

The Ω -Spectrum for Brown-Peterson Cohomology, Part III.

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ABSTRACT In this paper we give a description of the cohomology operations on the module of indecomposables for the cohomology of each space in the Ω -spectrum for Brown-Peterson cohomology.

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INTRODUCTION

Let $BP = \{BP_k\}$ be the Brown-Peterson spectrum given as an Ω -spectrum. In [9] the cohomologies of all the spaces BP_k were computed. In [6] the homology of these spaces is computed giving much more information, in particular, a technique for computing the comodule structure of H_*BP_k over the Steenrod algebra is given. (All coefficients in this paper are $\mathbb{Z}_p$, where p is the prime associated with the Brown-Peterson spectrum). In this paper an entirely different approach is taken to cohomology operations. We give a description of the structure of the indecomposables QH^*BP_k as a module over the Steenrod algebra. This may occasionally be easier to compute with than the results of [6].

Let R be a sequence of non-negative integers $(r_1, r_2, \dots)$. Define $d(R) = \sum 2r_i(p^i - 1)$, the degree of R . Define $\ell(R) = \sum r_i$. Let V_j be the graded group free over $\mathbb{Z}_{(p)}$ with a generator for each sequence R with $\ell(R) = j$ and grade $d(R)$. We can now form the generalized Eilenberg-MacLane spectrum $K(V_j)$ with Ω -spectrum $\{K(V_j, k)\}$. We have fundamental generators $i_R \in H^*K(V_j, k)$ for each R with $\ell(R) = j$ and i_R has degree $d(R) + k$. We have a map of modules over the Steenrod algebra

$$s^*(d_j) : QH^*K(V_j, k+1) \longrightarrow QH^*K(V_j, k)$$

defined by $s^*(d_j)(i_R) = \sum Q_i i_{R-\Delta_i}$ where Δ_i is the sequence with one in the i^{th} place and zeros everywhere else and Q_i is the Milnor primitive ([4]). Any element a of $QH^*K(V_j, k)$ can be written as $\sum a_R i_R (\ell(R) = j)$ where $a_R \in A_p$ the Steenrod algebra. If a can be written such that each a_R

is in the algebra of reduced p^{th} powers (see section 2 for $p = 2$) we say a is without Bocksteins.

Theorem A The kernel of the map $s^*(d_j)$ restricted to all $a \in QH^*K(V_j, 2k)$ with no Bocksteins is a module over the Steenrod algebra which is isomorphic to M_j/M_{j-1} in a filtration of QH^*BP_{2k}

$$QH^*BP_{2k} = M_\infty \supset \dots M_j \supset M_{j-1} \supset \dots M_0 \supset 0$$

by modules over the Steenrod algebra.

Remark 1. H^*BP_{2k} is a polynomial algebra and H^*BP_{2k-1} is an exterior algebra. (See [9].) Likewise for Y_{2k} and Y_{2k-1} below.

Remark 2. The filtration is finite on each degree.

Remark 3. From [9] we know that the cohomology suspension gives an isomorphism $s^* : QH^*BP_{2k} \longrightarrow QH^*BP_{2k-1}$ of modules over the Steenrod algebra so the theorem determines the structure for BP_{2k-1} as well.

Because the space BP_k can be broken up into the product of much smaller spaces ([10]) this computation can be made even more accessible.

In [10], irreducible spaces Y_k for $k > 0$ were constructed and the following homotopy equivalence was proved:

$$BP_k \simeq Y_k \times \prod_{j>n} Y_{k+2(p^j-1)}$$

when $2(p^{n-1} + \dots + p + 1) < k \leq 2(p^n + \dots + p + 1)$. For k in the above range we can now give a description of QH^*Y_k as a module over the Steenrod algebra.

Let $V(n)_j$ be the graded subgroup of V_j where we only use sequences R with $r_i = 0$ for $i > n$. We have the same map

$$s^*(d_j) : QH^*K(V(n)_j, k+1) \longrightarrow QH^*K(V(n)_{j-1}, k)$$

defined by $s^*(d_j)(i_R) = \sum Q_i i_{R-\Delta_i}$.

