi] \times \mathbb{R}$, $\theta \in (-\pi, \pi]$. The values at $\theta = 0, \pi$ correspond to helicoids, and the values at $\theta = \pm\frac{\pi}{2}$ correspond to these aformentioned catenoids.

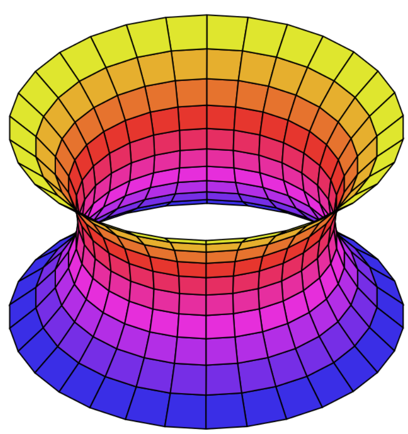


Figure 2: A Catenoid. It can be realized as the surface of revolution generated by a catenary curve.

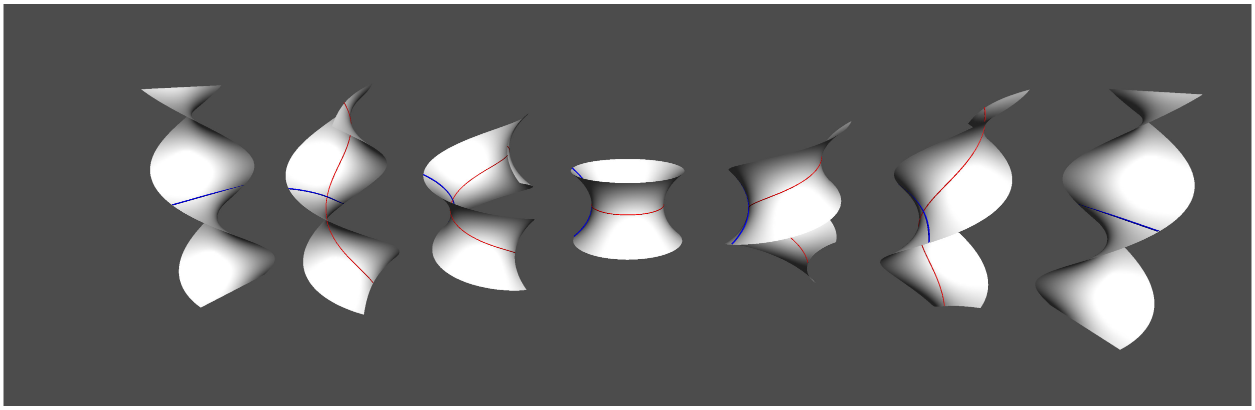
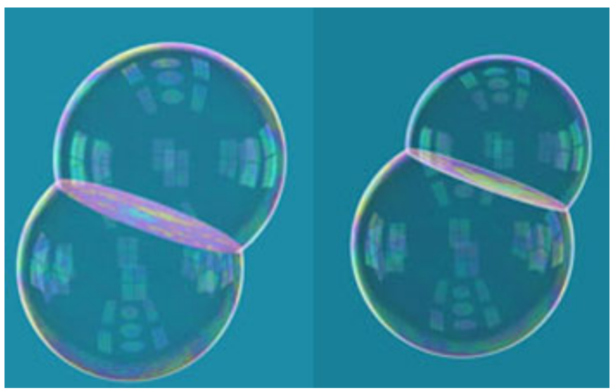


Figure 3: A picture of the deformation θ.

3 Soap Films and Plateau's Problem



A **soap film** is the surface you'd get from dipping a wire in some bubble soap.
The **Young-Laplace** equation

$$\Delta p = 2\gamma H$$

relates the surface tension γ and mean curvature H of an interface between two fluids (like a soap film) to the difference between the capillary pressures on either side of the interface. This equation physically captures why soap films should be minimal surfaces, just take the surrounding fluid to be air. The same pressure is exerted from either side of the film, so there is no pressure differential and hence the film arranges itself to minimize area. As an example, imagine that two soap bubbles have joined together. The interface along with they meet is an example of a soap film (where the "wire" is the boundary along which the bubbles meet), and the complex they form is called a **double bubble**.

