

E_∞ -STRUCTURES AND DIFFERENTIALS OF THE ADAMS SPECTRAL SEQUENCE

V.A.Smirnov

The Adams spectral sequence was invented by J.F.Adams [1] almost fifty years ago for calculations of stable homotopy groups of topological spaces and in particular of spheres. The calculation of differentials of this spectral sequence is one of the most difficult problem of Algebraic Topology. Here we consider an approach to solve this problem in the case of $\mathbb{Z}/2$ coefficients and find inductive formulas for the differentials. It is based on the A_∞ -structures [2], E_∞ -structures [3], [4], [5], [6] and functional homology operations [7], [8], [9]. This approach will be applied to the Kervaire invariant problem [10], [11].

CONTENTS

1. The Bousfield-Kan spectral sequences.
2. E_∞ -algebras and E_∞ -coalgebras.
3. E_∞ -structure on the Bousfield-Kan spectral sequence.
4. The homology of an E_∞ -operad and the Milnor coalgebra.
5. Homology operations for the operad E .
6. $\cup_\infty - A_\infty$ -Hopf algebras.
7. Differentials of the Adams spectral sequence and the Kervaire invariant problem.

1. The Bousfield-Kan spectral sequences

Consider the Bousfield-Kan spectral sequence [12], which is one of the most general spectral sequence of the homotopy groups.

Let R be a field. Given a simplicial set Z denote by RZ the free simplicial R -module generated by Z . There is the cosimplicial resolution

$$R^*Z : \quad Z @>\delta^0>> RZ @>\delta^0, \delta^1>> R^2Z \rightarrow \cdots \rightarrow R^n Z @>\delta^0, \dots, \delta^n>> R^{n+1}Z \rightarrow \dots$$

This resolution was used by Bousfield and Kan [12] to construct the spectral sequence of the homotopy groups of Z with coefficients in R .

The E^1 -term of this spectral sequence is expressed by the complex

$$H_*(Z; R) \rightarrow H_*(RZ; R) \rightarrow \cdots \rightarrow H_*(R^{n-1}Z; R) \rightarrow H_*(R^n Z; R) \rightarrow \dots$$

Higher differentials are expressed by the homology operations

$$d_m : H_*(R^{n-1}Z; R) \rightarrow H_*(R^{n+m-1}Z; R).$$

In [7], [8] the homology operations were defined as partial and multi-valued mappings. However there is a general method to choose the homology operations to be usual homomorphisms. The corresponding theory was developed in [9]. Recall the main definitions.

For a chain complex X denote by X_* its homology, $X_* = H_*(X)$. Fix chain mappings $\xi : X_* \rightarrow X$, $\eta : X \rightarrow X_*$ and chain homotopy $h : X \rightarrow X$ satisfying the relations

$$\eta \circ \xi = Id, \quad d(h) = \xi \circ \eta - Id, \quad h \circ \xi = 0, \quad \eta \circ h = 0, \quad h \circ h = 0.$$

Consider a sequence of mappings

$$f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$$

and define functional homology operations

$$H_*(f^n, \dots, f^1): X_*^1 \rightarrow X_*^{n+1}$$

putting

$$H_*(f^n, \dots, f^1) = \eta \circ f^n \circ h \circ \dots \circ f^1 \circ \xi.$$

Direct calculations show that the following relations are satisfied

$$(1) \quad \sum_{i=1}^{n-1} (-1)^{n-i+1} H_*(f^n, \dots, f^{i+1} \circ f^i, \dots, f^1) =$$

$$(2) \quad \sum_{i=1}^{n-1} (-1)^{n-i} H_*(f^n, \dots, f^{i+1}) \circ H_*(f^i, \dots, f^1).$$

Functional homology operations may be defined not only for the category of chain complexes but in some other situations, for example, for the category of simplicial modules.

Directly from the definition it follows that higher differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations

$$H_*(\delta, \dots, \delta): H_*(R^{n-1}Z; R) \rightarrow H_*(R^{n+m-1}Z; R).$$

So we have

Theorem 1. *The differentials of the Bousfield-Kan spectral sequence are expressed by the functional homology operations*

$$H_*(\delta, \dots, \delta): H_*(R^{n-1}Z, R) \rightarrow H_*(R^{n+m-1}Z, R).$$

These operations determine on the E^1 -term new differential. The homology of the corresponding complex is isomorphic to the E^∞ -term of the spectral sequence.

