AFFINE ÉTALE GROUP SCHEMES OVER TAMBARA FIELDS

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Abstract. We classify finite étale extensions and finite affine étale group schemes over the G-Tambara functor $\underline{\mathbb{F}}$, for $\mathbb{F}$ any algebraically closed field and G any finite group. This establishes G-Galois descent from the Tambara functor algebraic closure of $\underline{\mathbb{F}}$. In particular, we find new families of étale extensions of any G-Tambara functor and show that, together with one of the families discovered by Lindenstrauss–Richter–Zou, these give all finite étale extensions of $\underline{\mathbb{F}}$. Our arguments also show that the map $\underline{K} \to \mathrm{FP}(L)$ associated to any G-Galois extension L of K is étale, generalizing a result of Lindenstrauss–Richter–Zou when G is cyclic. Lastly, we classify flat finitely generated $\underline{\mathbb{F}}$-modules when $G = C_p$.

Contents

1. Introduction

Tambara functors are equivariant analogues of rings arising in representation theory, group cohomology, and equivariant homotopy theory. Recently, the commutative algebra of Tambara functors has been studied by homotopy theorists motivated in part by the hope of establishing equivariant analogues of classical applications of homological algebra and algebraic geometry to equivariant homotopy theory.

For example, [CMQ+24] study the Nakaoka spectrum (the equivariant analogue of the Zariski spectrum) of some free polynomial Tambara functors motivated in part by the hope of gaining insight into the algebraic geometry of the Tambara affine line. On the other hand, [HMQ23] observe that free polynomial Tambara functors often fail to be flat as modules, suggesting that new ideas may be necessary to approach some of the homological algebra which appears in equivariant stable homotopy theory.

One important concept in algebraic geometry is the notion of étaleness. For example, Grothendieck introduced étale cohomology and used it to prove three of the four Weil conjectures. In fact, Grothendieck was partially motivated by the observation that the Zariski topology was not always the correct topology. Equivariant algebra has the same issue: localization of Tambara functors can be very pathological, so the Zariski topology on the Nakaoka spectrum of a Tambara functor fails to have desirable properties.

In [Hil17], Hill introduced genuine Kähler differentials in the Tambara functor setting and showed that the genuine Kähler differentials support the universal Tambara functor derivation. Hill also proposed definition a *formally étale* Tambara functor morphism as map of Tambara functors which is both flat and for which the genuine Kähler differentials vanish. A Tambara functor morphism is *étale* if it is finitely presented and formally étale. Later in [LRZ24] Lindenstrauss, Richter, and Zou discovered interesting families of étale morphisms of C_n-Tambara functors.

To start with, we establish many familiar properties of étale morphisms: they are closed under composition, passing to projections, taking products and coinduction (a twisted product unique to the equivariant world), and they are preserved by base-change. Next, we find more examples of étale extensions.

Theorem A. *(cf. Theorem 4.7) For any G-Tambara functor k and subgroup $H \subset G$, the canonical map*

$$k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$$

is étale.

Previous work of Schuchardt, Spitz, and the author shows that the Tambara functors $\mathrm{CoInd}_H^G \mathbb{F}$, for $\mathbb{F}$ an algebraically closed field, should be viewed as the Tambara functor analogues of algebraically closed fields. Now $\mathrm{CoInd}_e^G \mathbb{F}$ carries a natural G-action, whose fixed-points are the constant Tambara functor $\underline{\mathbb{F}}$. Already we are encountering interesting new behavior in the equivariant algebra world: classically, the only group which acts faithfully on an algebraically closed field is C_2, in which case we must be in characteristic zero [AS27a] [AS27b].

As observed in [SSW25], CoInd_e^G gives a faithful embedding of the category of rings in the category of G-Tambara functors, so that the finite étale R-algebras correspond precisely to finite étale CoInd_e^G-algebras. In particular, an arbitrary $\mathrm{CoInd}_e^G \mathbb{F}$-algebra $\mathrm{CoInd}_e^G R$ is étale if and only if R is an étale $\mathbb{F}$-algebra.

Guided by the hope that Galois descent along $\underline{\mathbb{F}} \to \mathrm{CoInd}_e^G \mathbb{F}$ holds, we study finite affine étale group schemes, defined as follows.

Definition B. A finite affine étale group scheme over a ground Tambara functor k is a representable functor

$$\mathrm{Hom}_{k\text{-Alg}}(R, -) : k\text{-Alg} \to \mathsf{Grp}$$

such that $k \to R$ is finite and étale. We will write

$$\mathsf{Grp}_k^{\text{fét-aff}}$$

for the category of such objects (with morphisms the natural transformations).

With sufficient control over flat modules, we are able to completely classify finite étale G-Tambara functor maps out of the constant G-Tambara functor $\underline{\mathbb{F}}$, for $\mathbb{F}$ an algebraically closed field.

Theorem C. *(cf. Theorem 5.7) Let G be an arbitrary finite group and $\mathbb{F}$ any algebraically closed field. A finite $\underline{\mathbb{F}}$-algebra ℓ is étale if and only if it is a finite product of étale $\underline{\mathbb{F}}$-algebras*

$$\underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \underline{\mathbb{F}}.$$

Recall the topologist's notation $\mathcal{C}^{BG}$ for the category whose objects are the objects of a category $\mathcal{C}$ equipped with an action of the group G, and whose morphisms are G-equivariant morphisms in $\mathcal{C}$. We are able to establish G-Galois descent along $\underline{\mathbb{F}} \to \mathrm{CoInd}_e^G \mathbb{F}$ for finite affine étale group schemes.

Corollary D. *(cf. Corollary 5.8) Let G be an arbitrary finite group and $\mathbb{F}$ any algebraically closed field. Then*

$$\mathrm{ev}_{G/e} : \mathsf{Grp}_{\underline{\mathbb{F}}}^{\text{fét-aff}} \to (\mathsf{Grp}_{\mathbb{F}}^{\text{fét-aff}})^{BG}$$

is an equivalence of categories with inverse induced by the fixed-point construction FP.

Remark 1.1. From a homotopy theorist's point of view, $\mathrm{CoInd}_e^G \mathbb{F}$ is very uninteresting (for example, Bredon cohomology with $\mathrm{CoInd}_e^G R$ coefficients only detects the underlying space of a G-space). On the other hand, $\underline{\mathbb{F}}$ is an extremely common choice of coefficients–especially $\underline{\mathbb{F}_2}$ when $G = C_2$ (cf. [BW18, DHM24, HW20, Haz21, May20, Pet24]).

Galois descent in chromatic homotopy theory is an important tool, for example along the Galois extension $L_{K(n)}\mathbb{S} \to E_n$ with profinite Galois group the Morava stabilizer group. The topological analogue of Galois descent along $R \to \mathrm{CoInd}_e^G R$ is the homotopy fixed point spectral sequence, which is of fundamental importance.

When $|G|$ is invertible we can treat more general base Tambara functors.

Theorem E. *(cf. Theorem 5.4) Let ℓ be a flat finitely presented k-algebra in G-Tambara functors and assume either*

(1) *ℓ is cohomological and $|G|$ is invertible in $\ell(G/G)$, or*
(2) *all transfers in ℓ are surjective.*

Then ℓ is étale over k if and only if $\ell(G/e)$ is étale over $k(G/e)$.