Theorem B. For $2(p^{n-1} + \dots + p + 1) < 2k \leq 2(p^n + \dots + p + 1)$, the kernel of the map $s^*(d_j)$ restricted to all $a \in QH^*K(V(n)_j, 2k)$ with no Bocksteins is a module over the Steenrod algebra which is isomorphic to M_j/M_{j-1} in a filtration of QH^*Y_{2k}

$$QH^*Y_{2k} = M_\infty \supset \dots M_j \supset M_{j-1} \supset \dots M_0 \supset 0$$

by modules over the Steenrod algebra.

Remark 4. The cohomology suspension homomorphism gives an isomorphism of modules over A_p $s^* : QH^*Y_{2k} \longrightarrow QH^*Y_{2k-1}$.

Using this result, one can quite easily compute by hand the explicit Steenrod algebra structure of QH^*BP_k as far as one wants. (Try using [2].)

My apologies to anyone that is actually interested in seeing the proofs of the above theorems. You will learn quickly enough that complete intimacy with the notation and techniques of [9] is necessary for reading past the introduction. From your point of view, [9] and this paper should be one, however, the thoughts in this paper came somewhat later, and, it would be morally inexcusable to afflict the rather nice result of [9] with the presence of detailed proofs of the above theorems.

These results were motivated by three things; first, several people kept asking me what I knew about the cohomology operations on the spaces BP_k and Y_k . This paper is intended to supply an answer to

that question. Second, when the results were originally proven, serious applications existed. A much better approach to the applications has since been found. The short-lived applications will be discussed briefly in section 3. Third, I am convinced (through computations) that analogous theorems exist for the homology and homology operations. I had always hoped to intertwine these hypothetical theorems and the present ones in a much more informative proof of the main theorem of [9] ($H^*(BP_k, \mathbb{Z}_{(p)})$ has no torsion.) which would eliminate the messy details in [9] and here. However, I have made no progress here and the new proof given in [6] satisfies my personal quest. I do suggest that the analogous result dealing with homology operations could be a very interesting problem to anyone who solves it.

Technical propositions and a proof of theorem A appear in section 1. Theorem B is proven in section 2.

Remark 5. It is interesting to note that the results of this paper imply that every generator for H^*Y_k can be defined by some higher order co-homology operation on the k -dimensional class which arises because β is always zero as there is no torsion.

Section 1. Theorem A

Let us consider BP_{2k} . Here we have!

$$\begin{array}{ccc}
 K(V_j, 2k) & \xrightarrow{i_j} & X^j \\
 & \downarrow & \\
 & X^{j-1} & \xrightarrow{k_{j-1}} K(V_j, 2k+1)
 \end{array}$$

1.1

where $BP_{2k} = \text{inv. lim. } X^j$. There is a filtration on H^*X^j such that

1.2 $E_0 H^* X^j \simeq H^* X^{j-1} // k_{j-1}^* \otimes E[\dots] \otimes TP[\dots]$ where E denotes an exterior algebra on odd degrees and TP is a truncated polynomial algebra of height p on even dimensional generators. The generators of E and TP are both determined by $QH^* K(V_j, 2k+1) // (k_{j-1})^*$.

There is a natural map $PH \longrightarrow QH$, where P and Q denote the primitives and indecomposables of a Hopf algebra H . When this is onto, H is called primitive.

Lemma 1.3. (1.1 in [9]) $H' \subset H$ a sub-Hopf algebra over Z_p , H primitive, then H' is primitive.

If V is a graded module, let $s^{-1}V$ be the graded module with degrees lowered by one. TP is generated by s^{-1} of the odd degree part of $Q(H^* K(V_j, 2k+1) // (k_{j-1})^*)$, and E is generated by a quotient of the even degree part, $H^* K(V_j, 2k+1)$ is primitive so by 1.3, $H^* K(V_j, 2k+1) // (k_{j-1})^*$ is too. So, for $x \in Q(H^* K(V_j, 2k+1) // (k_{j-1})^*)$ we have $x' \longrightarrow x$, $x' \in P(H^* K(V_j, 2k+1) // (k_{j-1})^*)$. Let $x' \longrightarrow x'' \in PH^* K(V_j, 2k+1)$. $s^{-1}(x)$ is in the module of indecomposables for E or TP . It is possibly zero.

Lemma 1.4. (1.3 of [9])

$$(i_j)^*(s^{-1}(x)) = s^*(x'')$$

s^* the cohomology suspension homomorphism.