As it was proved in [5] instead of the Bousfield-Kan cosimplicial object we may consider the following cosimplicial object

$$F^*(C, RZ): \quad RZ @>\delta^0, \delta^1>> CRZ \rightarrow \dots \rightarrow C^{n-1}RZ @>\delta^0, \dots, \delta^n>> C^nRZ \rightarrow \dots,$$

where CRZ is the free commutative simplicial coalgebra generated by RZ .

The E^1 -term of the corresponding spectral sequence is expressed by the complex

$$H_*(Z; R) = \pi_*(RZ) \rightarrow \pi_*(CRZ) \rightarrow \dots \rightarrow \pi_*(C^{n-1}RZ) \rightarrow \pi_*(C^nRZ) \rightarrow \dots$$

Directly from the definition it follows that higher differentials of this spectral sequence are expressed by the functional homology operations

$$H_*[\delta, \dots, \delta]: \pi_*(C^nRZ) \rightarrow \pi_*(C^{n+m}RZ).$$

Moreover there is a cosimplicial mapping

$$\begin{array}{ccccccc} RZ & \longrightarrow & R^2Z & \longrightarrow & \dots & \longrightarrow & R^{n+1}Z & \longrightarrow & \dots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ RZ & \longrightarrow & CRZ & \longrightarrow & \dots & \longrightarrow & C^nRZ & \longrightarrow & \dots \end{array}$$

inducing the isomorphism of the corresponding spectral sequences. So we have

Theorem 2. *The differentials of the Bousfield-Kan spectral sequence of the cosimplicial object*

$$F^*(C, RZ): \quad RZ @>\delta^0, \delta^1>> CRZ \rightarrow \cdots \rightarrow C^{n-1}RZ @>\delta^0, \dots, \delta^n>> C^n RZ \rightarrow \cdots$$

are expressed by the functional homology operations

$$H_*(\delta, \dots, \delta): \pi_*(C^n RZ) \rightarrow \pi_*(C^{n+m} RZ).$$

These operations determine on the E^1 -term a new differential. The homology of the corresponding chain complex is isomorphic to the E^∞ -term of this spectral sequence.

2. E_∞ -algebras and E_∞ -coalgebras

Recall that an operad in the category of chain complexes is a family $E = \{E(j)\}_{j \geq 1}$ of chain complexes $E(j)$ together with given actions of symmetric groups Σ_j and operations

$$\gamma: E(k) \otimes E(j_1) \otimes \cdots \otimes E(j_k) \rightarrow E(j_1 + \cdots + j_k)$$

compatible with these actions and satisfying associativity relations [3], [4].

An operad $E = \{E(j)\}$ for which complexes $E(j)$ are acyclic and symmetric groups act on them freely is called an E_∞ -operad.

A chain complex X is called an algebra (a coalgebra) over an operad E or simply an E -algebra (an E -coalgebra) if there are given mappings

$$\mu: E(k) \otimes_{\Sigma_k} X^{\otimes k} \rightarrow X \quad (\tau: X \rightarrow \text{Hom}_{\Sigma_k}(E(k); X^{\otimes k})),$$

satisfying some associativity relations.

Algebras (coalgebras) over an E_∞ -operad are called E_∞ -algebras (E_∞ -coalgebras).

Any operad in the category of chain complexes determines a monad $\underline{E}$ and a comonad $\overline{E}$ by the formulas

$$(3) \quad \underline{E}(X) = \sum_j \underline{E}(j, X), \quad \underline{E}(j, X) = E(j) \otimes_{\Sigma_j} X^{\otimes j};$$

$$(4) \quad \overline{E}(X) = \prod_j \overline{E}(j, X), \quad \overline{E}(j, X) = \text{Hom}_{\Sigma_j}(E(j), X^{\otimes j}).$$

An operad structure γ induces natural transformations

$$\underline{\gamma}: \underline{E} \circ \underline{E} \rightarrow \underline{E}, \quad \overline{\gamma}: \overline{E} \rightarrow \overline{E} \circ \overline{E}.$$

An E -algebra (an E -coalgebra) structure on a chain complex X induces a mapping

$$\mu: \underline{E}(X) \rightarrow X \quad (\tau: X \rightarrow \overline{E}(X)).$$

So to give on a chain complex X an E -algebra (E -coalgebra) structure is the same as to give on X an algebra (coalgebra) structure over the monad $\underline{E}$ (the comonad $\overline{E}$).

One of the most important example of an E_∞ -algebra is the singular cochain complex $C^*(Y; R)$ of a topological space Y .

