

46. Extended p -th Powers of Complexes and Applications to Homotopy Theory

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1. Extended p -th power of a complex. Throughout this note p will denote an odd prime, $m=(p-1)/2$, $\pi=Z_p$ a cyclic group of order p , and the homology and cohomology groups will have the coefficient group Z_p . Let $W=W^\infty(=S^\infty)$ be a regular π -free acyclic CW -complex having one π -free basic cell e_i for each dimension i . The cells e_i are oriented such that in the infinite dimensional lens space W/π the dual $w_i \in H^i(W/\pi)$ of the class of e_i satisfies $w_{2i}=(w_i)^2$ and $\beta(w_2)=w_1$ for the cohomology Bockstein β .

For a finite CW -complex X , the product and the reduced join of p -copies of X will be denoted by $X^p=X \times \cdots \times X$ and $X^{(p)}=X \wedge \cdots \wedge X$ respectively. Let π acts on X^p and $X^{(p)}$ as cyclic permutations of the factors, and consider the quotient complexes

$$W^r \times_\pi X^p \quad \text{and} \quad ep^r(X)=(W^r \times_\pi X^{(p)})/(W^r/\pi),$$

where W^r indicates the r -skeleton of X and $W^r/\pi=W^r \times_\pi x_0^{(p)}$ for the base point x_0 of X . Let $x_0, x_1, x_2, \dots$ be a Z_p -basis of homogeneous elements of $H_*(X)$ which satisfies that if $\Delta x_j \neq 0$ for the homology Bockstein then $\Delta x_j = x_l$ for some l . A Z_p -basis of $H_*(W \times_\pi X^p)$ is given as the classes represented by the following cycles (cf. [2], [3]):

$$\begin{aligned} e_i \otimes_\pi x_j^p, \quad j=0, 1, 2, \dots, \quad x_j^p = x_j \otimes \cdots \otimes x_j \text{ (} p\text{-times)}, \\ e_0 \otimes_\pi (x_{j_1} \otimes \cdots \otimes x_{j_p}), \quad j_s \neq j_t \text{ for some } s, t, \end{aligned}$$

where $(j_1, \dots, j_p)$ runs through each representatives of the classes obtained by cyclic permutations of the indices. The same result holds for $H_*(W^r \times_\pi X^p)$ restricting e_i by $0 \leq i \leq r$ and by adding cycles of the form $\partial(e_{r+1} \otimes_\pi (x_{j_1} \otimes \cdots \otimes x_{j_p}))$.

By the natural projection $W^r \times_\pi X^p \rightarrow ep^r(X)$, a Z_p -basis of $\tilde{H}_*(ep^r(X))$ is obtained from that of $H_*(W^r \times_\pi X^p)$ by omitting the cycles containing x_0 .

Denote by $P_*^i: H_q \rightarrow H_{q-2i(p-1)}$ the dual of the Steenrod reduced power P^i , and let $P_*^i x_k = \sum_j a_{k,j}(i) x_j$ for $a_{k,j} \in Z_p$. Then the following relation has been established in [3].

Theorem 1. (Nishida).

$$P_*^n(e_{c+2n(p-1)} \otimes_\pi x_k^p) = \sum_{i,j} \binom{[c/2] + qm}{n-pi} a_{k,j}(i) (e_{c+2i(p-1)} \otimes_\pi x_j^p)$$

$$-\mu(q)\varepsilon(c+1)\Sigma_{i,j}\binom{[(c+1)/2]+qm-1}{n-pi-1}a_{i,j}(i)(e_{c+p+2ip(p-1)}\otimes_{\pi}x_j^p),$$

where c may be negative, $q=\deg x_k$, $m=(p-1)/2$, $\mu(t)=(-1)^{mt}/m!$, $\varepsilon(s)=1$ if s is even, $\varepsilon(s)=0$ if s is odd, $x_i=\Delta x_k$ if $\Delta x_k \neq 0$, and the second term is omitted if $\Delta x_k=0$.

As is easily seen, $\Delta(e_s \otimes_{\pi} x_j^p) = \varepsilon(s)e_{s-1} \otimes_{\pi} x_j^p$.