Using Theorem 5.4 we may generalize [LRZ24, Theorem 4.4]. Specifically, [LRZ24, Theorem 4.4] is the special case of the following result in which K is a C_n-Kummer extension of a field L.

Corollary F. *(cf. Corollary 5.5) Let L be a G-Galois extension of a field K. Then*

$$\underline{K} \to \mathrm{FP}(L)$$

is formally étale. Under a mild technical condition ($\underline{K}$ satisfies the Hilbert basis theorem) it is étale.

In a different direction, Theorem 5.4 allows us to establish G-Galois descent for affine étale group schemes along $\underline{R} \to \mathrm{CoInd}_e^G R$.

Corollary G. *(cf. Corollary 5.6) Let R be a ring such that $|G|$ is a unit in R. Under a mild technical condition ($\underline{R}$ satisfies the Hilbert basis theorem), the functor*

$$\mathrm{ev}_{G/e} : \mathsf{Grp}_{\underline{R}}^{\text{fét-aff}} \to (\mathsf{Grp}_{R}^{\text{fét-aff}})^{BG}$$

is an equivalence of categories with inverse induced by FP.

For example, if $\mathcal{A}_G$ satisfies the Hilbert basis theorem and R is Noetherian, then $\underline{R}$ satisfies the Hilbert basis theorem [Sun25]. $\mathcal{A}_G$ is known to satisfy the Hilbert basis theorem for many groups: C_p [CMQ+24], Dedekind groups [Sun25], and more (see [Sun25] for a more complete list).

On the other hand, in modular characteristic and when $G = C_p$ we obtain the following. It is of independent interest, and it may also be useful in the study of étale $\underline{\mathbb{F}}$-algebras when $\mathbb{F}$ has characteristic p.

Theorem H. *(cf. Theorem 3.6) Let $\mathbb{F}$ be any field of characteristic p and $G = C_p$. The following conditions are equivalent for a finitely generated $\underline{\mathbb{F}}$-module M:*

(1) *M is flat*
(2) *M is projective*
(3) *M is free.*

Besides flatness results, the other two ingredients in the proofs of Theorem 5.7 and Corollary 5.8 are the classification of finite étale extensions of algebraically closed fields, and [Wis25a, Propositions 3.6 and 3.8], which allow one to deduce some of the structure of a Tambara functor k from its bottom level $k(G/e)$. As the results of [Wis25a] that we use are false for Green functors, we do not expect any variant of our argument to produce similar theorems for Green functors in modular characteristic.

Acknowledgments. The author thanks Mike Hill for a wealth of insightful suggestions. Additionally, the author thanks David Chan, Ayelet Lindenstrauss, Jackson Morris, and Birgit Richter for helpful conversations. Lastly, the author thanks Yuchen Liu for teaching a wonderful course in algebraic geometry that was very inspirational.

2. Recollections on the commutative algebra of Tambara functors

In this section we collect some known results in equivariant algebra and review the definition of étaleness. We assume the reader is familiar with Mackey, Green, and Tambara functors. A module over a Tambara functor is a module over the underlying Green functor.

Throughout, G is a finite group. We start by collecting some results on modules.

Lemma 2.1. *Let ℓ be a Tambara functor such that either*

(1) *ℓ is cohomological and $|G|$ is invertible in $\ell(G/G)$, or*
(2) *all transfers in ℓ are surjective*

then all restrictions in every ℓ-module are injective.

Proof. If $|G|$ is invertible in $\ell(G/G)$, then it is invertible in each ring $\ell(G/H)$. The cohomological assumption then implies that all transfers are surjective, so the first hypothesis is a special case of the second hypothesis.

Assume all transfers in ℓ are surjective and let $H \subset K$ be arbitrary. Then $1 \in \ell(G/K)$ is the transfer of some $y \in \ell(G/H)$. If M is an ℓ-module, Frobenius reciprocity implies that the restriction Res_H^K in M is injective:

$$m = 1 \cdot m = \mathrm{Tr}_H^K(y) \cdot m = \mathrm{Tr}_H^K(y \cdot \mathrm{Res}_H^K(m)).$$

□

Definition 2.2 ([CW25]). A Tambara functor k is *relatively finite dimensional* if each restriction Res_H^G is a finite ring map ($k(G/H)$ is a finitely generated $k(G/G)$-module).

Definition 2.3. We say that a map $k \to \ell$ of Tambara functors is *finite* if ℓ is a finitely generated k-module.

If k is relatively finite dimensional, then by [CW25, Proposition 3.31] $k \to \ell$ is finite if and only if it is levelwise finite.

Definition 2.4. Let k be a Tambara or Green functor and M a k-module. We say M is flat if the functor $M \boxtimes_k -$ is exact.

If X is a finite G-set, We will use the notation ev_X to describe either the functor $k \mapsto k(X)$ from Tambara functors to rings, or from Tambara functors to rings with an action of the Weyl group $\mathrm{Aut}(X)$. It will be clear from context which of these two we mean.

Definition 2.5. Let k be a Tambara functor. The free polynomial k-algebra on a finite G-set X is the representing object of the functor

$$\mathrm{ev}_X : k\text{-Alg} \to \mathrm{Ring}.$$

We denote this k-algebra by $k[y_X]$ and say y_X is a generator in level X.

Definition 2.6. We say that a k-algebra R is *finitely presented* if R is isomorphic to a coequalizer

$$\mathrm{Coeq}(k[x_i]_{i\in I} \rightrightarrows k[x_j]_{j\in J})$$

of free k-algebras on finitely many generators.

Equivalently (by [SSW25, Proposition 4.3]) R is finitely presented if and only if it is *compact* in the category of k-algebras, i.e. whenever $F : \mathsf{D} \to k\text{-Alg}$ is a filtered diagram in k-algebras, the canonical map

$$\mathrm{Hom}_k(R, \mathrm{Colim}_{d\in\mathsf{D}} F(d)) \to \mathrm{Colim}_{d\in\mathsf{D}} \mathrm{Hom}_k(R, F(d))$$

is a bijection of sets. From this description it follows that free polynomial k-algebras are finitely presented, and any finite colimit of finitely presented k-algebras is finitely presented.

Definition 2.7. We say that a Tambara functor k satisfies the *Hilbert basis theorem* if every ideal of every free polynomial k-algebra on finitely many generators is finitely generated (equivalently, every free finitely generated polynomial k-algebra satisfies the ascending chain condition on ideals, i.e. is Noetherian).

[CMQ+24, Corollary 3.12] and [Sun25, Theorems A and C] establish fairly general sufficient criteria for a Tambara functor to satisfy the Hilbert basis theorem. However, there exists finite groups G such that it is not currently known whether or not the Burnside Tambara functor $\mathcal{A}_G$ satisfies the Hilbert basis theorem.

If a k-algebra R receives a surjection from a free polynomial k-algebra on finitely many generators, we say that R is finitely generated over k. Finitely presented implies finitely generated; the converse is true if and only if k satisfies the Hilbert basis theorem.