The **Double bubble theorem**, proved in 2002, asserts that the smallest surface enclosing and separating two given volumes is exactly the "double bubble" described above. Similar to the isoperimetric inequality.

Soap films were first studied extensively in the early 19th century by the Belgian physicist Joseph Plateau, who proposed that soap films always exhibited certain phenomena when taken in groups. These properties were called **Plateau's Laws**, and they are as follows:

- Soap films are smooth surfaces.
- Soap films have constant mean curvature everywhere.
- Soap films always meet in threes, at a mutual dihedral angle of $120°$.
- The four lines along which soap films meet intersect at a common vertex, forming a tetrahedron in the sense that, taken pairwise, the lines meet at tetrahedral angles. ($109.47°$)

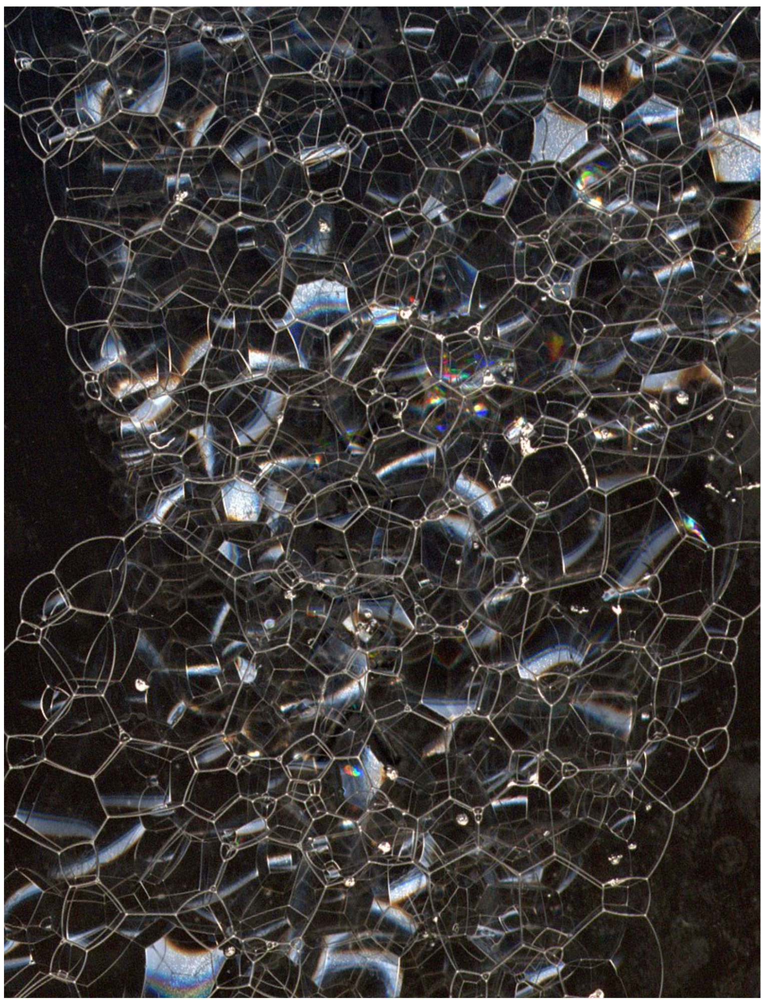


Figure 4: A foam of soap films demonstrating the above phenomena

It is a very amazing fact that these conditions can in fact be proven mathematically by modeling soap films in a way which reflects a certain area minimizing principle for surfaces. This was done in 1976 by Frederick Almgren and Jean E. Taylor. The principle they chose to model was that a physical system would remain in a given geometric configuration only if it cannot readily change to a configuration which requires less energy to maintain. Ignoring the effects of gravitational potential energy on the configuration of the surface, the only other thing that contributes is the compressional energy supplied by the surrounding air (a static fluid), and the surface energy of the suspended fluid (soap in our case). Thus, when there is no contribution from the air pressure, the surface energy per unit surface area must be minimized. Among other things, this has the effect of minimizing the area of the surface along the boundary, i.e. these soap films are minimal surfaces.