Dually, the singular chain complex $C_*(Y; R)$ of a topological space Y and the chain complex $N(RZ)$ of a simplicial set Z are examples of E_∞ -coalgebras.

The homotopy theory of E_∞ -coalgebras was constructed in [5]. There were defined the homotopy groups of E_∞ -coalgebras. For the chain complex $N(RZ)$ of a simplicial set Z these homotopy groups are isomorphic to the homotopy groups of Z with coefficients in R .

For an E -coalgebra X , using cosimplicial resolution

$$F^*(\bar{E}, \bar{E}, X) : X @>\tau>> \bar{E}(X) \rightarrow \cdots \rightarrow \bar{E}^{n-1}(X) \rightarrow \bar{E}^n(X) \rightarrow \dots,$$

there was constructed the spectral sequence of the homotopy groups of the E -coalgebra X , [5].

Denote by X_* the homology of the complex X and by $\bar{E}_*$ the homology of a comonad $\bar{E}$. X_* will be $\bar{E}_*$ -coalgebra. There is the cosimplicial resolution

$$F^*(\bar{E}_*, \bar{E}_*, X_*) : \bar{E}_*(X_*) \rightarrow \bar{E}_*^2(X_*) \rightarrow \cdots \rightarrow \bar{E}_*^n(X_*) \rightarrow \bar{E}_*^{n+1}(X_*) \rightarrow \dots$$

The E^1 term of the spectral sequence is expressed by the cobar construction

$$F(\bar{E}_*, X_*) : X_* \rightarrow \bar{E}_*(X_*) \rightarrow \cdots \rightarrow \bar{E}_*^n(X_*) \rightarrow \bar{E}_*^{n+1}(X_*) \rightarrow \dots,$$

obtained from the resolution by taking primitive elements. So there is the inclusion $F(\bar{E}_*, X_*) \rightarrow F(\bar{E}_*, \bar{E}_*, X_*)$.

The functional homology operations

$$H_*(\delta, \dots, \delta) : \bar{E}_*^n(X_*) \rightarrow \bar{E}_*^{n+m}(X_*)$$

determine new differential in the resolution and in the cobar construction. The corresponding complexes denote by $\tilde{F}(\bar{E}_*, \bar{E}_*, X_*)$, $\tilde{F}(\bar{E}_*, X_*)$.

Note that the complex $\tilde{F}(\bar{E}_*, \bar{E}_*, X_*)$ is a resolution of the complex X_* and there is the inclusion $\tilde{F}(\bar{E}_*, X_*) \rightarrow \tilde{F}(\bar{E}_*, \bar{E}_*, X_*)$.

Theorem 3. *Differentials of the spectral sequence of the homotopy groups of an E -coalgebra X are determined by the functional homology operations*

$$H_*(\delta, \dots, \delta) : \bar{E}_*^n(X_*) \rightarrow \bar{E}_*^{n+m}(X_*).$$

The homology of $\tilde{F}(\bar{E}_*, X_*)$ is isomorphic to the E^∞ -term of the spectral sequence.

If X is the normalized chain complex of a simplicial set Z , i.e. $X = N(RZ)$, then there is a mapping of cosimplicial objects

$$\begin{array}{ccccccc} N(RZ) & \longrightarrow & N(CRZ) & \longrightarrow & \dots & \longrightarrow & N(C^n RZ) \longrightarrow \dots \\ \downarrow & & \downarrow & & & & \downarrow \\ X & \longrightarrow & \bar{E}(X) & \longrightarrow & \dots & \longrightarrow & \bar{E}^n(X) \longrightarrow \dots \end{array}$$

inducing the isomorphism of the corresponding spectral sequences. So we have

Theorem 4. *The differentials of the Bousfield-Kan spectral sequence of the homotopy groups of a simplicial set Z are determined by the functional homology operations*

$$H_*(\delta, \dots, \delta) : \bar{E}_*^n(Z_*) \rightarrow \bar{E}_*^{n+m}(Z_*).$$

The homology of $\tilde{F}(\bar{E}_*, Z_*)$ is isomorphic to the E^∞ -term of the spectral sequence.