Next, we review of the definition and first properties of étale morphisms of Tambara functors, following the original definition due to Hill [Hil17] and the treatment by Lindenstrauss, Richter, and Zou [LRZ24]. We start by working towards the definition of genuine Kähler differentials of a morphism of Tambara functors.

Definition 2.8 ([Hil17])**.** Let I be a Tambara ideal of a Tambara functor R. Define $I^{>1}$ to be the sub-ideal of I generated by all nontrivial norms (including products) of elements of I. Explicitly, $I^{>1}(G/H)$ is generated by $I(G/H)^2$ along with the images of all norms $I(G/K) \to I(G/H)$ for K conjugate to a proper subgroup of H.

Definition 2.9 ([Hil17])**.** Let $k \to R$ be a morphism of Tambara functors, let I be the kernel of the map

$$R \boxtimes_k R \to R,$$

and define the genuine Kähler differentials by

$$\Omega^{1,\mathrm{Tamb}}_{R/k} := I/I^{>1}.$$

Definition 2.10 ([Hil17])**.** A morphism $k \to R$ of Tambara functors is *formally étale* if $\Omega^{1,\mathrm{Tamb}}_{R/k} = 0$ and R is flat as a k-module.

As in the classical case, the genuine Kähler differentials support the universal derivation.

Definition 2.11 ([Hil17, Definition 4.1])**.** Let R be a k-algebra and M an R-module. A k-module morphism $d : R \to M$ is a *genuine k-derivation* if

(1) for all $H \subset G$, $r_1, r_2 \in R(G/H)$,

$$d(r_1 \cdot r_2) = r_1 \cdot d(r_2) + d(r_1) \cdot r_2,$$

(2) for all $H \subset K \subset G$, $r \in R(G/H)$,

$$d(\mathrm{Nm}_H^K r) = \mathrm{Tr}_H^K \mathrm{Nm}_{d_2} \mathrm{Res}_{d_1}(r) \cdot d(r)$$

where d_i is the restriction to the compliment of the diagonal of the projection onto the ith factor of $K/H \times K/H$, and

(3) d vanishes on the image of k in R.

The set of genuine k-derivations from R to M is denoted $\mathrm{Der}_k(R, M)$.

Theorem 2.12 ([Hil17, Theorem 5.7])**.** *There is a natural isomorphism*

$$\mathrm{Der}_k(R, -) \cong \mathrm{Hom}_R(\Omega^{1,\mathrm{Tamb}}_{R/k}, -)$$

of functors.

Definition 2.13. A morphism $k \to R$ of Tambara functors is *étale* if it is formally étale and finitely presented.

3. Flat modules are free

Let $\mathbb{F}$ be a field of characteristic p, $G = C_p$, and form the constant Tambara functor $\underline{\mathbb{F}}$. When $p = 2$ and $\mathbb{F} = \mathbb{F}_2$, [DHM24] prove that a finitely generated $\underline{\mathbb{F}_2}$-module is flat if and only if it is free. This is a consequence of a structure theorem in [DHM24] for $\underline{\mathbb{F}_2}$-modules which is known to fail at odd primes. In this section we show that this flatness result holds in greater generality.

Lemma 3.1. *Let k be a G-Green functor. The endofunctors*

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G : k\text{-Mod} \to k\text{-Mod}$$

and

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \boxtimes_k - : k\text{-Mod} \to k\text{-Mod}$$

are naturally isomorphic.

Proof. By [Wis25a, Theorem F] we can choose a $\mathrm{Res}_H^G k$-module N naturally in M so that we have a natural isomorphism

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \boxtimes_k M \cong \mathrm{CoInd}_H^G N$$

of $\mathrm{CoInd}_H^G \mathrm{Res}_H^G k$-modules (a fortiori of k-modules). It suffices to show $\mathrm{Res}_H^G M$ and N are naturally isomorphic.

We apply the strong symmetric monoidal functor Res_H^G, obtaining

$$\prod_{g \in H \backslash G / H} \mathrm{CoInd}^H_{H \cap {}^g H} \mathrm{Res}^{{}^g H}_{H \cap {}^g H} {}^g(\mathrm{Res}_H^G k) \boxtimes_{\mathrm{Res}_H^G k} \mathrm{Res}_H^G M \cong \prod_{g \in H \backslash G / H} \mathrm{CoInd}^H_{H \cap {}^g H} \mathrm{Res}^{{}^g H}_{H \cap {}^g H} {}^g N.$$

One checks straghtforwardly that this isomorphism restricts to an isomorphism

$$\mathrm{Res}_H^G k \boxtimes_{\mathrm{Res}_H^G k} \mathrm{Res}_H^G M \cong N$$

of $\mathrm{Res}_H^G k$-modules on the identity double coset factor. This is the desired isomorphism. □

In fact, this admits a multiplicative refinment which will be useful later.

Lemma 3.2. *Let $k \to R$ a map of G-Green or G-Tambara functors. Then*

$$\begin{array}{ccc} k & \longrightarrow & R \\ \downarrow & & \downarrow \\ \mathrm{CoInd}_H^G \mathrm{Res}_H^G k & \longrightarrow & \mathrm{CoInd}_H^G \mathrm{Res}_H^G R \end{array}$$

is a pushout square of k-algebras.

Proof. We must construct a natural isomorphism

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \boxtimes_k R \cong \mathrm{CoInd}_H^G \mathrm{Res}_H^G R$$

of k-algebras. By [Wis25a, Theorem F and Proposition 5.8] it suffices to show that we obtain naturally isomorphic H-Tambara functors upon applying Res_H^G. Now the claim follows from strong symmetric monoidality of Res_H^G, the double coset formula for the restriction of a coinduction, and the fact that $- \boxtimes_k R$ commutes with finite products. □

It is not true that projective modules are flat in an arbitrary abelian category equipped with bilinear symmetric monoidal structure. Fortunately, this result is true in the world of equviariant algebra.

Lemma 3.3. *Let k be a G-Green functor. All finitely generated projective k-modules are flat.*

Proof. Since sums and summands of flat modules are easily seen to be flat, it suffices to show that each $k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$ is flat. Now we have natural isomorphisms

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \boxtimes_k - \cong \mathrm{CoInd}_H^G \mathrm{Res}_H^G (-)$$

of endofunctors of k-modules. Res_H^G and CoInd_H^G are exact (as they are each other's left and right adjoints) so the claim follows. □

Guided by the heuristic that flat modules are "torsion-free" and viewing the kernel of restriction as torsion elements, we obtain the following.