We need just a couple more lemmas before we get started.

Lemma 1.5. (1.4 of [9]) Let $a \in PH^m K(V_j, 2k+1)$. If $s^*(a) = 0$, then $a = P^t x_{2t}$ or $a = \beta P^t x_{2t+1}$ where $P^i \in A$ is the i^{th} reduced p^{th} power, A is the Steenrod algebra and x_i has degree i .

Lemma 1.6. (2.2 of [9])

If $a \in H^{2m}K(V_j, 2k+1)$ and $(i_{j-1})^* \circ (k_{j-1})^*(a) = 0$, then there exists $b \in H^{2m-1}K(V_{j+1}, 2k+1)$ such that $s^*(a) = (i_j)^* \circ (k_j)^*(b)$.

Now we can get to work and prove our main technical proposition.

Proposition 1.7 (j)

$$a(j) \quad H^*X^j \setminus (i_j)^* = H^*X^{j-1} // (k_{j-1})^*$$

$$b(j) \quad H^*X^{j-1} // (k_{j-1})^* \subset H^*X^j \text{ injects into } H^*X^{j+1}$$

$$c(j) \quad H^*X^j // (k_j)^* \text{ is a polynomial algebra on even dimensional generators.}$$

$$d(j) \quad \text{The maps in } b(j) \text{ are split as algebras.}$$

Proof of a(j). Assume for $j-1$. The induction is easy to start. We have

1.2. $H^*X^{j-1} // (k_{j-1})^*$ is clearly in $H^*X^j \setminus (i_j)^*$, so all we need to show for a(j) is that the generators of E and TP inject into $H^*K(V_j, 2k)$. By lemmas 1.4 and 1.5 applied to our description of generators for TP we see that they inject. By the same argument, the only generators of E that can possibly go to zero must be of the form $\beta P^t x_{2t+1}$ or $P^t x_{2t}$ in $Q(H^*K(V_j, 2k+1) \setminus (k_{j-1})^*)$.

Case 1. $\beta P^t x_{2t+1}$

If $(k_{j-1})^*(\beta P^t x_{2t+1}) = 0$, then $(k_{j-1})^*(x_{2t+1}) = 0$. Proof: If $(k_{j-1})^*(x_{2t+1}) \neq 0$, then by b(j-1), it is not in $H^*X^{j-2} // (k_{j-2})^* \subset H^*X^{j-1}$ so since it is an odd dimensional primitive it must land in E (for X^{j-1}). By a(j-1) it goes on through to $H^*K(V_{j-1}, 2k)$ and there βP^t is injective on

$2t+1$ dimensional classes ([7]) so $(i_{j-1})^* \circ (k_{j-1})^* (\beta P^t x_{2t+1}) \neq 0$.

This contradicts our assumption that $(k_{j-1})^* (\beta P^t x_{2t+1}) = 0$, so

$(k_{j-1})^* (x_{2t+1}) = 0$. By [8] there is a differential in the Eilenberg-

Moore spectral sequence that gives 1.2 which goes from $\gamma_p(s^{-1} x_{2t+1})$ to

$s^{-1}(\beta P^t x_{2t+1})$, i.e., $\beta P^t x_{2t+1}$ is zero in the quotient of

$s^{-1}Q(H^*K(V_{j,2k+1}) \searrow (k_{j-1})^*)$ which gives the generators of E .

Case 2 $P^t x_2$.

The same type of argument as above shows that $(k_{j-1})^* (P^t x_{2t}) = 0$ implies $(k_{j-1})^* (x_{2t}) = 0$. In that case, $P^t x_{2t}$ is a p^{th} power in $H^*K(V_{j,2k+1}) \searrow (k_{j-1})^*$ and therefore zero in the indecomposables.

Proof of b(j) If $b(j)$ is not true, then there must be some

$a \in H^{2m} K(V_{j+1,2k+1})$ with $(k_j)^*(a) \in H^* X^{j-1} / (k_{j-1})^*$. a must be of even degree by $c(j-1)$. Now by $a(j)$, $(i_j)^* \circ (k_j)^*(a) = 0$. If $s^*(a) \neq 0$, by lemma 1.6 there must be a b with $(i_{j+1})^* \circ (k_{j+1})^*(b) = s^*(a)$. But if this is so, then $s^{-1}(a)$ must be in $E_0 H^* X^{j+1}$ for $(k_{j+1})^*(b)$ to hit, thus $(k_j)^*(a) = 0$. This is a contradiction, so $s^*(a) = 0$. By lemma 1.5 $a = P^t x_{2t}$ or $\beta P^t x_{2t+1}$.

