

$\pi_\star H\underline{\mathbb{Z}}$ for C_{p^n} and Homological Algebra

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Tuesday 12th September, 2017

Introduction

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- ▶ Relation to homological algebra of $\mathbb{Z}$ -modules.
- ▶ Application in slice spectral sequences.

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With some compatibility condition.

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- ▶ $\underline{\mathbb{Z}}$ is the Mackey functor with $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ and all restrictions are isomorphism.

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- ▶ Because it plays the role of abelian groups in equivariant stable homotopy theory.
- ▶ A G -spectrum X is (not) a spectrum X with G action.
- ▶ And it comes with $RO(G)$ -graded homotopy Mackey functor $\underline{\pi}_\star(X)$, as the fundamental algebraic invariant. (Think about $\pi_\star(X)$ in the classical case)

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- ▶ Given a virtual representation $V \in RO(G)$, let S^V be the one-point compactification of V , thus a sphere with G -action.
- ▶ $\pi_V(X)(G/H) := [S^V \wedge_H G_+, X]$ can be made into a Mackey functor.

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- ▶ But how about the $RO(G)$ -grading?
- ▶ It is coming!

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- ▶ How about a little bit more about $\underline{\mathbb{Z}}$?
- ▶ $\underline{\mathbb{Z}}$ is a monoid, and $\pi_{\star}(H\underline{\mathbb{Z}})$ is an $RO(G)$ -graded $\underline{\mathbb{Z}}$ -module.

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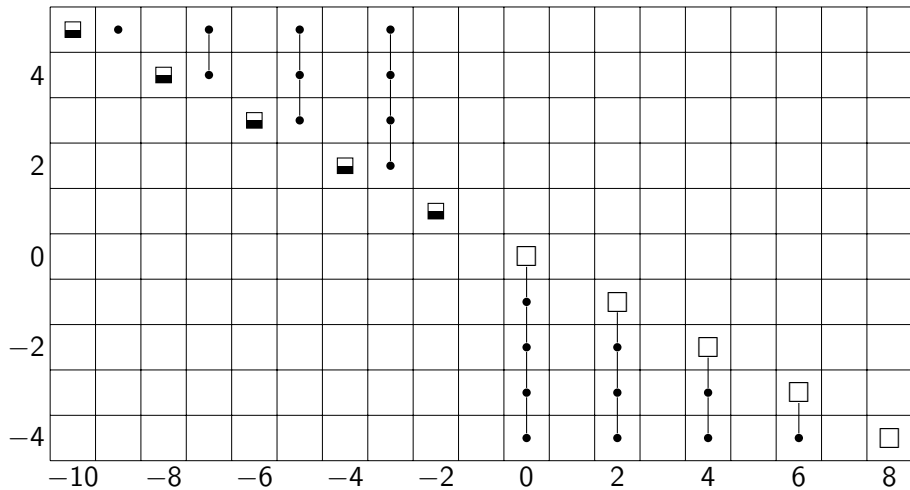
- ▶ $G = C_p$ and $p > 2$, $RO(G) = \mathbb{Z}\langle 1, \lambda \rangle$, where 1 is the trivial representation and λ is the $\mathbb{R}^2$ -representation given by rotation by $\frac{2\pi}{p}$.

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- ▶ But what does it look like?

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- ▶ When $G = C_{p^2}$, $RO(G) = \mathbb{Z}\langle 1, \lambda_1, \lambda_0 \rangle$. Where λ_i is the representation by rotating by $\frac{2\pi}{p^2-i}$ on $\mathbb{R}^2$.