Lemma 3.4. *Let k be a G-Green functor such that all restrictions in k are injective. If M is a flat k-module, then all restrictions in M are injective.*

Proof. Our assumption on k implies that whenever $L \subset H$, the canonical k-algebra map

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \to \mathrm{CoInd}_L^G \mathrm{Res}_L^G k$$

is injective. By Lemma 3.1 and flatness of M we deduce that

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G M \to \mathrm{CoInd}_L^G \mathrm{Res}_L^G M$$

is injective. The composition

$$M(G/H) \cong (\mathrm{CoInd}_H^G \mathrm{Res}_H^G M)(G/G) \to (\mathrm{CoInd}_L^G \mathrm{Res}_L^G M)(G/G) \cong M(G/L)$$

is the restriction Res_L^H in M. Since monics in k-Mod are detected levelwise, the claim follows. □

Remark 3.5. Let $G = C_p$ and $\mathbb{F}$ a field of characteristic p. If M is any $\underline{\mathbb{F}}$-module, then $M(C_p/e)$ is a $\mathbb{F}[C_p]$-module. Our characteristic assumption implies $\mathbb{F}[C_p] \cong \mathbb{F}[\sigma]/(\sigma-1)^p$, which is a principal ideal domain. If M is finitely generated, then $M(C_p/e)$ is a finitely generated $\mathbb{F}[C_p]$-module, and consequently the structure theorem for finitely generated modules over principal ideal domains applies. In particular, it supplies an isomorphism

$$M(C_p/e) \cong \oplus_i \mathbb{F}[\sigma]/(\sigma-1)^{a_i}$$

where i ranges through a finite indexing set and $1 \leq a_i \leq p$.

We required Theorem 3.6 to prove Corollary 5.8 in an earlier version of the article, although it is now no longer necessary. However, besides being of intrinsic interest, it may be useful in the study of finite étale $\underline{\mathbb{F}}$-algebras when one drops the assumption that $\mathbb{F}$ is algebraically closed.

Theorem 3.6. *Let $G = C_p$ and $\mathbb{F}$ a field of characteristic p. A finitely generated $\underline{\mathbb{F}}$-module is flat if and only if it is free.*

Proof. Free modules are flat by Lemma 3.3. Conversely, let M be a flat (finitely generated) $\underline{\mathbb{F}}$-module. Let $x \in M(C_p/e)$ be an element whose C_p-orbits are linearly independent; such an element exists if and only if in the direct sum decomposition of Remark 3.5 there is a summand with $a_i = p$. Then x generates a submodule of M isomorphic to $\mathrm{CoInd}_e^{C_p} \underline{\mathbb{F}}$. This is an injective $\underline{\mathbb{F}}$-module (cf. [CW25, Lemma 4.8]) hence splits off as a summand. Since $\mathrm{CoInd}_e^{C_p} \underline{\mathbb{F}}$ is flat by the previous paragraph, M is flat if and only if the complimentary summand is flat. Splitting off more such summands, we reduce to the case that no terms with $a_i = p$ appear in the direct sum decomposition of Remark 3.5.

Next, recall from Lemma 3.4 that the restriction in M is injective. The transfer is therefore determined by the sum over C_p-orbits. In $M(C_p/e)$, this corresponds to multiplication by

$$1 + \sigma + \cdots + \sigma^{p-1} = \frac{\sigma^p - 1}{\sigma - 1} = \frac{(\sigma - 1)^p}{\sigma - 1} = (\sigma - 1)^{p-1}$$

which is zero by the fact that we have reduced to the case

$$M(C_p/e) \cong \oplus_i \mathbb{F}[\sigma]/(\sigma - 1)^{a_i}$$

with $1 \leq a_i \leq p - 1$.

Finally, let D be the $\underline{\mathbb{F}}$-module specified by $D(C_p/C_p) = 0$ and $D(C_p/e) = \mathbb{F}$. There is an injection $D \to \underline{\mathbb{F}}$, and, since M is flat, it follows that $D \boxtimes M \to M$ is injective. We compute

$$(D \boxtimes M)(C_p/C_p) \cong M(C_p/e)_{C_p}/\mathrm{Res}_e^{C_p}(M(C_p/C_p))$$

using the description of the box product in [Maz13], and the map to $M(C_p/C_p)$ is given by sending the class $[x]$ of $x \in M(C_p/e)$ to its transfer, which is zero. Since $D \boxtimes M \to M$ is injective, we deduce $M(C_p/e)_{C_p}/\mathrm{Res}_e^{C_p}(M(C_p/C_p)) \cong 0$.

Unwinding definitions, if follows that the C_p-orbits of the image of the restriction in M generate $M(C_p/e)$. Since the restriction lands in the fixed points, the C_p-orbits of the image of the restriction are equal to the image of restriction. Thus the restriction in M is surjective, hence an isomorphism. We have thus shown that M is isomorphic to a sum of copies of $\underline{\mathbb{F}}$, which is free, as desired. □

Corollary 3.7. *Let $G = C_p$ and $\mathbb{F}$ a field of characteristic p. The following three conditions on a finitely generated $\underline{\mathbb{F}}$-module are equivalent:*

(1) *flat*
(2) *projective*
(3) *free.*

Proof. By Theorem 3.6 flat and free are equivalent, and by [CW25, Theorem 4.7] projective and free are equivalent (since $\underline{\mathbb{F}}$ is a relatively finite dimensional Green meadow). □

Morally, the most pathological behavior of field-like Tambara functors tends to be captured by $\underline{\mathbb{F}_p}$ when $G = C_p$, as these fail to be field-like Green functors. We therefore expect Corollary 3.7 to be true for all field-like G-Tambara functors regardless of G.

4. Fundamental properties of étale morphisms

In this section we establish that compositions, products, and base-changes of étale morphisms are étale. We begin with compositions.

Proposition 4.1. *Let $f : k \to R$ and $g : R \to S$ be Tambara functor morphisms.*

(1) *If S is finitely presented over R and R is finitely presented over k, then S is finitely presented over k.*
(2) *If S is flat over R and R is flat over k, then S is flat over k.*
(3) *If $\Omega^{1,\mathrm{Tamb}}_{S/R} = 0$ and $\Omega^{1,\mathrm{Tamb}}_{R/k} = 0$, then $\Omega^{1,\mathrm{Tamb}}_{S/k} = 0$.*

Consequently the class of étale morphisms of Tambara functors is closed under composition.

Proof. If S is finitely presented over R and R is finitely presented over k, then S is a finite colimit of free polynomial R-algebras $SR[x_{H_i}]$. Since base-change along $k \to k[x_{H_i}]$ preserves colimits and takes free algebras to free algebras, we deduce that each $R[x_{H_i}]$ is a finite colimit of free polynomial k-algebras. Thus S is a finite colimit of free polynomial k-algebras.

Second, we have a natural isomorphism

$$S \boxtimes_R R \boxtimes_k - \cong S \boxtimes_k -$$

of functors, so that flatness of S over R and of R over k imply flatness of S over k.

Finally, we establish the claim about genuine Kähler differentials. Assume every genuine k-derivation of R is zero and that every genuine R-derivation of S is zero. Let $d : S \to M$ be a genuine k-derivation. Then $d \circ g$ is a genuine k derivation of R, hence is zero. Thus d is a genuine R-derivation of S, hence is zero. □

Now we may move on to studying products and étaleness.

Lemma 4.2. *Let R_1 and R_2 be k-algebras. Then we have*

$$\Omega^{1,\mathrm{Tamb}}_{R_1 \times R_2/k} \cong \Omega^{1,\mathrm{Tamb}}_{R_1/R} \oplus \Omega^{1,\mathrm{Tamb}}_{R_2/R}$$

as $R_1 \times R_2$-modules (where we view an R_i-module as an $R_1 \times R_2$-module by restriction along the projection).