Note that the suspension SX over an E -coalgebra X is an SE -coalgebra and the following diagrams commute

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\bar{\gamma}} & \bar{E} \circ \bar{E} \\ \downarrow & & \downarrow \\ \overline{SE} & \xrightarrow{\overline{S\gamma}} & \overline{SE} \circ \overline{SE} \end{array} \quad \begin{array}{ccc} SX & \xrightarrow{\tau} & \overline{SE}(SX) \\ \downarrow & & \downarrow \\ SX & \xrightarrow{S\tau} & S(\bar{E}(X)) \end{array}$$

Moreover from the expression of the homology $\bar{E}_*$ of the comonad $\bar{E}$ (see below) it follows that the mappings $\xi: \bar{E}_* \rightarrow \bar{E}$, $\eta: \bar{E} \rightarrow \bar{E}_*$, $h: \bar{E} \rightarrow \bar{E}$ may be chosen permutable with the suspension homomorphism $\overline{SE} \rightarrow \bar{E}$. Therefore constructed functional homology operations permute with the suspension homomorphism. Hence the following theorem is taken place.

Theorem 5. *Functional homology operations giving higher differentials of the Bousfield-Kan spectral sequence permute with the suspension and hence are stable. They induce the differentials of the Adams spectral sequence of stable homotopy groups of a topological space.*

3. The homology of an E_∞ -operad and the Milnor coalgebra

Let E be an E_∞ -operad, M be a graded module (over $\mathbb{Z}/2$). As it is known (see for example [6]), the homology $\underline{E}_*(M)$ of the complex $\underline{E}(M)$ is the polynomial algebra generated by the elements $e_{i_1} \dots e_{i_k} x_m$, $1 \leq i_1 \leq \dots \leq i_k$, $x_m \in M$ of dimensions $i_1 + 2i_2 + \dots + 2^{k-1}i_k + 2^k m$.

The elements $e_{i_1} \dots e_{i_k} x_m$ of $\underline{E}_*(M)$ may be rewritten in the form

$$Q^{j_1} \dots Q^{j_k} \otimes x_m; \quad j_1 \leq 2j_2, \dots, j_{k-1} \leq 2j_k, m \leq j_k,$$

where

$$(5) \quad j_k = i_k + m,$$

$$(6) \quad j_{k-1} = i_{k-1} + i_k + 2m,$$

$$(7) \quad \dots$$

$$(8) \quad j_1 = i_1 + i_2 + 2i_3 + \dots + 2^{k-2}i_k + 2^{k-1}m.$$

The sequences $Q^{j_1} \dots Q^{j_k}$ give the elements of the Dyer-Lashof algebra $\mathcal{R}$, [14], [15].

Given a graded module M denote by $\mathcal{R} \times M$ the quotient module of the tensor product $\mathcal{R} \otimes M$ under the submodule generated by the elements $Q^{j_1} \dots Q^{j_k} \otimes x_m$, $j_k < m$. The correspondence $M \mapsto \mathcal{R} \times M$ determines the monad in the category of graded modules.

A graded module M is called an unstable module over the Dyer-Lashof algebra if it is an algebra over the corresponding monad.

Dually, the homology $\bar{E}_*(M)$ of the complex $\bar{E}(M)$ is the free commutative coalgebra generated by the elements

$$e_{i_1} \dots e_{i_k} x^m, \quad 1 \leq i_1 \leq \dots \leq i_k, \quad x^m \in M$$

of dimensions $2^k m - (i_1 + 2i_2 + \dots + 2^{k-1}i_k)$.

Regrading of the elements of $\bar{E}_*(M)$ leads to the Milnor coalgebra $\mathcal{K}$. By definition $\mathcal{K}$ is the polynomial algebra generated by the elements ξ_i , $i \geq 0$ of dimensions

$2^i - 1$. A comultiplication

$$\nabla: \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K}$$

on the generators ξ_i is given by the formula

$$\nabla(\xi_i) = \sum_k \xi_{i-k}^{2^k} \otimes \xi_k.$$

On the other elements the comultiplication is determined by the Hopf relation.

Define the grading $\deg(x)$ of elements $x \in \mathcal{K}$ putting $\deg(\xi_i) = 1$ and the grading of the product equal to the sum of the gradings of factors.

Given graded module M denote by $\mathcal{K} \times M$ the submodule of the tensor product $\mathcal{K} \otimes M$ generated by the elements $x \otimes y$, $\deg(x) = \dim(y)$. The correspondence $M \mapsto \mathcal{K} \times M$ determines the comonad $\mathcal{K}$ in the category of graded modules.

A graded module M is called an unstable comodule over the Milnor coalgebra if it is a coalgebra over the corresponding comonad.