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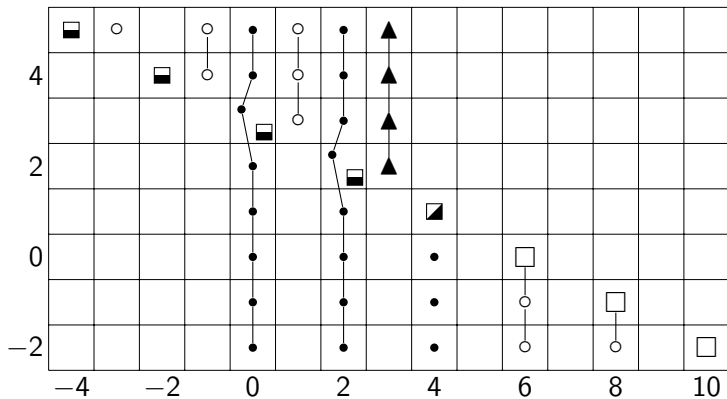
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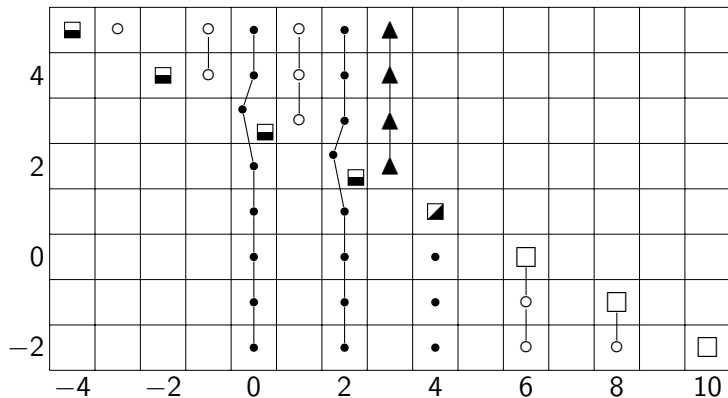
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- ▶ How about a 2D slice of it?

$$\pi_{m\lambda_0-3\lambda_1+i}(H\underline{\mathbb{Z}})$$

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We see ∞ -towers of multiplication and division by element a_{λ_0} .

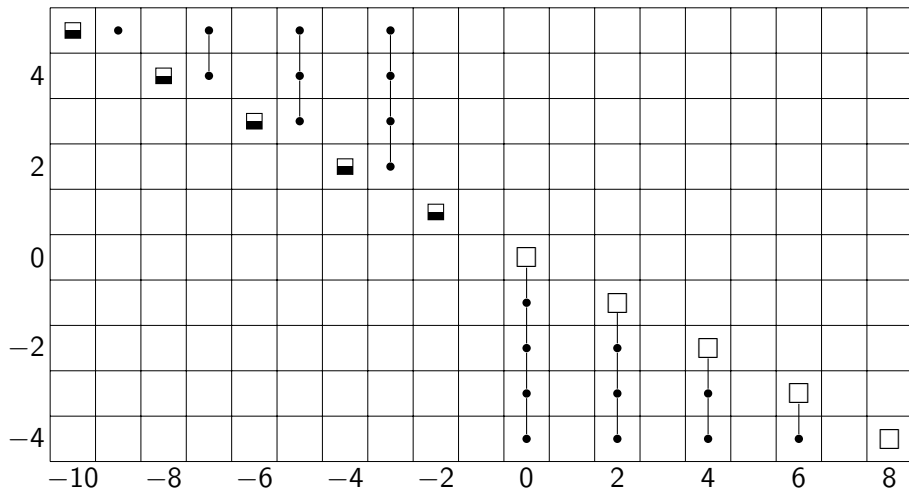
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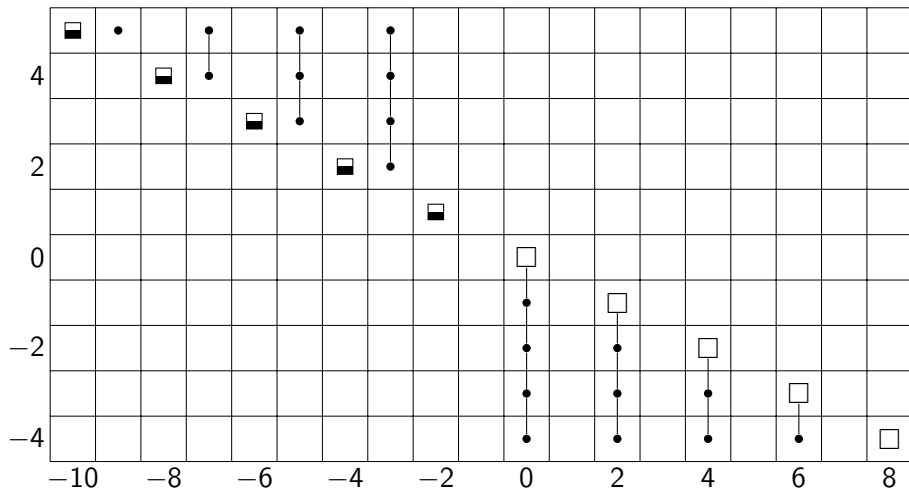
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You see duality with a strange shift.