Proof. One straightforwardly checks that there is a natural isomorphism

$$\mathrm{Der}_k(R_1 \times R_2, M) \cong \mathrm{Der}_k(R_1, M_1) \oplus \mathrm{Der}_k(R_2, M_2).$$

where $M \cong M_1 \oplus M_2$ is the isomorphism of [Wis25a, Proposition 5.3]. The result follows from Yoneda's lemma and [Hil17, Theorem 5.7]. □

Proposition 4.3. *Let R_1 and R_2 be k-algebras.*

1. *$R_1 \times R_2$ is flat over k if and only if R_1 and R_2 are flat over k.*
2. *$\Omega^{1,\mathrm{Tamb}}_{R_1 \times R_2/k} = 0$ if and only if $\Omega^{1,\mathrm{Tamb}}_{R_1/k} = 0$ and $\Omega^{1,\mathrm{Tamb}}_{R_2/k} = 0$.*
3. *$R_1 \times R_2$ is finitely presented over k if and only if R_1 and R_2 are.*

Consequently R_1 and R_2 are étale k-algebras if and only if $R_1 \times R_2$ is an étale k-algebra.

Proof. First, we note that products and direct sums are the same thing for modules. Since $\boxtimes_k$ is additive, the direct sum of flat k-modules is flat, and summands of flat k-modules are flat. Second, Lemma 4.2 implies that $\Omega^{1,\mathrm{Tamb}}_{R_1 \times R_2/k} \cong 0$ if and only if $\Omega^{1,\mathrm{Tamb}}_{R_1/k} \cong 0$ and $\Omega^{1,\mathrm{Tamb}}_{R_2/k} \cong 0$.

Lastly, we establish the claim about finite presentability. If $R_1 \times R_2$ is a compact k-algebra, then $k \to R_1$ is the coequalizer of the two maps from $k[x_G]$ from the free polynomial k-algebra on a generator in level G/G respectively classifying the choice of zero and the choice of the idempotent generating the kernel of the projection

$$(R_1 \times R_2)\,(G/G) \cong R_1(G/G) \times R_2(G/G) \to R_1(G/G).$$

Therefore R_1 is a finite colimit of finitely presented k-algebras, hence is finitely presented. By symmetry R_2 is also finitely presented.

Conversely, suppose R_1 and R_2 are finitely presented k-algebras. In $(R_1 \times R_2)(G/e)$, let $x_1 = (1, 0)$ and $x_2 = (0, 1)$. If S is any k-algebra, the set $\mathrm{Hom}_k(R_1 \times R_2, S)$ decomposes as the disjoint union indexed by G-fixed idempotents $y \in S(G/e)$ of the set of k-algebra morphisms $R_1 \times R_2 \to S$ which send x_1 to y. Writing $S = yS_1 \times (1 - y)S_2$ (using [Wis25a, Proposition 3.6]), we thus have

$$\mathrm{Hom}_k(R_1 \times R_2, S) \cong \bigsqcup_y \mathrm{Hom}_k(R_1, yS) \sqcup \mathrm{Hom}_k(R_2, (1 - y)S). \tag{1}$$

Let $F : \mathsf{D} \to k$-Alg be a filtered diagram and y a G-fixed idempotent of $\mathrm{Colim}_{\mathsf{D}} F(d)$. Since filtered colimits are computed levelwise, and filtered colimits of rings commute with the choice of a finite list of elements satisfying a finite list of equations (by compactness of all quotients of $\mathbb{Z}[x_1, \ldots, x_n]$ in the category of rings), by passing to a cofinal diagram, we may assume each $F(d)(G/e)$ contains a G-fixed idempotent y_d mapping to y in the colimit. By [Wis25a, Proposition 3.6] we are entitled to write $F(d) \cong y_d F(d) \times (1 - y_d)F(d)$ as k-algebras.

In light of Equation (1) and the fact that filtered colimits commute with all finite products and all coproducts, it suffices to show for each y that we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_k(R_1, y\cdot \mathrm{Colim}_{\mathsf{D}} F(d)) \sqcup \mathrm{Hom}_k(R_2, (1-y)\cdot \mathrm{Colim}_{\mathsf{D}} F(d)) \\ \cong \mathrm{Colim}_{\mathsf{D}}(\mathrm{Hom}_k(R_1, y_d F(d)) \sqcup \mathrm{Hom}_k(R_1, (1-y_d)F(d))).\end{aligned}$$

But this follows immediately from compactness of R_1 and R_2 over k. □

Next, we show that flat base-change preserves étaleness. We start by showing that genuine Kähler differentials enjoy the expected base-change property for flat morphisms.

Proposition 4.4. *Let $f : k \to \ell$ be a flat map of Tambara functors, R a k-algebra, and $S := R \boxtimes_k \ell$. Then we have a natural isomorphism*

$$\Omega^{1,\mathrm{Tamb}}_{S/\ell} \cong \Omega^{1,\mathrm{Tamb}}_{R/k} \boxtimes_k \ell$$

of S-modules.

Proof. Let I be the kernel of $R \boxtimes_k R \to R$ and J the kernel of $S \boxtimes_\ell S \to S$. We have a commutative diagram

$$\begin{array}{ccc}(R \boxtimes_k R) \boxtimes_k \ell & \xrightarrow{\cong} & S \boxtimes_\ell S \\ \downarrow & & \downarrow \\ R \boxtimes_k \ell & \xrightarrow{\cong} & S\end{array}$$

so that by flatness of ℓ over k, $I \boxtimes_k \ell \cong J$. Note that this is an isomorphism of non-unital Tambara functors. It follows that $I^{>1} \boxtimes_k \ell^{>1} \cong J^{>1}$. Since ℓ is unital, $\ell^{>1} = \ell$, whence we compute

$$\Omega^{1,\mathrm{Tamb}}_{S/\ell} = J/J^{>1} \cong (I \boxtimes_k \ell)/(I^{>1} \boxtimes_k \ell) \cong (I/I^{>1}) \boxtimes_k \ell = \Omega^{1,\mathrm{Tamb}}_{R/k} \boxtimes_k \ell$$

as desired. □

When G is the trivial group Proposition 4.4 is known to hold without the flatness assumption. One way to remove the flatness assumption in the Tambara setting would be to mimic the proof of [Sta25, Lemma 10.131.12]. The only difficulty one enounters is showing that, if $d : R \to M$ is a genuine k-differential, then $d \otimes_k 0 : S \to M \boxtimes_k \ell$ is a genuine ℓ-differential. Checking the first and third conditions in Definition 2.11 for $d \otimes_k 0$ to be a differential is straightforward, although checking the second condition amounts to a very complicated computation involving the exponential formula for norms. When $G = C_p$, the formula simplifies enough to check by hand, so that Proposition 4.4 can be seen to hold without the flatness assumption.

Proposition 4.5. *Let $k \to \ell$ be a map of G-Tambara functors. The base-change functor $- \boxtimes_k \ell$ enjoys the following properties:*

1. *if R is finitely presented over k, then $R \boxtimes_k \ell$ is finitely presented over ℓ.*
2. *if R is flat over k, then $R \boxtimes_k \ell$ is flat over ℓ.*
3. *if $k \to \ell$ is flat and $\Omega^{1,\mathrm{Tamb}}_{R/k} = 0$, then $\Omega^{1,\mathrm{Tamb}}_{R\boxtimes_k \ell/\ell} = 0$.*

Hence flat base-change preserves étale morphisms.