4 Plateau's Problem

Perhaps a natural question to question to ask at this point is whether there is always a minimal surface with a given boundary. After all, our intuition about soap films would support this being the case.

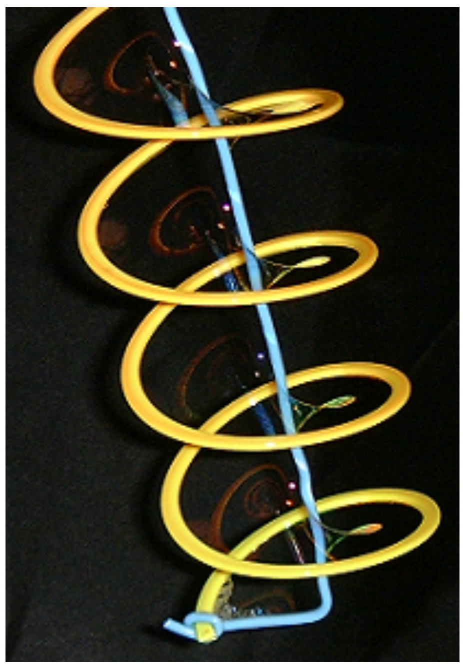


Figure 5: A helicoid is the minimal surface bounded by a helix. We see a soap film taking on this shape.

This is known as **Plateau's problem**. The answer is in the affirmative, and mathematician Jesse Douglas won a Fields medal for his solution. It was shown by Gauss that any regular parametrized surface $r : B(0, 1) \subset \mathbb{C} \to \mathbb{R}^n$ admits a conformal parametrization (i.e. the Jacobian matrix is a scalar times a rotation matrix, so that the parametrization preserves angles). The first step was to realize that for a conformal parametrization (so that $E = G$ and $F = 0$ for the coefficients of I_p), the minimality condition is equivalent with having the component functions r_i all harmonic. Thus the Plateau problem may be formulated as follows:

Problem 1. Given a contour $\Gamma \subset \mathbb{R}^n$, find a regular parametrization $r : \overline{B(0,1)} \to \mathbb{R}^n$ which is harmonic (i.e. all co-ordinate functions are harmonic) and conformal on $B(0,1)$ and for which the restriction to $\partial B(0,1)$ is a regular parametrization for Γ.

An important fact about harmonic functions defined on open subsets of the complex plane is the following: suppose $u : \Omega - \{p\} \subset \mathbb{C} \to \mathbb{R}$ is harmonic except at $p \in \Omega$ (Ω a domain), and u is bounded in some neighborhood of p, then u extends to a harmonic map on all of Ω. This is called a *harmonic extension* of u to Ω. Thus

Douglas' idea was to find a parametrization g^* of the contour Γ whose harmonic extension was conformal, giving a minimal surface bounded by Γ and thus solving Plateau's problem. He showed if $g : \partial B(0,1) \to \Gamma \subset \mathbb{R}^3$ parametrized Γ and $\phi : \partial B(0,1) \to \partial B(0,1)$ was any homeomorphism solving the following integral equation

$$\int_\Gamma \frac{g'(t) \cdot g'(\tau)}{\phi(t) - \phi(\tau)} d\tau = 0,$$

that the harmonic extension of $g \circ \phi^{-1}$ to the upper half plane by means of the Poisson Kernel is the required harmonic and conformal extension. He found this ϕ to be the minimizer of a a certain mysterious functional, called his A-functional, whose definitional motivation remains largely unknown.

5 Further Direction

I am interested to look more closely at how the theory generalizes to general Riemannian manifolds, and in the future am looking to develop a better understand of the methods of Douglas and Taylor to motivate further study of the more generalized theory of minimal submanifolds. A very special thanks to Professor Sema Salur, without whose direction, prudence, and kindness, this project would not have happened.