Case 1 $a = \beta P^t x_{2t+1}$

If $(k_j)^*(x_{2t+1}) \neq 0$, then by $c(j-1)$ it must lie in E and by $a(j)$, $(i_j)^* \circ (k_j)^*(x_{2t+1}) \neq 0$. By the injectivity of βP^t in $H^*K(V_{j,2k})$ we have that $(i_j)^* \circ (k_j)^* (\beta P^t x_{2t+1} = a) \neq 0$. This contradicts our assumption so $(k_j)^*(x_{2t+1}) = 0$ and this implies $(k_j)^* (\beta P^t x_{2t+1} = a) = 0$.

Case 2 $a = P^t x_{2t}$

Assume $P^t x_{2t} = a$ is the element of minimal degree with $(k_j)^* (a) \in H^*(x^{j-1}) // (k_{j-1})^*$. Clearly $(k_j)^* (x_{2t}) \neq 0$ and so lies in TP by our minimality condition. By $a(j)$, x_{2t} goes on to $H^* K(V_j, 2k)$ non-zero but $P^t x_{2t}$ cannot. This contradicts the fact that the p^{th} power of the image of x_{2t} in $H^* K(V_j, 2k)$ will be non-zero.

Proof of $c(j)$

First we must prove that everything in E for $E_0 H^* X^j$ gets killed by $(k_j)^*$. Use $a(j)$ and lemma 1.6 with techniques similar to those already repeated several times.

For TP, I claim that in $H^* X^j$ (as opposed to $E_0 H^* X^j$) this is a polynomial algebra. A generator for TP comes from $x_{2t+1} \in Q(H^* K(V_j, 2k+1) \setminus (k_{j-1})^*)$ and by $a(j)$, $(i_j)^* (s^{-1} (x_{2t+1})) \neq 0$. $P^t ((i_j)^* (s^{-1} (x_{2t+1}))) \neq 0$ because it is a p^{th} power in a product of Eilenberg-MacLane spaces. By $a(j)$, $P^t (s^{-1} (x_{2t+1})) = (s^{-1} (x_{2t+1}))^p$ must be $s^{-1} (P^t x_{2t+1})$.

The problem is to show that if $(k_j)^*$ gets a p^{th} power, say $s^{-1} (P^t x_{2t+1}) = (k_j)^* (a)$, then it also hits $s^{-1} (x_{2t+1})$. Using the techniques of [9] one can show that x_{2t+1} must be "with Bocksteins". This can be used to show the necessary result.

The proof of $d(j)$ is nothing exciting and is left to the diligent reader of this paper and [9].

We have the following corollaries to the proposition.

Corollary 1.8 The diagram 1.1 gives rise to an exact sequence in the category of Hopf algebras.

Corollary 1.9

$$H^* X^j // (k_j)^* \longrightarrow H^* BP_{2k}$$

is injective and splits as a map of algebras.

Corollary 1.10

$$Q(H^* X^j // (k_j)^*) \longrightarrow Q H^* BP_{2k}$$

is injective.

We can now define the filtration:

1.11 $QH^* BP_{2k} = M_\infty \supset \dots \supset M_j \supset M_{j-1} \supset \dots \supset M_0 \supset 0$ by setting $M_j = \text{image } Q(H^* X^j // (k_j)^*)$. By corollary 1.10 and the construction 1.1 of BP_{2k} this is well-defined. We have also already computed M_j/M_{j-1} although not yet in quite the nice form stated in the theorem in the introduction.

If $a \in PH^* K(V_j, k)$, we say a is with Bocksteins if $a = \sum a_R i_R$ ($\ell(R) = j$) where each $a_R \in (Q_0)$ the left-right ideal generated by the Bockstein $Q_0 = \beta$. We say a is with no Bocksteins if each a_R is in the algebra of reduced p^{th} powers. (For $p = 2$, if each a_R can be written in terms of the Milnor basis $Sq^{(I)}$ with all sequences made of even integers.) We have the map

$$s^*(d_j) : H^* K(V_j, k+1) \longrightarrow H^* K(V_{j-1}, k)$$

defined by $s^*(d_j)(i_R) = \sum Q_i i_{R-\Delta_i}$.