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If $G = C_{p^n}$, $I_{\mathbb{Z}}(H\mathbb{Z}) \simeq H\mathbb{Z}^ \simeq S^{2-\lambda_0} \wedge H\mathbb{Z}$*

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- ▶ It is a duality with centre of symmetry $(-1, \frac{\lambda_0}{2})$.

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$$H\underline{M} \simeq S^V \wedge H\underline{\mathbb{Z}}$$

for a $V \in RO(G)$.

Furthermore, it gives a one-to-one correspondence between all such $\underline{M}$ (Forms of $\underline{\mathbb{Z}}$) and all $V \in RO(G)$ with $S^V \wedge H\underline{\mathbb{Z}}$ Eilenberg-Mac Lane.

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- This lemma also gives an algebraic interpretation of the "gap" in the upper-left part.

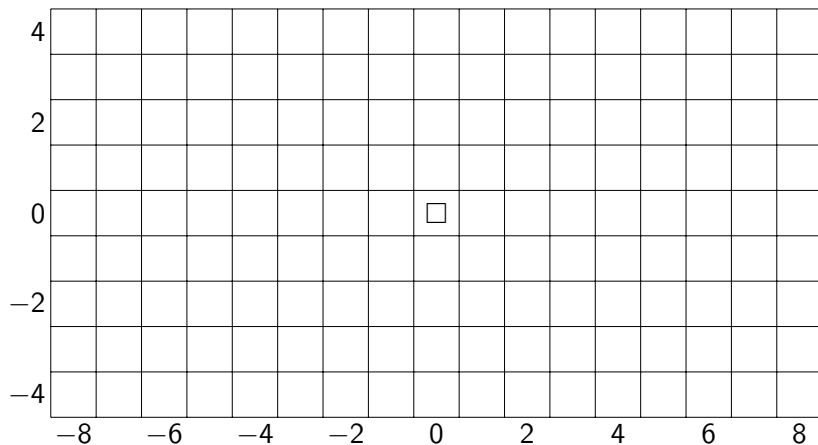
Duality, $\therefore$

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If you start with $\pi_*(H\underline{\mathbb{Z}})$, the integer grading. By using these two dualities, you can actually recover the whole figure.

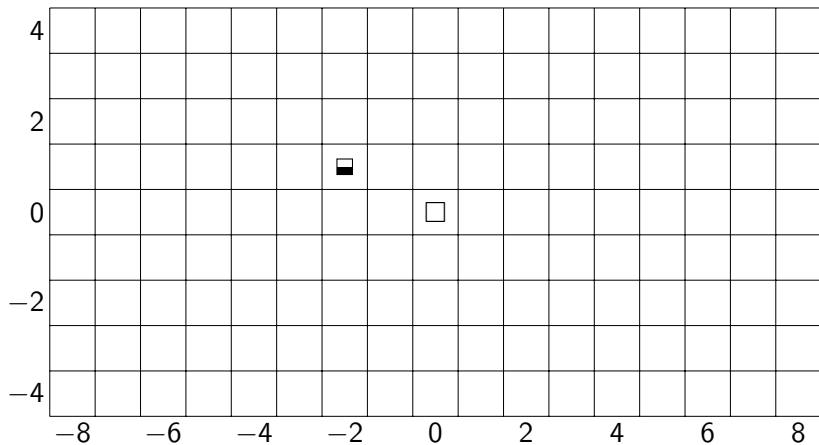
Duality, :||

You start with $\underline{\mathbb{Z}}$ in $(0, 0)$.