Let M be an unstable comodule over the Milnor coalgebra. There is a cosimplicial resolution

$$F^*(\mathcal{K}, \mathcal{K}, M) : M \rightarrow \mathcal{K} \times M \rightarrow \cdots \rightarrow \mathcal{K}^{\times n} \times M \rightarrow \mathcal{K}^{\times n+1} \times M \rightarrow \cdots$$

If Y is a "nice" (in the sense of Massey-Peterson) space then the Bousfield-Kan spectral sequence turns to the Massey-Peterson spectral sequence. The E^1 -term of this spectral sequence may be written in the form

$$F^*(\mathcal{K}, Y_*) : Y_* \rightarrow \mathcal{K} \times Y_* \rightarrow \cdots \rightarrow \mathcal{K}^{\times n} \times Y_* \rightarrow \cdots,$$

where $Y_* = H_*(Y; \mathbb{Z}/2)$.

The functional homology operations determine on the Milnor coalgebra the structure of an A_∞ -coalgebra [9]. On the homology Y_* of a topological space there is the structure of an A_∞ -comodule over the A_∞ -coalgebra $\mathcal{K}$. The corresponding cobar construction denote $\tilde{F}(\mathcal{K}, Y_*)$.

From the previous theorems it follows the next theorem.

Theorem 6. *If Y is a "nice" space then the homology of the cobar construction $\tilde{F}(\mathcal{K}, Y_*)$ is isomorphic to the E^∞ term of the Massey-Peterson spectral sequence.*

Besides the Milnor coalgebra $\mathcal{K}$ we will consider the stable Milnor coalgebra $\mathcal{K}_s$ for which $\xi_0 = 1$.

Given comodule M over the stable Milnor coalgebra there is a cosimplicial resolution

$$F^*(\mathcal{K}_s, \mathcal{K}_s, M) : \mathcal{K}_s \otimes M \rightarrow \mathcal{K}_s^{\otimes 2} \otimes M \rightarrow \cdots \rightarrow \mathcal{K}_s^{\otimes n} \otimes M \rightarrow \cdots$$

Stabilization of the Bousfield-Kan spectral sequence leads to the Adams spectral sequence of stable homotopy groups of a topological space Y . E^1 -term of this spectral sequence may be written in the form

$$F^*(\mathcal{K}_s, Y_*) : Y_* \rightarrow \mathcal{K}_s \otimes Y_* \rightarrow \cdots \rightarrow \mathcal{K}_s^{\otimes n} \otimes Y_* \rightarrow \cdots$$

Since the functional homology operations are stable, the corresponding A_∞ -coalgebra structure on the Milnor coalgebra $\mathcal{K}$ also will be stable. Hence it induces an A_∞ -coalgebra structure on the stable Milnor coalgebra $\mathcal{K}_s$. Thus we have

Theorem 7. *For a topological spaces Y the homology of the cobar construction $\tilde{F}(\mathcal{K}_s, Y_*)$ is isomorphic to the E^∞ term of the Adams spectral sequence of stable homotopy groups of the topological space Y .*

4. E_∞ -structure on the Bousfield-Kan spectral sequence

Our aim here is to define E_∞ -structure on the Bousfield-Kan spectral sequence. To do it consider the following additional property of the operad E .

Theorem 8. *Given E_∞ -operad E there is a permutation mapping*

$$T: \underline{E} \circ \overline{E} \rightarrow \overline{E} \circ \underline{E},$$

commuting the following diagrams

$$\begin{array}{ccccc} \underline{E}^2 \circ \overline{E} & \xrightarrow{T\underline{E} \circ \overline{E}T} & \overline{E} \circ \underline{E}^2 & \underline{E} \circ \overline{E} & \xrightarrow{T} & \overline{E} \circ \underline{E} \\ \gamma \overline{E} \downarrow & & \downarrow \overline{E} \gamma & \underline{E} \gamma \downarrow & & \downarrow \gamma \underline{E} \\ \underline{E} \circ \overline{E} & \xrightarrow{T} & \overline{E} \circ \underline{E} & \underline{E} \circ \overline{E}^2 & \xrightarrow{\overline{E}T \circ T\underline{E}} & \overline{E}^2 \circ \underline{E} \end{array}$$

Proof. Given an E_∞ -operad E we may construct an operad mapping $\nabla: E \rightarrow E \otimes E$ consisting of mappings $\nabla(j): E(j) \rightarrow E(j) \otimes E(j)$. This mapping give on E a Hopf operad structure. Denote $\nabla(j, i): E(j) \rightarrow E(j)^{\otimes i}$ the iterations of these mappings $\nabla(j, 2) = \nabla(j)$. They are Σ_j -mappings, i.e.

$$\nabla(j)(x\sigma) = \nabla(j)(x)\sigma^{\otimes j}, \quad \sigma \in \Sigma_j$$

but are not commuting with permutations of factors of $E(j)^{\otimes i}$. However they may be extended till the mappings $\nabla(j, i): E(i) \otimes E(j) \rightarrow E(j)^{\otimes i}$, compatible with the actions of symmetric groups Σ_i and Σ_j .