6 Other Examples

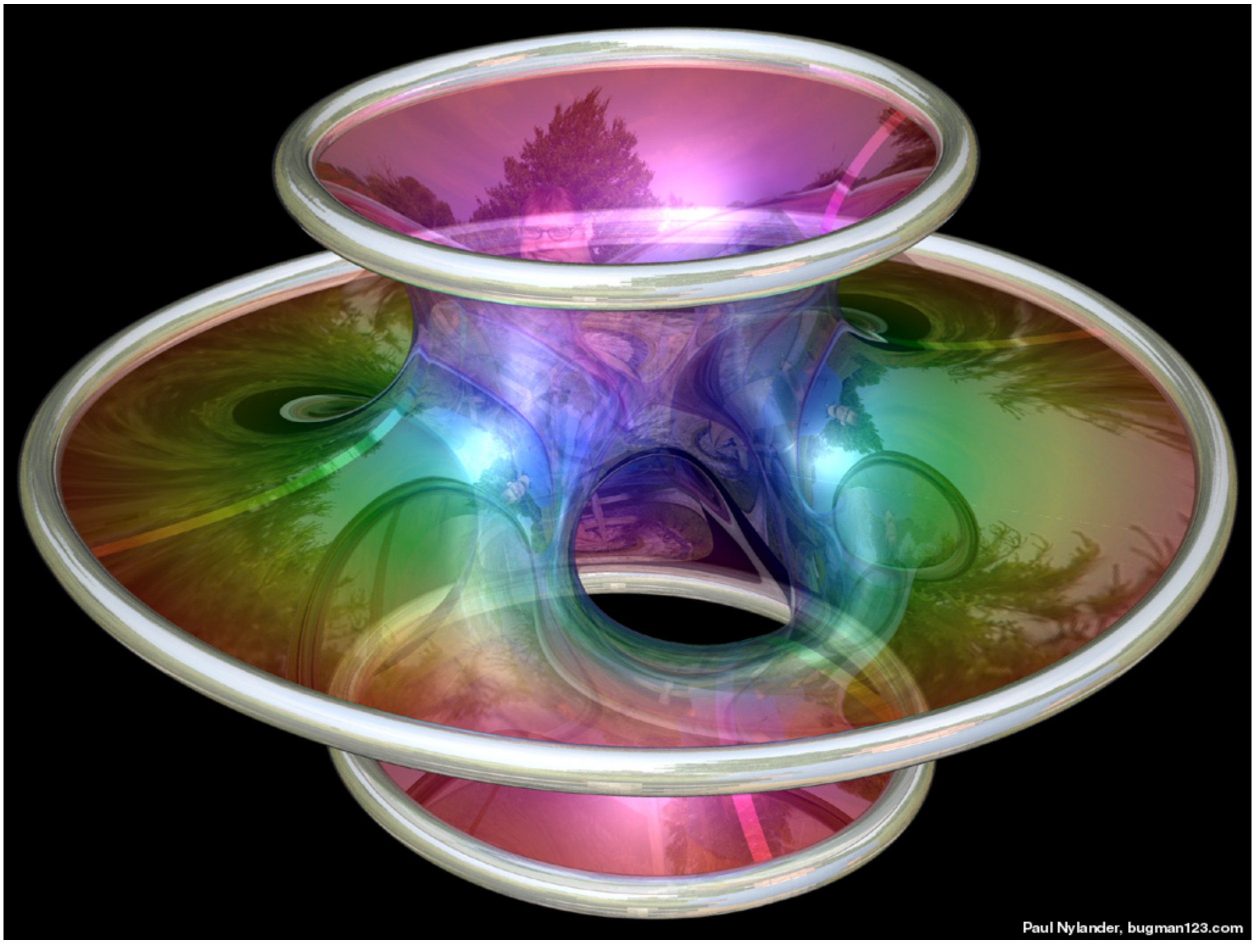


Figure 6: This is an example of an embedded minimal surface with 3 ends (3 ways to "reach infinity.") It is homeomorphic to a triply-punctured torus.