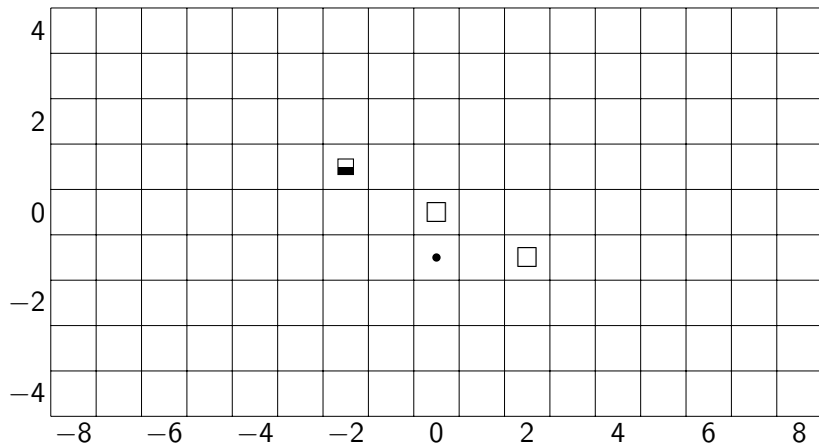
Duality, :||

Apply Anderson duality.



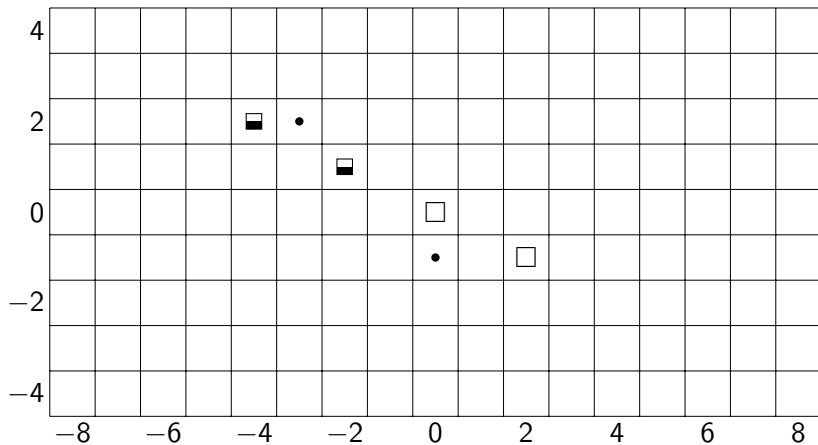
Duality, $\vdash$

Compute the universal coefficient spectral sequence.



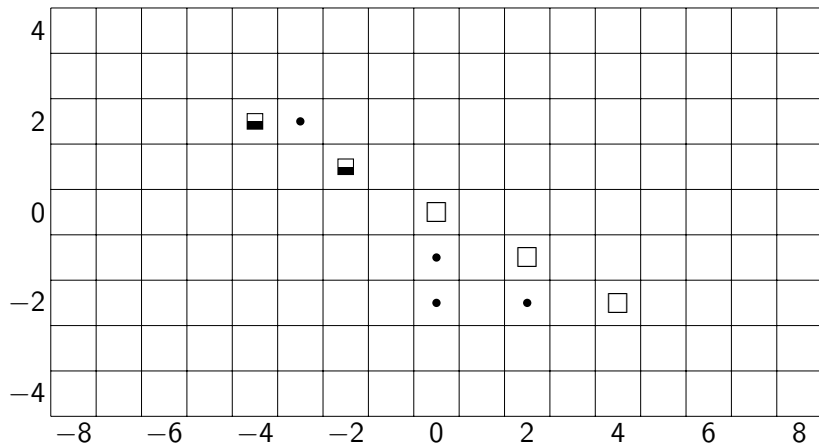
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Apply Anderson duality again.