Rewrite these mappings in the form

$$\nabla(j, i): E(j) \rightarrow \overline{E}(i) \otimes E(j)^{\otimes i}.$$

If the operad E is chosen freely then they may be done compatible with an operad structure.

Passing to the dual mappings we obtain mappings

$$\overline{\nabla}(j, i): E(i) \otimes \overline{E}(j)^{\otimes i} \rightarrow \overline{E}(j).$$

Define mappings

$$T(j, i): E(j) \otimes \overline{E}(i)^{\otimes j} \rightarrow E(i) \otimes \overline{E}(j)^{\otimes i}$$

as the compositions

$$(9) \quad E(j) \otimes \overline{E}(i)^{\otimes j} @>> \nabla(j) \otimes 1^{\otimes j} >> E(j) \otimes E(j) \otimes \overline{E}(i)^{\otimes j} \rightarrow$$

$$(10) \quad \rightarrow E(j) \otimes \overline{E}(i)^{\otimes j} \otimes E(j) @>> \overline{\nabla}(j, i) \otimes 1 >> \overline{E}(i) \otimes E(j) \rightarrow$$

$$(11) \quad @>> 1 \otimes \nabla(j, i) >> \overline{E}(i) \otimes \overline{E}(i) \otimes E(j)^{\otimes i} @>> \overline{\nabla}(i) \otimes 1^{\otimes i} >> \overline{E}(i) \otimes E(j)^{\otimes i}.$$

The family $T(j, i)$ determines the required permutation mapping

$$T: \underline{E} \circ \overline{E} \rightarrow \overline{E} \circ \underline{E}.$$

A chain complex X will be called an E_∞ -Hopf algebra if there are given an E_∞ -algebra structure $\mu: \underline{E}(X) \rightarrow X$ and an E_∞ -coalgebra structure $\tau: X \rightarrow \overline{E}(X)$

such that the following diagram commutes

$$\begin{array}{ccccc}
\underline{E}(X) & \xrightarrow{\mu} & X & \xrightarrow{\tau} & \overline{E}(X) \\
\underline{E}(\tau) \downarrow & & & & \downarrow = \\
\underline{E}\overline{E}(X) & \xrightarrow{T} & \overline{E}\underline{E}(X) & \xrightarrow{\overline{E}(\mu)} & \overline{E}(X)
\end{array}$$

If a topological space Y is an E_∞ -space then its singular chain complex $C_*(Y; R)$ will be an E_∞ -Hopf algebra. For example the singular chain complex of the infinite loop space is an E_∞ -Hopf algebra.

Let X is an E_∞ -Hopf algebra. Then there is a mapping of augmented cosimplicial objects

$$\begin{array}{ccccccc}
\underline{E}(X) & \longrightarrow & \underline{E}\overline{E}(X) & \longrightarrow & \dots & \longrightarrow & \underline{E}\overline{E}^n(X) \longrightarrow \dots \\
\downarrow & & \downarrow & & & & \downarrow \\
X & \longrightarrow & \overline{E}(X) & \longrightarrow & \dots & \longrightarrow & \overline{E}^n(X) \longrightarrow \dots
\end{array}$$

Then the complex $F(\overline{E}, \overline{E}, X)$ will be E_∞ -algebra. Passing to the homology we obtain E_∞ -algebra structure on the complex $\tilde{F}(\overline{E}_*, \overline{E}_*, X_*)$. Thus we have

Theorem 9. *If X is an E_∞ -Hopf algebra, then the complex $\tilde{F}(\overline{E}_*, \overline{E}_*, X_*)$ possesses E_∞ -algebra structure.*

Corollary. *If Y is a "nice" E_∞ -space then the complex $\tilde{F}(\mathcal{K}, \mathcal{K}, Y_*)$ possesses E_∞ -algebra structure.*

This structure will be used in the further calculations of higher differentials of the Bousfield-Kan spectral sequence.