For a base point preserving cellular map $f: X \rightarrow Y$, the product $1 \times f^p: W^r \times X^p \rightarrow W^r \times Y^p$ defines a cellular map

$$ep^r(f): ep^r(X) \rightarrow ep^r(Y).$$

Obviously, $ep^r(f)|ep^s(X) = ep^s(f)$, $s \leq r$, $ep^0(f) = f \wedge \cdots \wedge f$ (p -times), $ep^r(1) = 1$, $ep^r(g) \circ ep^r(f) = ep^r(g \circ f)$, and if $f \simeq f'$ (homotopic) then $ep^r(f) \simeq ep^r(f')$.

Denote by $C_f = Y \cup_f CX$ the mapping cone of f and represents each point of CX by (x, t) , $x \in X$, $t \in I = [0, 1]$, with $(x, 1) = (x_0, t) = y_0$, and $(x, 0) = f(x)$. Then the natural inclusion of $ep^r(Y)$ into $ep^r(C_f)$ can be extended over a map (not cellular)

$$D_f: C_{ep^r(f)} = ep^r(Y) \cup C(ep^r(X)) \rightarrow ep^r(C_f)$$

by sending $(w \times (x_1 \wedge \cdots \wedge x_p), t)$ to $w \times ((x_1, t) \wedge \cdots \wedge (x_p, t))$. Assume that the induced chain map $f_*: C_*(X) \otimes Z_p \rightarrow C_*(Y) \otimes Z_p$ is trivial, hence so is $ep^r(f)_*$. Then there is a canonical splitting $H_{q+1}(C_f) = H_q(X) + H_{q+1}(Y)$. Denote by $\hat{x} = x^{\wedge} \in H_{q+1}(C_f)$ the element corresponding to $x \in H_q(X)$, and similarly for $C_{ep^r(f)}$. Then we have

Theorem 2. $D_{f*}((e_i \otimes_{\pi} x^p)^{\wedge}) = -\mu(q+1)(e_{i-p+1} \otimes_{\pi} (\hat{x})^p) \quad (=0 \text{ if } i < p-1).$

In order to prove this, consider the diagonal map $d: W^r \times I \rightarrow W^r \times I^p$. Leaving fix d on $W^r \times \partial I$, we can deform d equivariantly to a cellular map d' . Put $D'_f(w \times (x_1 \wedge \cdots \wedge x_p), t) = w' \times ((x_1, t_1) \wedge \cdots \wedge (x_p, t_p))$ for $d'(w, t) = (w', t_1, \dots, t_p)$. Then $D_f \simeq D'_f$. Let $d'_*(e_i \otimes_{\pi} I) = \lambda \cdot e_{i-p+1} \otimes_{\pi} I^p + \cdots$, where each of the rest terms contains a face of I^p . Then, by use of the assumption $f_* = 0$, we have $D_{f*}((e_i \otimes_{\pi} x^p)^{\wedge}) = D'_{f*}((e_i \otimes_{\pi} x^p)^{\wedge}) = \pm \lambda \cdot e_{i-p+1} \otimes_{\pi} (\hat{x})^p$. Here the sign $\pm = (-1)^{qp(p-1)/2} = (-1)^{mq}$ is caused of the permutation $(X \times I)^p \rightarrow X^p \times I^p$ and the coefficient λ is $(-1)^{mq}$ by Lemma 5.3 of [4, VII] by considering the case $C_f = S^1$. This proves Theorem 2.

2. Special cases. In the following, n will be sufficiently large so that complexes and maps considered are in stable range. S^n denotes an n -sphere, $M_p^{n+1} = S^n \cup_p e^{n+1}$ a Moore space of type (Z_p, n) , and $i: S^n \rightarrow M_p^{n+1}$ and $\pi: M_p^n \rightarrow S^n$ the natural maps. We shall write sometimes the stable homotopy class of a map $f: X \rightarrow Y$ by the same symbol $f \in \{X, Y\} = \lim [S^n X, S^n Y]$. For example, $\delta = i\pi \in \{M_p^n, M_p^{n+1}\}$, and a generator α of $\{M_p^{n+2p-2}, M_p^n\} \approx Z_p$ is characterized by the relation $P_*^1 e^{n+2p-1} = e^{n+1}$ in the mapping cone C_{α} of α . $G_t = \{S^{n+t}, S^n\}$ is the

t -stem group, and $\alpha_1 = \pi\alpha i \in G_{2p-3}$ is the first element of order p .