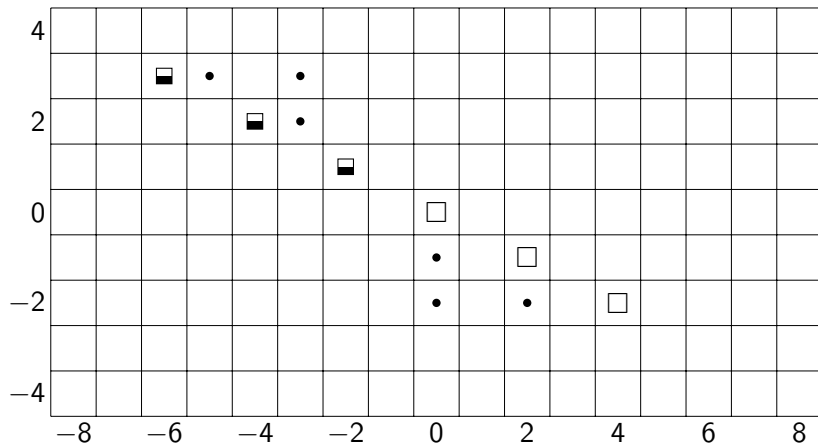
Duality, $\mathbb{Z}/2$

$\mathbb{Z}/2$: Universal coefficient spectral sequence. Anderson Duality $\mathbb{Z}/2$