Note that on the cobar constructions $\tilde{F}(\overline{E}_*, X_*)$, $\tilde{F}(\mathcal{K}, Y_*)$ there are no E_∞ -algebra structure.

Let us calculate the E_∞ -algebra structure on the Milnor coalgebra. As it was pointed out above for an E_∞ -operad E there is the permuting mapping $T: \underline{E} \circ \overline{E} \rightarrow \overline{E} \circ \underline{E}$. It induces the permuting mapping $T_*: \underline{E}_* \circ \overline{E}_* \rightarrow \overline{E}_* \circ \underline{E}_*$ commuting diagrams

$$\begin{array}{ccccccc}
\underline{E}_*^2 \circ \overline{E}_* & \xrightarrow{T_* \underline{E}_* \circ \overline{E}_* T_*} & \overline{E}_* \circ \underline{E}_*^2 & \underline{E}_* \circ \overline{E}_* & \xrightarrow{T_*} & \overline{E}_* \circ \underline{E}_* \\
\downarrow \gamma_* \overline{E}_* & & \downarrow \overline{E}_* \gamma_* & \downarrow \underline{E}_* \overline{\gamma}_* & & \downarrow \overline{\gamma}_* \underline{E}_* \\
\underline{E}_* \circ \overline{E}_* & \xrightarrow{T_*} & \overline{E}_* \circ \underline{E}_* & \underline{E}_* \circ \overline{E}_*^2 & \xrightarrow{\overline{E}_* T_* \circ T_* \overline{E}_*} & \overline{E}_*^2 \circ \underline{E}_*
\end{array}$$

The permuting mapping T_* induces the action $\mu_*: \underline{E}_* \circ \overline{E}_* \rightarrow \overline{E}_*$ and the dual coaction $\tau_*: \underline{E}_* \rightarrow \overline{E}_* \circ \underline{E}_*$.

Denote by $e_i: \mathcal{K} \rightarrow \mathcal{K}$ the operation on the Milnor coalgebra inducing by the restriction of μ_* on the elements e_i . From the commutative diagrams for the permuting mapping T_* it follows

Theorem 10. *The operations $e_i: \mathcal{K} \rightarrow \mathcal{K}$ satisfy the relations:*

1. $e_0(x) = x^2$.
2. $e_i(xy) = \sum e_k(x)e_{i-k}(y)$.
3. $\nabla e_i(x) = \sum \xi_0^{-k} e_{i-k}(\xi_0^k x') \otimes e_k(x'')$, where $\sum x' \otimes x'' = \nabla(x)$.

Using these relations to calculate the operations e_i it is sufficient to calculate only $e_1(\xi_0)$. Direct calculations show that $e_1(\xi_0) = \xi_1 \xi_0$. From the third relation it follows

Theorem 11. *There are the following formulas*

$$e_i(\xi_k) = \begin{cases} \xi_{m+k}\xi_k, & i = 2^{m+k} - 2^k; \\ \xi_{m+k}\xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}; \\ \dots & \dots \\ \xi_{m+k}\xi_0, & i = 2^{m+k} - 2^k - \dots - 1; \\ 0, & \text{in other cases.} \end{cases}$$

Using the second relation we may obtain formulas for the operations e_i on the products of the elements ξ_k .

Passing from the elements e_i to the elements of the Dyer-Lashof algebra we obtain the action of the Dyer-Lashof algebra on the Milnor coalgebra. On the generators ξ_i it is given by the formulas

$$Q^{i+2^k-1}(\xi_k) = \begin{cases} \xi_{m+k}\xi_k, & i = 2^{m+k} - 2^k; \\ \xi_{m+k}\xi_{k-1}, & i = 2^{m+k} - 2^k - 2^{k-1}; \\ \dots & \dots \\ \xi_{m+k}\xi_0, & i = 2^{m+k} - 2^k - \dots - 1; \\ 0, & \text{in other cases.} \end{cases}$$

On the other elements this action is determined by the Hopf relations

$$Q^i(xy) = \sum Q^k(x)Q^{i-k}(y).$$

Besides the action of the Dyer-Lashof algebra on the Milnor coalgebra $\mathcal{K}$ there are $\cup_i$ -products and an E_∞ -algebra structure. On the generators $x \in \mathcal{K}$ it is defined by the formulas

$$x \cup_i x = e_i(x).$$

On the other elements $\cup_i$ -products are defined by the relations

$$(12) \quad x \cup_i y = y \cup_i x,$$

$$(13) \quad (x_1 x_2) \cup_i y = x_1(x_2 \cup_i y) + (x_1 \cup_i y)x_2.$$

Note that the stable Milnor coalgebra $\mathcal{K}_s$ has no action of the Dyer-Lashof algebra and has no E_∞ -algebra structure.