First consider the complex $ep^{2r}(S^n)$ which consists of a vertex x_0 and cells e^{pn+j} , $0 \leq j \leq 2r$, with $\partial(e^{pn+2i}) = p \cdot e^{pn+2i-1}$. Up to homotopy type, $ep^{2r}(S^n)$ is a mapping cone of a map $M_p^{pn+2r-1} \rightarrow ep^{2r-2}(S^n)$. Using the results on the stable groups, we have

Lemma 1. *$ep^{4p-6}(S^n)$ has the same homotopy type as the bouquet of some mapping cones $S^{pn} \cup CM_p^{pn+2p-3}$ and $M_p^{pn+2i} \cup CM_p^{pn+2i+2p-3}$, $1 \leq i \leq p-2$. In particular, S^{pn} is a retract of $ep^{p-1}(S^n)$ and there exists a map of M_p^{pn+p-1} into $ep^{p-1}(S^n)$ inducing a monomorphism of the homology.*

Here the attaching maps of the above mapping cones are determined by P_*^1 . By Theorem 1 we have $P_*^1(e^{pn+j+2p-2}) = ([j/2] + n(p-1)/2)e^{pn+j}$. In particular the attaching map of the first mapping cone is a multiple of $\pi\alpha\partial$ and it is trivial if and only if $n \equiv 0 \pmod{p}$. Thus we have

Lemma 2. *There exists a map of $ep^{4p-6}(S^n)$ into $C_{\alpha_1} = C_{i\alpha\pi} = S^{pn} \cup e^{pn+2p-2}$ which is identical on $ep^0(S^n) = S^{pn}$. If $n \equiv 0 \pmod{p}$ then we can replace C_{α_1} by S^{pn} and there exists a map of $M_p^{pn+2p-2}$ into $ep^{2p-2}(S^n)$ inducing a monomorphism of the homology.*

Next consider $ep^r(M_p^{n+1})$. For $x \in H_{n+1}(M_p^{n+1})$, we have by Theorem 1 $P_*^1(e_{p-2} \otimes_\pi x^p) = -\mu(n+2)e_0 \otimes_\pi (\Delta x)^p$ and $P_*^1(e_{p-1} \otimes_\pi x^p) = 0$. Thus we have

Lemma 3. *There exists a map of $C_{\pi\alpha} = S^{pn} \cup {}_\pi C M_p^{pn+2p-2}$ into $ep^{p-1}(M_p^{n+1})$ which is identical on $S^{pn} = ep^0(S^n) \subset ep^0(M_p^{n+1})$.*

Consider $\alpha_1: S^{n+2p-3} \rightarrow S^n$ and the induced map $ep^r(S^{n+2p-3}) \rightarrow ep^r(S^n)$ for $r < 2p(p-1)$. In $ep^r(C_{\alpha_1})$ we see by Theorem 1 that $P_*^p(e_s \otimes_\pi x^p) = -\mu(n+2)\varepsilon(s+1)(e_{s-p} \otimes_\pi (P_*^1 x))$, $x \in H_{n+2p-2}(C_{\alpha_1})$. By Theorem 2, this gives a non-triviality of the functional P^p -operation for $ep^r(\alpha_1)$. In particular, we have

Lemma 4. *Let $\bar{\beta}: M_p^{pn+2p(p-1)-1} \rightarrow ep^{p-1}(S^{n+2p-3}) \rightarrow ep^{p-1}(S^n) \rightarrow S^{pn}$ be the composition of the map of Lemma 1 to $ep^{p-1}(S^{n+2p-3})$, $ep^{p-1}(\alpha_1)$ and the retraction of Lemma 1. Then $\bar{\beta}|_{S^{pn+2p(p-1)-2}} = \beta_1$ is a generator of the p -component of $G_{2p(p-1)-2}$.*

Finally consider $\alpha i: S^{n+2p-2} \rightarrow M_p^{n+1}$ and $ep^{2p-2}(\alpha i)$ for the case $n \equiv 2 \pmod{p}$. Let $j: M_p^a \rightarrow ep^{2p-2}(S^{n+2p-2})$ be the map of Lemma 2, $a = pn + 2(p+1)(p-1)$. Denote by β_s a generator of the p -component of $G_{2(sp+s-1)(p-1)-2}$, $1 \leq s \leq p-1$.