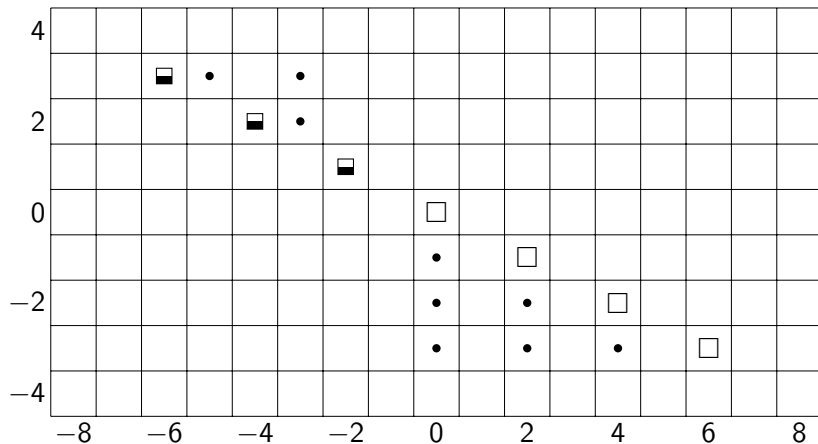
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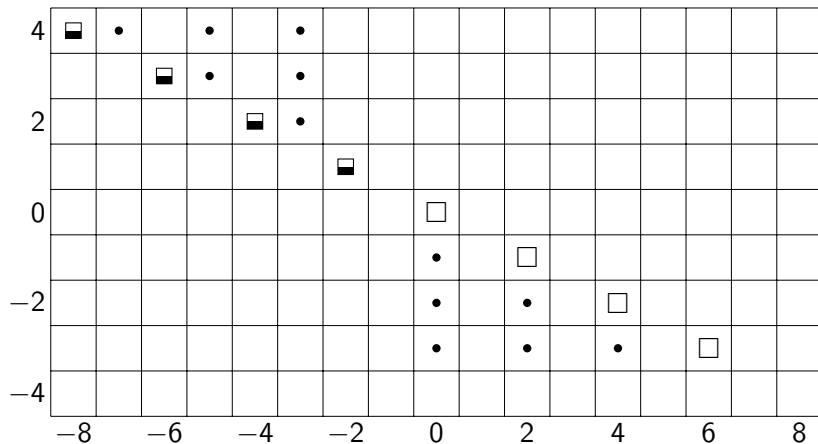
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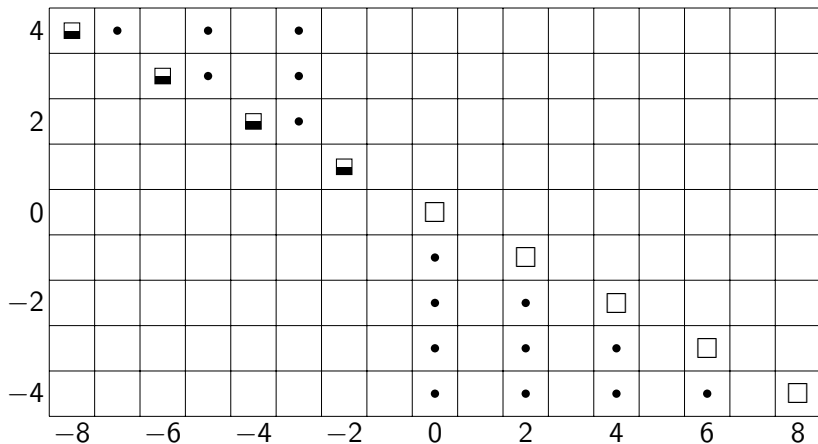
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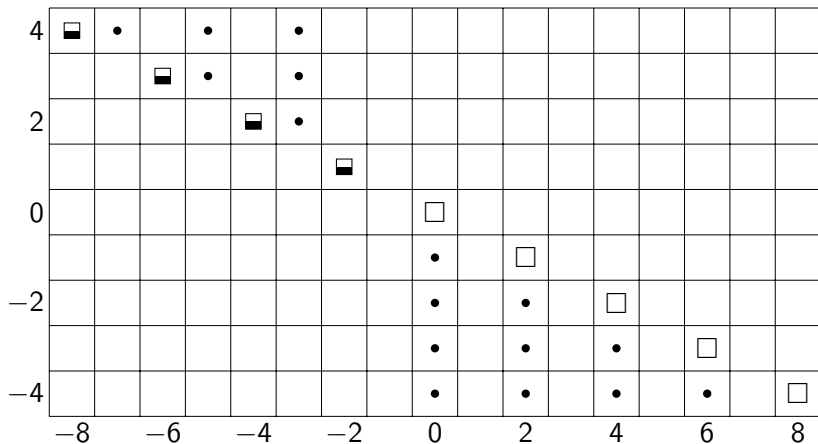
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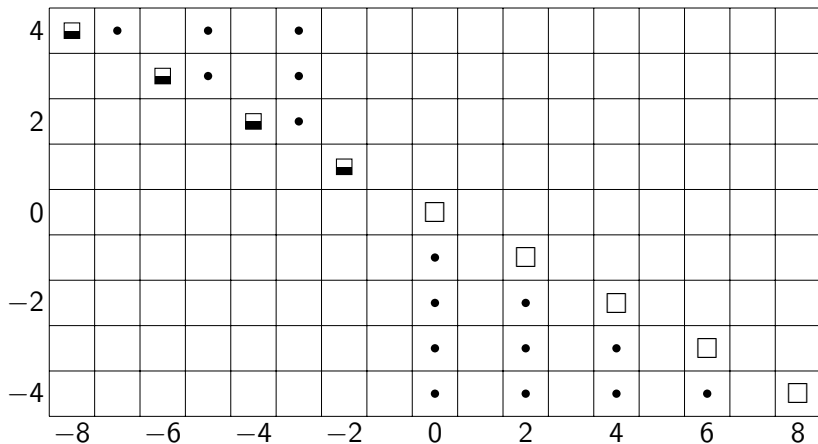
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- ▶ However as a n -dimensional grid.

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- ▶ The multiplicative structure can help in deducing differentials in these gradings from those known slice differentials.

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- ▶ What do these dualities look like for other groups? Especially for those the miracle fails.
- ▶ Tate diagram can also effectively compute $H\underline{\mathbb{F}}_p$ in all cyclic p -groups. Then what does A_* look like? Say, C_4 ?

The End

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Thank you!