Each $a \in PH^*K(V_j, k)$ can be written as $a = b + c$ where b is with Bocksteins and c is with no Bocksteins. In [9, p.50] we showed:

Lemma 1.12 $s^*(d_j)(a) = 0$ if and only if both $s^*(d_j)(b) = 0$ and $s^*(d_j)(c) = 0$.

We also proved a much stronger lemma than we quoted in 1.6.

Lemma 1.13 (2.2 of [9])

If $a \in PH^*K(V_j, k)$ with Bocksteins and $s^*(d_j)(a) = 0$, then there exists $b \in PH^*K(V_{j+1}, k)$ such that $s^*(d_{j+1})(b) = s^*(a)$.

Note 1.14 $s^*(d_j)$ and $(i_{j-1})^* \circ (k_{j-1})^*$ are the same.

Note 1.15 We really needed 1.13 in the proof of $c(j)$ when we referred the reader to techniques of [9].

Now, by the definition of the filtration M_* (1.11), and the proposition 1.7, we can compute M_j/M_{j-1} . We know that $(k_j)^*$ kills the odd dimensional generators in 1.2 and $Q(H^*X^j/(k_j)^*)$ is even dimensional. Even dimensional generators of H^*X^j which do not come from $H^*X^{j-1}/(k_{j-1})^*$ come from TP in 1.2. Everything in $Q(TP)$ comes from an $a \in PH^*K(V_j, 2k+1)$ (for BP_{2k}) with $s^*(d_j)(a) = 0$. From 1.12, $a = b + c$ and $s^*(d_j)(b) = 0 = s^*(d_j)(c)$. By 1.5 $s^*(b) \neq 0$ (if $b \neq 0$) and by 1.13, $s^{-1}(b)$ is killed by $(k_j)^*$ and so we need only concern ourselves with $a = c$, i.e., a with no Bocksteins. Furthermore, since anything in the image of $s^*(d_{j+1})$ is with Bocksteins, we have essentially computed M_j/M_{j-1} . Many of the generators of TP are not generators of $H^*X^j/(k_j)^*$ because they are p^{th} powers, but that can be detected in $H^*K(V_j, 2k)$. We have shown that M_j/M_{j-1} is the set of all $s^*(a) \in QH^*K(V_j, 2k)$ where $a \in PH^*K(V_j, 2k+1)$ is with no Bocksteins and $s^*(d_j)(a) = 0$. This is not in the form we give in the introduction yet either. We need the following diagram and lemma.

$$\begin{array}{ccc}
 1.16 & QH^*K(V_{j-1}, 2k-1) & \xleftarrow{s^*(d_j)} QH^*K(V_j, 2k) \\
 & \uparrow & \uparrow \\
 & PH^*K(V_{j-1}, 2k-1) & \xleftarrow{s^*(d_j)} PH^*K(V_j, 2k) \\
 & \uparrow s^* & \uparrow s^* \\
 & QH^*K(V_{j-1}, 2k) & \xleftarrow{s^*(d_j)} QH^*K(V_j, 2k+1) \\
 & \uparrow & \uparrow \\
 & PH^*K(V_{j-1}, 2k) & \xleftarrow{s^*(d_j)} PH^*K(V_j, 2k+1)
 \end{array}$$

In the above diagram the map $P \rightarrow Q$ is the standard map.

Lemma 1.17. The set of all $a \in QH^*K(V_j, 2k)$ with no Bocksteins such that $s^*(d_j)(a)$ is the same as the set of all $s^*(a') \in QH^*K(V_j, 2k)$, $a' \in PH^*K(V_j, 2k+1)$ with no Bocksteins and $s^*(d_j)(a') = 0$.