Proof. Base-change preserves colimits and free polynomial algebras, hence preserves being finitely presented. If R is flat over k, then

$$R \boxtimes_k \ell \boxtimes_\ell - \cong R \boxtimes_k -$$

so that $R \boxtimes_k \ell$ is a flat ℓ-module. Finally, if $\Omega^{1,\mathrm{Tamb}}_{R/k} \cong 0$ and ℓ is flat over k, then by Proposition 4.4 we have

$$\Omega^{1,\mathrm{Tamb}}_{R\boxtimes_k \ell/\ell} \cong \Omega^{1,\mathrm{Tamb}}_{R/k} \boxtimes_k \ell \cong 0.$$

□

Remark 4.6. The Green functor analogue of Proposition 4.4 is true without a flatness assumption. It is also true that a Green formally étale morphism of Tambara functors is Tambara formally étale. Therefore the (possibly non-flat) base-change of a Green étale morphism of Tambara functors is Tambara étale.

Finally, we study the interaction between coinduction and étaleness.

Theorem 4.7. *Let k be any G-Tambara functor. Then the adjunction unit*

$$k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$$

is étale.

Proof. Our proof is by induction on $|G|$. If $H = G$, then there is nothing to prove, so we assume $|H| < |G|$.

First, we show $\mathrm{CoInd}_H^G \mathrm{Res}_H^G k$ is finitely presented by showing that it is compact. Let $F : \mathsf{D} \to k\text{-Alg}$ be a filtered diagram. If $\mathrm{Colim}_{\mathsf{D}} F(d)$ is not in the image of CoInd_H^G, then no $F(d)$ is in the image of CoInd_H^G, using the fact that $\mathrm{Colim}_{\mathsf{D}} F(d)$ is an $F(d)$-algebra for each object d of D and that fact that any Tambara functor receiving a map from a coinduced Tambara functor is coinduced (by [Wis25a, Corollary G]). Consequently

$$\mathrm{Colim}_{\mathsf{D}} \mathrm{Hom}_k(\mathrm{CoInd}_H^G \mathrm{Res}_H^G k, F(d)) \cong \mathrm{Colim}_{\mathsf{D}} \emptyset \cong \emptyset \cong \mathrm{Hom}_k(\mathrm{CoInd}_H^G \mathrm{Res}_H^G k, \mathrm{Colim}_{\mathsf{D}} F(d)).$$

On the other hand, if $\mathrm{Colim}_{\mathsf{D}} F(d)$ is in the image of CoInd_H^G, then since filtered colimits are computed levelwise, we see that for some d in D, the G-ring $F(d)(G/e)$ contains a type H-idempotent whose distinct G-orbits form a complete set of orthogonal idempotents (since filtered colimits commute with the existence of finitely many elements satisfying finite lists of equations by compactness of all quotients of $\mathbb{Z}[x_1, \ldots, x_n]$ in the category of rings). Thus there is some x in D such that $F(x) \cong \mathrm{CoInd}_H^G R$, and by passing to a cofinal subdiagram we may assume by [Wis25a, Corollary G] that F lands in $\mathrm{CoInd}_H^G R$-algebras.

The map $k \to \mathrm{CoInd}_H^G R$ is adjoint to $\mathrm{Res}_H^G k \to R$, so that applying CoInd_H^G makes $\mathrm{CoInd}_H^G R$ into a $\mathrm{CoInd}_H^G \mathrm{Res}_H^G k$-algebra. Thus we may view F as landing in $\mathrm{CoInd}_H^G \mathrm{Res}_H^G k$-algebras. As filtered colimits are computed levelwise, changing the target category of F in this way does not change the colimit. By [Wis25a, Corollary G], F factors through CoInd_H^G in the sense that we may write $F = \mathrm{CoInd}_H^G \circ E$ for some functor $E : \mathsf{D} \to \mathrm{Res}_H^G k\text{-Alg}$.

By inductive hypothesis, for each $g \in H \backslash G / H$,

$$\mathrm{Res}_H^G k \to \mathrm{CoInd}_{H \cap {}^g H}^H \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k$$

is finitely presented. Thus, by Proposition 4.3 and the double coset formula for the restriction of a coinduction,

$$\mathrm{Res}_H^G k \to \mathrm{Res}_H^G \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$$

is finitely presented. Since CoInd_H^G commutes with filtered colimits, we compute

$$\begin{aligned}
\mathrm{Hom}_k(\mathrm{CoInd}_H^G \mathrm{Res}_H^G k, \mathrm{Colim}_{\mathsf{D}} F(d)) &\cong \mathrm{Hom}_k(\mathrm{CoInd}_H^G \mathrm{Res}_H^G k, \mathrm{CoInd}_H^G \mathrm{Colim}_{\mathsf{D}} E(d)) \\
&\cong \mathrm{Hom}_{\mathrm{Res}_H^G k}(\mathrm{Res}_H^G \mathrm{CoInd}_H^G \mathrm{Res}_H^G k, \mathrm{Colim}_{\mathsf{D}} E(d)) \\
&\cong \mathrm{Colim}_{\mathsf{D}} \mathrm{Hom}_{\mathrm{Res}_H^G k}(\mathrm{Res}_H^G \mathrm{CoInd}_H^G \mathrm{Res}_H^G k, E(d)) \\
&\cong \mathrm{Colim}_{\mathsf{D}} \mathrm{Hom}_k(\mathrm{CoInd}_H^G \mathrm{Res}_H^G k, F(d)).
\end{aligned}$$

We have thus shown that $\mathrm{CoInd}_H^G \mathrm{Res}_H^G k$ is finitely presented over k. Next, by Lemma 3.3 free k-modules are flat. Finally, we check that the genuine Kähler differentials vanish.

Let I be the kernel of

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G k \boxtimes_k \mathrm{CoInd}_H^G \mathrm{Res}_H^G k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k.$$

Via Lemma 3.2, the isomorphism

$$\mathrm{CoInd}_H^G \mathrm{Res}_H^G \mathrm{CoInd}_H^G \mathrm{Res}_H^G k \cong \prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^G \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k,$$

and [Wis25b, Theorem A], I is the coinduction of the ideal J defined as the kernel of

$$\prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^H \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k \boxtimes_{\mathrm{Res}_H^G k} \prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^H \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k$$

$$\to \prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^H \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k$$

Additionally, one straightforwardly checks $I^{>1} = \mathrm{CoInd}_H^G(J^{>1})$ (ultimately because $\mathrm{res}_H^G : G\text{-Set} \to H\text{-Set}$ preserves cardinality). Since coinduction preserves quotients we deduce

$$\begin{aligned}
\Omega^{1,\mathrm{Tamb}}_{\mathrm{CoInd}_H^G \mathrm{Res}_H^G k/k} &= I/I^{>1} \\
&\cong \mathrm{CoInd}_H^G (J/J^{>1}) \\
&\cong \mathrm{CoInd}\left(\Omega^{1,\mathrm{Tamb}}_{\prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^H \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \mathrm{Res}_H^G k/\mathrm{Res}_H^G k}\right) \\
&\cong \mathrm{CoInd}_H^G 0 \cong 0
\end{aligned}$$

where the second-to-last isomorphism follows from the third statement of Proposition 4.3 and the inductive hypothesis. □

Proposition 4.8. *Let $k \to R$ be a morphism of H-Tambara functors. Then we have an isomorphism*

$$\Omega^{1,\mathrm{Tamb}}_{\mathrm{CoInd}_H^G R/\mathrm{CoInd}_H^G k} \cong \mathrm{CoInd}_H^G \Omega^{1,\mathrm{Tamb}}_{R/k}$$

of $\mathrm{CoInd}_H^G R$-modules.