Lemma 5. *For an element $\tilde{\beta}_{s-1}$ of $\pi_b(M_p^a)$, $b = pn + 2(sp+s-1) \times (p-1) - 2$, such that $\pi\tilde{\beta}_{s-1}\beta_s = -1$, we have $ep^{2p-2}(\alpha i)_* j_* \beta_{s-1} \equiv i_* \beta_s$, $i: S^{pn} \subset ep^{2p-2}(S^n)$, modulo the images of $\pi_b(M_p^{pn+2})$ and $\pi_b(M_p^{pn+3})$, $2 \leq s \leq p-1$.*

The proof is based on the methods in [8], [9], but the details are

too long to describe here.

3. Relations in stable homotopy. As before $\alpha_1 \in G_{2p-3}$ and $\beta_s \in G_{2(sp+s-1)(p-1)-2}$, $1 \leq s \leq p-1$, are elements of order p .

Theorem 3. *If $p \cdot \gamma = 0$ for $\gamma \in G_t$, then $\alpha_1 \gamma^p = 0$ and $\{\gamma^p, \alpha_1, p\iota\} \equiv 0$.*

Proof. By the assumption there exists a map $f: M_p^{n+1} \rightarrow S^{n-t}$ such that $f|S^n$ represents γ . Consider the composition of the map $C_{\pi\alpha} \rightarrow ep^{p-1}(M_p^{n+1})$ of Lemma 3, the induced map $ep^{p-1}(f): ep^{p-1}(M_p^{n+1}) \rightarrow ep^{p-1}(S^{n-t})$ and the retraction $ep^{p-1}(S^{n-t}) \rightarrow S^{pn-pt}$ of Lemma 1. Its restriction on S^{pn} represents γ^p . The existence of such a map is equivalent to $\gamma^p \pi \alpha = 0$ which indicates the last assertion, and $\alpha_1 \gamma^p = \gamma^p \alpha_1 = \gamma^p \pi \alpha i = 0$.

Theorem 4. *If $\alpha_1 \gamma = 0$ for $\gamma \in G_t$, then $\beta_1 \gamma^p = 0$ and $\{\gamma^p, \beta_1, p\iota\} \equiv 0$.*

Proof. By the assumption, $ep^{p-1}(\gamma) \circ ep^{p-1}(\alpha_1)$ is homotopic to zero. Let $j: M_p^{pn+2p(p-1)-1} \rightarrow ep^{p-1}(S^{n+2p-3})$ be the map of Lemma 1. Since $\{M_p^{pn+2p(p-1)-1}, M_p^{pn+2i}\} = 0$ for $1 \leq i < p-1$, Lemma 1 and Lemma 4 show that $ep^{p-1}(\alpha_1) \circ j$ is homotopic to $\bar{\beta}$. Then applying the retraction $ep^{p-1}(S^{n-t}) \rightarrow S^{pn-pt}$ of Lemma 1 we have that $\gamma^p \circ \bar{\beta}$ is homotopic to zero, and the theorem follows.