Sketch proof. We have the commutative diagram 1.16 to aid us in the visualization of the proof. By commutativity, every such $s^*(a')$ is contained in the first set and our only problem is to go the other way. All the vertical maps are onto, in fact, for $a \in QH^*K(V_j, 2k)$ with no Bocksteins there exists $a' \in PH^*K(V_j, 2k+1)$ with no Bocksteins that goes down to it. All we need to show is that $s^*(d_j)(a') = 0$. a' is an odd dimensional element and $P \rightarrow Q$ is an isomorphism on odd dimensional elements. Also, s^* is injective on odd dimensional elements by 1.5 so we really have $s^*(d_j)(a') = 0$ iff the lifting of a to $a'' \in PH^*K(V_j, 2k)$ has $s^*(d_j)(a'') = 0$. By commutativity, $s^*(d_j)(a'') = (x_{2t})^P = P^t x_{2t}$ as this is the only type of element in the kernel of the map $P \rightarrow Q$

and we must have zero in the upper left hand corner of 1.16. Now, using the techniques of [9] it is easy to show that x_{2t} is with Bocksteins and $s^*(d_{j-1})(x_{2t}) = 0$. Using lemma 2.2(j) of [9] we can find a $y_{2t} \in PH^*K(V_j, 2k)$ with no Bocksteins such that $s^*(d_j)(y_{2t}) = x_{2t}$. If we alter our choice for a'' by adding $-P^t y_{2t}$ we obtain the desired result.

This concludes the proof of theorem A stated in the introduction.

Section 2 The Y_k and Theorem B

As the BP_k can be built up out of smaller spaces (see the theorem of [10] mentioned in the introduction), it is to our advantage to be able to compute with the Y_k instead of the much larger spaces BP_k .

Y_k for $2(p^{n-1} + \dots + p + 1) < k \leq 2(p^n + \dots + p + 1)$ is the k^{th} space in the Ω -spectrum for $BP < n >$, a BP module spectrum with $\pi_* BP < n > = \pi_* BP / (v_{n+1}, v_{n+2}, \dots)$ where $\pi_* BP = Z_{(p)}[v_1, v_2, \dots]$ with degree $v_i = 2(p^i - 1)$. $H^* BP = A/A(Q_0, Q_1, Q_2, \dots)$ and $H^* BP < n > = A/A(Q_0, \dots, Q_n)$.

The theories $BP < n >$ have their applications in [1] and [3] as well as in [10].

For k in the above range, $BP_k = Y_k \times X = BP < n >_k \times X$. The map $BP_k \rightarrow Y_k$ comes from the map of BP to $BP < n >$. Since we have this splitting, the "Adams-Postnikov" construction for BP_k (1.1)

$$\begin{array}{ccc} K(V_j, k) & \longrightarrow & X_j \\ & & \downarrow \\ & & X_{j-1} \longrightarrow K(V_j, k+1) \end{array}$$

must also split at each stage. So, all we really need to do in order to determine the split off system for Y_k is to identify the homotopy groups of Y_k in terms of those for BP_k . This follows easily from the fact that $v^I = v_1^{i_1} v_2^{i_2} \dots$ in $\pi_* BP$ corresponds to the I generator in $\pi_* K(V_{\ell(I)}, k)$ modulo higher order products. A little thought (or a little work with the Adams spectral sequence for BP and $BP < n >$) will verify this and convince the reader that the splitting is as in the introduction, i.e., Y_k is built from a system

$$\begin{array}{ccc}
 & i(n)_j & \\
 K(V(n)_{j,k}) & \xrightarrow{\quad} & X(n)_j \\
 & | & \\
 & k(n)_{j-1} & \\
 X(n)_{j-1} & \xrightarrow{\quad} & K(V(n)_{j,k+1})
 \end{array}$$

where $V(n)_j$ is the graded group, free over $Z_{(p)}$ with a generator for each $R = (r_1, r_2, \dots)$ with $\ell(R) = j$ and $r_i = 0$ for $i > n$. We have the same maps

$$(i(n)_{j-1})^* \circ (k(n)_{j-1})^* (i_R) = s^*(d(n)_j) (i_R) = \sum Q_i i_{R-\Delta_i}$$

Now, by the fact that each stage splits off from that for BP_k we have injections on all of our cohomology and we can just read off theorem B.

As a novel application we have $BU_{(p)} \sim \prod_{i=1}^p Y_{2i}$ and we have therefore described the Steenrod algebra structure of QH^*BU .

Section 3. Past applications.