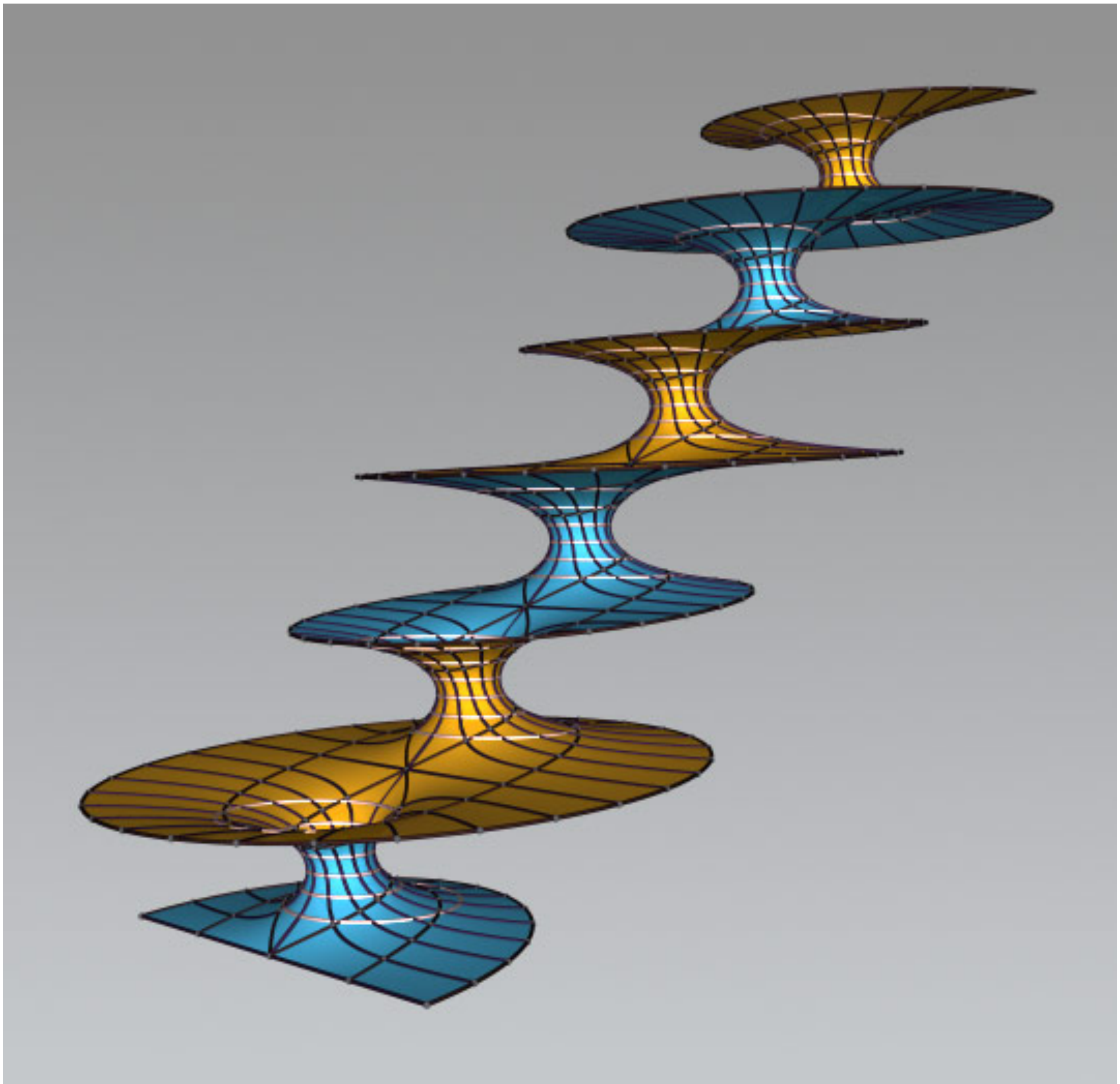


Figure 7: The Riemann examples are a parametrized family of minimal surfaces which are foliated by circles. They characterize all planar minimal surfaces (boundary is a 2 circles between two parallel planes.