Theorem 5. *If $\{\alpha_1, p\iota, \gamma\} \equiv 0$ for $\gamma \in G_t$ and $2 \leq s \leq p-1$, then $\beta_s \gamma^p \equiv 0 \pmod{\alpha_1 \cdot G_c}$, $c = pt + 2(sp + s - 2)(p - 1) - 1$.*

Proof. Remark that in Lemma 5 the generators of $\pi_b(M_p^{pn+2})$ and $\pi_b(M_p^{pn+3})$ are of the form $\xi \alpha_1$. By the assumption $\bar{\gamma} \circ \alpha i \simeq 0$ for an extension $\bar{\gamma}: M_p^{n+1} \rightarrow S^{n-t}$ of γ . Let $r: ep^{2p-2}(S^{n-t}) \rightarrow C_{\alpha_1} = S^{pn-pt} \cup ep^{pn-pt+2p-2}$ be the map of Lemma 2. Then $r_* ep^{2p-2}(\bar{\gamma})_* ep^{2p-2}(\alpha i)_* j_* \beta_{s-1}$ and $r_* ep^{2p-2}(\pi_b(M_p^{pn+i}))$, $i = 2, 3$, vanish. Thus Lemma 5 shows $i_*(\gamma^p \beta_s) = r_* ep^{2p-2}(\bar{\gamma})_* i_* \beta_s = 0$, i.e., $\beta_s \gamma^p = \gamma^p \beta_s \equiv 0 \pmod{\alpha_1 \cdot G_c}$.

For the case $\gamma = \beta_1$ and $s = 2$, we know that the p -component of G_c vanishes [7]. Thus

Corollary 1. $\beta_2 \beta_1^p = 0$, and the p -component of the $(2(p^2 + 2p) \times (p - 1) - 4)$ -stem group vanishes.

By Theorems 3 and 4, we have $\beta_1^{p^2+1} = 0$, but this is not best possible since $\beta_1^6 = 0$ for $p = 3$. If $p \geq 5$ and $2 \leq s \leq p-1$, then $\{\alpha_1, p\iota, \beta_s\} \equiv 0$. It follows from Theorems 5 and 3 that $\beta_s^{p^2+1} = 0$. If $p = 3$ we have $\{\alpha_1, 3\iota, \beta_2\} = \pm \beta_1^3$, hence $\{\alpha_1, 3\iota, \beta_2^2\} \equiv \pm \beta_2 \beta_1^3 = 0$. Thus we have $\beta_2^{10} = 0$.

Corollary 2. *The elements β_s , $1 \leq s \leq p-1$, are nilpotent.*

Here we make some remarks. As in Lemma 4, for a map $p: S^n \rightarrow S^n$ of degree p , the map $ep^{2p-2}(p)$ composed with maps of Lemma 2 gives $\pi\alpha$. The composition of the map of Lemma 3 and $ep^{p-1}(\beta)$ for the map β of Lemma 4 has a non-trivial functional P^{p^2} -operation. This proves the main theorem of [7]. Further discussions give a complex $S^n \cup e^{n+a} \cup e^{n+b} \cup e^{n+b+1} \cup e^{n+c} \cup e^{n+c+1}$, $a = p^2(2p(p-1))$

$-2)+1$, $b=a+2p(p-1)-1$, $c=b+2p-1=2p^3(p-1)-1$, with $P^{p^3}H^n = \beta P^1 P^p H^{n+a} = H^{n+c+1}$ and $\beta_1^{p^2}$ as the attaching map of e^{n+a} .

4. Non-associativity in mod 3 generalized cohomology theory. Let h^* be a generalized (reduced cohomology theory equipped with a commutative and associative multiplication μ . For an integer $q > 1$, mod q cohomology theory $h^*(\ ; Z_q)$ is defined by $h^i(X; Z_q) = h^{i+2} \times (X \wedge M_q), M_q$ being a Moore space of type $(Z_q, 1)$. If $q \not\equiv 2 \pmod{4}$, $M_q \wedge M_q$ is stably homotopy equivalent to $SM_q \vee S^2 M_q$. Let $\psi: S^2 M_q \rightarrow M_q \wedge M_q$ and $\varphi: M_q \wedge M_q \rightarrow SM_q$ be the natural maps composed the above equivalence. Then $\mu_q = \sigma^{-2}(1 \wedge \psi)^* \mu$ gives an admissible multiplication in $h^*(\ ; Z_q)$ in the sense of [1].