Proof. This follows from flat base-change along $k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$ using Lemmas 3.1 and 3.2 and Proposition 4.4. □

Proposition 4.9. *Let $k \to R$ be a morphism of H-Tambara functors.*

1. *If R is finitely presented over k, then $\mathrm{CoInd}_H^G R$ is finitely presented over $\mathrm{CoInd}_H^G k$.*
2. *R is flat over k if and only if $\mathrm{CoInd}_H^G R$ is flat over $\mathrm{CoInd}_H^G k$.*
3. *$\Omega^{1,\mathrm{Tamb}}_{R/k} = 0$ if and only if $\Omega^{1,\mathrm{Tamb}}_{\mathrm{CoInd}_H^G R/\mathrm{CoInd}_H^G k} = 0$.*

Thus if R is étale over k, then $\mathrm{CoInd}_H^G R$ is étale over $\mathrm{CoInd}_H^G k$.

Proof. The forwards direction is the special case of Proposition 4.5 applied to flat base-change along $k \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G k$, using Lemma 3.2. The backwards direction on the flatness statement follows from the fact that CoInd_H^G is an exact symmetric monoidal equivalence of abelian categories [Wis25a, Theorem F], and the backwards direction on the statement on vanishing of the genuine Kähler differentials follows from Proposition 4.8 and the fact that CoInd_H^G reflects zero. □

5. Classification theorems

We begin by showing that $\mathrm{ev}_{G/e}$ preserves finite affine étale group schemes. Throughout this section we will sometimes implicitly use the identification $\mathrm{Res}_H^G \underline{R} \cong \underline{R}$ of constant H-Tambara functors at a ring R.

Lemma 5.1. *Let $k \to R$ be a map of Tambara functors. We have a natural isomorphism*

$$\Omega^{1,\mathrm{Tamb}}_{R/k}(G/e) \cong \Omega^1_{R(G/e)/k(G/e)}$$

of $k(G/e)$-modules.

Proof. Let I denote the kernel of $R \boxtimes_k R \to R$. In level G/e this map has the form

$$R(G/e) \otimes_{k(G/e)} R(G/e) \to R(G/e)$$

so it suffices to check $I^{>1}(G/e) = I(G/e)^2$. Since the only norms landing in the G/e-level of a Tambara functor are honest multiplications, the claim follows. □

Proposition 5.2. *Fix a relatively finite dimensional G-Tambara functor k. The functor* $\mathrm{ev}_{G/e}$ *preserves the following:*

1. *finitely generated k-modules,*
2. *flat k-modules,*
3. *finitely generated k-algebras,*
4. *finitely presented k-algebras, and*
5. *cogroup structures over k.*

In particular, it preserves finite affine étale group schemes.

Proof. $\mathrm{ev}_{G/e}$ preserves finitely generated modules by relative finite dimensionality of k. Since $\mathrm{ev}_{G/e}$ is strong symmetric monoidal with respect to the box product over k and tensor product over $k(G/e)$, it takes flat k-modules to flat $k(G/e)$-modules.

By [Bru05, Theorem A] all free k-algebras on finitely many generators are sent by $\mathrm{ev}_{G/e}$ to free $k(G/e)$-algebras on finitely many generators. Thus $\mathrm{ev}_{G/e}$ preserves finite generation. Since $\mathrm{ev}_{G/e}$ (viewed as landing in the category of rings) is left adjoint to CoInd_e^G, it preserves coequalizers, hence finite presentation. Finally, since $\mathrm{ev}_{G/e}$ is strong symmetric monoidal, it also preserves cogroup structures. □

Corollary 5.3. *The functor*

$$\mathrm{ev}_{G/e} : \mathsf{Grp}_k^{\text{fét-aff}} \to \mathsf{Grp}_{k(G/e)}^{\text{fét-aff}}$$

naturally factors through the forgetful functor

$$(\mathsf{Grp}_{k(G/e)}^{\text{fét-aff}})^{BG} \to \mathsf{Grp}_{k(G/e)}^{\text{fét-aff}}$$

from finite affine étale group schemes with G-action to finite affine étale group schemes.

Proof. Let R represent any finite affine étale group scheme over k. The Weyl action defines a G-action on the ring $\mathrm{ev}_{G/e}R = R(G/e)$. The G-action is by cogroup homomorphisms as the G/e-level of the box product is the tensor product with the diagonal G-action, so it determines a G-action on the group scheme $R(G/e)$ represents. □

Theorem 5.4. *Let k be a G-Tambara functor and ℓ a flat k-algebra. Assume either*

1. *ℓ is cohomological and $|G|$ is invertible in $\ell(G/G)$, or*
2. *all transfers in ℓ are surjective*

then ℓ is formally étale over k if and only if $\ell(G/e)$ is formally étale over $k(G/e)$. If ℓ is finitely presented over k, then ℓ is étale over k if and only if $\ell(G/e)$ is étale over $k(G/e)$.

Proof. By Lemma 5.1 we have

$$\Omega^{1,\mathrm{Tamb}}_{\ell/k}(G/e) \cong \Omega^1_{\ell(G/e)/k(G/e)}.$$

Since all restrictions in the ℓ-module $\Omega^{1,\mathrm{Tamb}}_{\ell/k}$ are injective Lemma 2.1, $\Omega^{1,\mathrm{Tamb}}_{\ell/k} \cong 0$ if and only if $\Omega^{1,\mathrm{Tamb}}_{\ell/k}(G/e) \cong 0$. □

Corollary 5.5. *Let L be a G-Galois extension of a field K. Then*

$$\underline{K} \to \mathrm{FP}(L)$$

is formally étale. If $\underline{K}$ satisfies the Hilbert basis theorem, then it is étale.

Proof. First, we observe $\mathrm{FP}(L)$ is flat. Observe the $K[G]$-module isomorphic $L \cong K[G] \cong (\mathrm{CoInd}_e^G K)(G/e)$. Since L and $\mathrm{CoInd}_e^G K$ are fixed-point $\underline{K}$-modules, we have a chain of k-module isomorphisms

$$\mathrm{FP}(L) \cong \mathrm{FP}((\mathrm{CoInd}_e^G K)(G/e)) \cong \mathrm{CoInd}_e^G K.$$

Thus $\mathrm{FP}(L)$ is a free $\underline{K}$-module, so that it is flat by Lemma 3.3. Additionally, since all transfers in $\mathrm{CoInd}_e^G K$ are surjective, we deduce that all transfers in $\mathrm{FP}(L)$ are also surjective, so that we are in the situation of Theorem 5.4.