In [6] we consider the monster space

$$BP = \text{dir. lim. } (n \rightarrow \infty) \prod_{k \geq n} BP_{2k}$$

Not only is this an H-space because it is a loop space ($\Omega^2 BP = BP$), but there is another product which comes from the fact that BP is a ring spectrum

$$BP_{2k} \wedge BP_{2n} \rightarrow BP_{2(k+n)}.$$

$\Pi_0 BP = \Pi_* BP = Z_{(p)}[v_1, v_2, \dots]$. This gives rise to elements

$$[v_n] \in H_0 BP_{-2(p^n-1)} \subset H_0 BP.$$

These elements generate $H_0 BP$. We have another select set of elements.

$BP^* \cap CP^\infty = BP^*[[T]]$ where $T \in BP^{2p} CP^\infty$. $BP^k(X) = [X, BP_k]$, so let $CP^\infty \rightarrow BP_2$ represent T . Now, we have generators $\beta_i \in H_{2i} CP^\infty$ which we can push into $H_{2i} BP_2$. In particular we let

$$b_{(i)} \in H_{2pi} BP_2 \subset H_{2pi} BP$$

be the image of $\beta_{\frac{i}{p}}$. The $b_{(i)}$ generate the stable homology of BP using the second product mentioned above. In [6] we construct a "Hopf-ring" completely algebraically which is isomorphic to $H_* BP$. The proof that they are isomorphic depends on the fact that the $[v_i]$ and $b_{(i)}$ generates $H_* BP$ if both products are allowed. In [6] we solve that problem by computing $H_* BP$ directly in terms of the $[v_i]$ and $b_{(i)}$. However, originally we proved it using the technical proposition 1.7. What follows is a very brief sketch of how we approached the problem then.

If we consider $H_* BP$ as an algebra using the loop space product, it is enough to show that the elements of $QH_* BP$ are linear combinations of elements

$$v_i^{I_j} = \dots [v_2]^{oi_2} \circ [v_1]^{oi_1} \circ b_{(o)}^{oj_o} \circ b_{(1)}^{oj_1} \dots$$

where the circle denotes the second product we discussed. The element

$v_i \in \pi_* BP$ gives rise to a map

$$v_i : S^{2(p^i-1)} BP \rightarrow BP$$

which is just multiplication by v_i . This also carries over to

$BP \rightarrow BP$ and gives a map

$$(v_i)_* : H_* BP \rightarrow H_* BP$$

In fact, $v^I b^J$ is just $(v^I)_*(b^J)$. By a simple induction we need only prove that the b^J generate QH_*BP_{2k} modulo the image of all the $(v^I)_*$ from the $QH_*BP_{2k+d(I)}$. By duality, it will be enough to show that either $x \in PH^*BP_{2k}$ is in the image of the cohomology suspension from H^*BP or $(v^I)^*(x) \neq 0$ for some I . We know from [9] and [5] that if x is a primitive it is either a primitive generator or a p^{th} power of one. By the fact that H^*BP_{2k} is a polynomial algebra, the p^{th} powers will follow if we can prove it for primitive generators. So, $x \in QH^*BP_{2k}$. Find j such that $x \in M_j$ but $x \notin M_{j-1}$. By the definition of the M_j and proposition 1.7 we have $(i_j)^*(x) \neq 0$ and is in fact an element with no Bocksteins in $QH^*K(V_j, 2k)$. Find an $i_R \in H^*K(V_j, 2k)$ such that the coefficient in $(i_j)^*(x) = \sum a_R i_R$ has $a_R, i_R \neq 0$. A little hand-waving should convince the reader that the map

$$v^{R'}: BP_{2k+d(R')} \longrightarrow BP_{2k}$$

actually factors through $K(V_j, 2k)$ and i_R pulls back to the generator y of $H^{2k+d(R')}BP_{2k+d(R')}$. This is just the fact mentioned in section 2 that the homotopy of BP_{2k} given by $K(V_j, 2k)$ is just the v^I with $\ell(I) = j$ modulo higher order products. In particular, $(v^{R'})^*(x) = a_R y$ which is non-zero and in the image of H^*BP .

Remark: The above notation allows us to define $M_j \subset QH^*BP_{2k}$ directly.

$$M_j = \bigcap_{\ell(I) > j} \ker(v^I)^*$$

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