We consider the case that q is odd, then $\psi \simeq T\psi$ for the switching permutation T of $M_q \wedge M_q$. This implies the commutativity of μ_q . Also, the associativity is deduced from $(1 \wedge \psi) S^2 \psi \simeq (\psi \wedge 1)(1 \wedge T) S^2 \psi$. By $\psi \simeq T\psi$ we have $(\psi \wedge 1)(1 \wedge T) S^2 \psi = P(1 \wedge \psi)(S^2 T)(S^2 \psi) \simeq P(1 \wedge \psi) S^2 \psi$ for a cyclic permutation P of $M_q^{(3)} = M_q \wedge M_q \wedge M_q$. The stable group $\{S^4 M_q, M_q^{(3)}\}$ is generated by the element $(1 \wedge \psi) S^2 \psi$ of order q and the composition $i\alpha_1 \pi: S^4 M_q \rightarrow S^6 \rightarrow S^3 \rightarrow M_q^{(3)}$ of order $(q, 3)$. By the arguments of [1] we have $(1 \wedge \psi) S^2 \psi - P(1 \wedge \psi) S^2 \psi = k \cdot i\alpha_1 \pi$ for some $k \in Z_{(q, 3)}$ and this implies the relation $x(yz) - (xy)z = k \cdot \alpha_1^{**}(\delta_q x)(\delta_q y)(\delta_q z)$ in $h^*(\ ; Z_q)$.

Similarly, $S\varphi(1 \wedge \varphi) - S\varphi(1 \wedge \varphi)P = k' \cdot i'\alpha_1 \pi'$ for some $k' \in Z_{(q, 3)}$ and $i'\alpha_1 \pi': M_q^{(3)} \rightarrow S^6 \rightarrow S^3 \rightarrow S^2 M_q$, and this implies $x(yz) - (xy)z = k' \cdot \alpha_1(\partial_q x) \times (\partial_q y)(\partial_q z)$ for the multiplication in the stable group $\pi_*(M_q)$ given by φ . By use of $\varphi\psi = 0$, $\pi = \pi'(1 \wedge \psi) S^2 \psi$, $i' = S\varphi(1 \wedge \varphi)i$, we have $k \cdot i'\alpha_1 \pi = -S\varphi(1 \wedge \varphi)P(1 \wedge \psi) S^2 \psi = k' \cdot i'\alpha_1 \pi$, and this implies $k \equiv k' \pmod{(q, 3)}$. Obviously $h^*(\ ; Z_q)$ and $\pi_*(M_q)$ are associative if $q \not\equiv 0 \pmod{3}$ or if $q \equiv k \equiv 0 \pmod{3}$.

Now assume $q \equiv 0 \pmod{3}$, then $k \cdot i\alpha_1$ is an obstruction to extend the map $S\varphi(1 \wedge \varphi)$ over $W^1 \times_\pi M_q^{(3)} \supset M_q^{(3)}$ since $W^1 \times_\pi M_q^{(3)}$ is obtained from $I \times M_q^{(3)}$ identifying $0 \times M_q^{(3)}$ with $1 \times M_q^{(3)}$ by the permutation P . It follows without difficulty that $k \not\equiv 0 \pmod{3}$ if and only if $P_*(e_1 \otimes_\pi x^3) \neq 0$ for a generator x of $H_2(M_q)$. By Theorem 1, $P_*(e_1 \otimes_\pi x^3) = e_0 \otimes_\pi (\Delta x)^3$. Thus $k \not\equiv 0 \pmod{3}$ if and only if $\Delta x \neq 0$, i.e., $q \not\equiv 0 \pmod{9}$. Consequently we have

Theorem 6. *Let q be odd > 1 . If $q \not\equiv 0 \pmod{3}$ or $q \equiv 0 \pmod{9}$ then $h^*(\ ; Z_q)$ and $\pi_*(M_q)$ are associative. If $q \equiv 0 \pmod{3}$ and $q \not\equiv 0 \pmod{9}$, then we have $x(yz) - (xy)z = \pm \alpha_1^{**}(\delta_q x)(\delta_q y)(\delta_q z)$ in $h^*(\ ; Z_q)$ and $= \pm \alpha_1(\partial_q x)(\partial_q y)(\partial_q z)$ in $\pi_*(M_q)$.*

Note that $\pi_*(M_3)$ is not associative since $\alpha_1 \beta_1^2 \beta_2 \neq 0$.

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