If $\underline{K}$ satisfies the Hilbert basis theorem, then since $\mathrm{FP}(L)$ is levelwise finite over $\underline{K}$, it is finitely generated over $\underline{K}$, hence finitely presented. Since Galois extensions of fields are étale, Theorem 5.4 applies. □

If k is cohomological and $|G|$ is invertible in $k(G/G)$, then every k-algebra is fixed point by Lemma 2.1. Since fixed-point Mackey functors are cohomological, we deduce that every k-algebra R is cohomological and that $|G|$ is invertible in $R(G/G)$. It would be interesting to have an application of Theorem 5.4 in the case that k is not cohomological and/or $|G|$ is not invertible in $k(G/G)$.

Corollary 5.6. *Let R be a ring such that $|G|$ is a unit in R. If $\underline{R}$ satisfies the Hilbert basis theorem, then*

$$\mathrm{ev}_{G/e} : \mathsf{Grp}_{\underline{R}}^{\text{fét-aff}} \to (\mathsf{Grp}_R^{\text{fét-aff}})^{BG}$$

is an equivalence of categories with inverse induced by FP.

Proof. Let L be the representing ring of a finite affine étale group scheme over R with G-action. Then L is a cogroup equipped with a G-action by cogroup automorphisms. Our assumption on R implies that

$$\mathrm{FP} : R[G]\text{-Mod} \to \underline{R}\text{-Mod}$$

is strong symmetric monoidal (it's the inverse equivalence to the strong symmetric monoidal functor $\mathrm{ev}_{G/e}$ by Lemma 2.1), so we deduce that $\mathrm{FP}(L)$ represents a finite affine group scheme over $\underline{R}$. By Theorem 5.4, $\mathrm{FP}(L)$ is étale over $\underline{R}$. We have also used that each L^H is a finite R-algebra, so that $\mathrm{FP}(L)$ is finitely generated over $\underline{R}$, hence finitely presented by the assumption that $\underline{R}$ satisfies the Hilbert basis theorem.

It remains to check that $\mathrm{ev}_{G/e}$ and FP are mutually inverse, up to natural isomorphism. If ℓ represents an affine étale group scheme over k, then the adjunction unit $\ell \to \mathrm{FP}(\mathrm{ev}_{G/e}\ell)$ is an isomorphism by Lemma 2.1. Conversely, if L is the representing ring of an affine étale group scheme over $k(G/e)$ with G-action, then $\mathrm{ev}_{G/e}\mathrm{FP}(L)$ clearly returns L. □

For example, Corollary 5.6 applies to the Tambara functor $\underline{R}$ whenever $\mathbb{Q} \subset R$, at least for the list of finite groups G appearing in [Sun25, Theorem A].

We now turn our attention to the case in which the characteristic possibly divides the order of G.

Theorem 5.7. *Let G be an arbitrary finite group and $\mathbb{F}$ any algebraically closed field. Then a finite $\underline{\mathbb{F}}$-algebra ℓ is étale if and only if it is a finite product of $\underline{\mathbb{F}}$-algebras*

$$\underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \underline{\mathbb{F}}.$$

Proof. By Proposition 5.2 $\ell(G/e)$ is a finite étale $\mathbb{F}$-algebra. Thus $\ell(G/e) \cong \prod_i \mathbb{F}$ for a finite indexing set; G acts by permuting factors. Since $\ell(G/e)$ is Noetherian, [Wis25a, Theorem D] supplies a G-Tambara functor isomorphism

$$\ell \cong \prod_i \mathrm{CoInd}_{H_i}^G \ell_i$$

where ℓ_i is an H_i-Tambara functor under $\underline{\mathbb{F}}$ such that $\ell_i(G/e) \cong \mathbb{F}$. By Proposition 4.3 each ℓ_i is étale, so without loss of generality we may take $\ell = \ell_i$.

Now we induct on $|G|$. The base case is $G = \{e\}$, wherein the result is classical. In the inductive step, we have two things to show: first, if $\ell \cong \mathrm{CoInd}_H^G \ell'$, then ℓ' is an étale $\mathrm{CoInd}_H^G \underline{\mathbb{F}}$-algebra, so that by inductive hypothesis

$$\ell \cong \mathrm{CoInd}_H^G \mathrm{CoInd}_K^H \underline{\mathbb{F}} \cong \mathrm{CoInd}_K^G \underline{\mathbb{F}}.$$

Second, we must show that if $\ell(G/e) \cong \mathbb{F}$, then $\ell \cong \underline{\mathbb{F}}$

For the first point, by Lemma 3.2 the base-change of $\underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \ell'$ along the flat map $\underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \underline{\mathbb{F}}$ is an étale map

$$\mathrm{CoInd}_H^G \underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \mathrm{Res}_H^G \mathrm{CoInd}_H^G \ell' \cong \prod_{g \in H\backslash G/H} \mathrm{CoInd}_{H \cap {}^g H}^G \mathrm{Res}_{H \cap {}^g H}^{{}^g H} {}^g \ell'$$

By Proposition 4.3 the projection onto the identity double coset factor describes an étale map

$$\mathrm{CoInd}_H^G \underline{\mathbb{F}} \to \mathrm{CoInd}_H^G \ell'$$

of $\underline{\mathbb{F}}$-algebras.

For the second point, observe that all restrictions in ℓ are injective by Lemma 3.4. For each $H \subset G$ we have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{F} \xrightarrow{=} & \underline{\mathbb{F}}(G/H) & \longrightarrow & \ell(G/H) & \\
& \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Res}_e^H} & \\
& \underline{\mathbb{F}}(G/e) & \xrightarrow{\cong} & \ell(G/e) & \xrightarrow{\cong} \mathbb{F}
\end{array}$$

from which it follows that $\underline{\mathbb{F}}(G/H) \to \ell(G/H)$ is an isomorphism. This establishes the claim. $\square$

We are finally able to establish G-Galois descent for finite affine étale group schemes along $\underline{\mathbb{F}} \to \mathrm{CoInd}_e^G \mathbb{F}$ in the case of modular characteristic.

Corollary 5.8. *Let $\mathbb{F}$ be an algebraically closed field and G an arbitrary finite group. Then*

$$\mathrm{ev}_{G/e} : \mathsf{Grp}_{\underline{\mathbb{F}}}^{\text{fét-aff}} \to (\mathsf{Grp}_{\mathbb{F}}^{\text{fét-aff}})^{BG}$$

is an equivalence of categories with inverse induced by FP.

Proof. Clearly $\mathrm{ev}_{G/e} \circ \mathrm{FP} \cong \mathrm{Id}$. On the other hand, by Theorem 5.7, if ℓ represents a finite affine étale group scheme, then the adjunction unit $\ell \to \mathrm{FP}(\ell(G/e))$ is an isomorphism. Thus $\mathrm{FP} \circ \mathrm{ev}_{G/e} \cong \mathrm{Id}$ as well. $\square$

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