5. Homology operations for the operad E .

Let $\Delta^* = \{\Delta^n\}$ denotes the cosimplicial object of the category of chain complexes, consisting of the chain complexes of the standard n -dimensional simplices. Let further F be a functor in the category of chain complexes for which there are given transformations

$$\Delta^n \otimes F(X) \rightarrow F(\Delta^n \otimes X),$$

permuting with coface and codegeneracy operators. Such functor F will be called a chain functor.

A transformation $\alpha: F' \rightarrow F''$ of chain functor is a transformations of functors, commuting the diagrams

$$\begin{array}{ccc} \Delta^n \otimes F'(X) & \longrightarrow & F'(\Delta^n \otimes X) \\ 1 \otimes \alpha \downarrow & & \downarrow \alpha \\ \Delta^n \otimes F''(X) & \longrightarrow & F''(\Delta^n \otimes X) \end{array}$$

Given chain functor F we may consider mappings

$$F(f): F(X) \rightarrow F(Y),$$

induced not only by chain mappings $f: X \rightarrow Y$ of dimension zero but dimension n also. Namely given mapping $f: X \rightarrow Y$ of dimension n we represent as the restriction of the mapping $\tilde{f}: \Delta^n \otimes X \rightarrow Y$ on the n -dimensional generator $u_n \in \Delta^n$. Then the required mapping

$$F(f): F(X) \rightarrow F(Y)$$

of dimension n will be the restriction of the composition

$$\Delta^n \otimes F(X) \rightarrow F(\Delta^n \otimes X) @>\tilde{f}>> F(Y)$$

on the n -dimensional generator $u_n \in \Delta^n$.

Given chain functor F denote by F_* the functor, corresponding to a chain complex X the graded module of its homology $F_*(X) = H_*(F(X))$. The functor F_* is not only a functor. There are functional homology operations which assigns to sequences of chain mappings $f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$ the mapping

$$F_*(f^n, \dots, f^1) = H_*(F(f^n), \dots, F(f^1)): F_*(X^1) \rightarrow F_*(X^{n+1}),$$

of dimension $n - 1$.

Theorem 12. *Let F be a chain functor. Then for any sequence of chain mappings $f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$ the following formula is taken place*

$$H_*(F(f^n), \dots, F(f^1)) = \sum (-1)^\epsilon F_*(H_*(f^n, \dots, f^{n_m+1}), \dots, H_*(f^{n_1}, \dots, f^1)),$$

where the sum is taken over m and $n_1, \dots, n_m$ such that $1 \leq n_1 < \dots < n_m < n$.

Proof. Let X be chain complex. We take the mapping $F_*(X_*) \rightarrow F(X)$ of choosing representatives as the composition

$$\xi(F): F_*(X_*) \rightarrow F(X_*), \quad F(\xi): F(X_*) \rightarrow F(X).$$

Similary we take the projection $F(X) \rightarrow F_*(X_*)$ as the composition

$$F(\eta): F(X) \rightarrow F(X_*), \quad \eta(F): F(X_*) \rightarrow F_*(X_*).$$

We take the homotopy $H: F(X) \rightarrow F(X)$ as the sum

$$F(\xi) \circ h(F) \circ F(\eta) + F(h).$$

Substituting these mappings to the formula of functional homology operation we obtain the required formula.

Show that the functors $\underline{E}$, $\overline{E}$ corresponding to an E_∞ -operad E are chain. To do it we define the family of mappings

$$\Delta^n \otimes \underline{E}(j; X) \rightarrow \underline{E}(j; \Delta^n \otimes X)$$

to be the compositions

(14)

$$\Delta^n \otimes \underline{E}(j; X) = \Delta^n \otimes E(j) \otimes_{\Sigma_j} X^{\otimes j} @>1 \otimes \nabla \otimes 1>> \Delta^n \otimes E(j) \otimes E(j) \otimes_{\Sigma_j} X^{\otimes j}$$

(15)

$$@>\tau \otimes 1 \otimes 1>> \Delta^{n \otimes j} \otimes E(j) \otimes_{\Sigma_j} X^{\otimes j} \rightarrow E(j) \otimes_{\Sigma_j} (\Delta^n \otimes X)^{\otimes j} = \underline{E}(j; \Delta^n \otimes X),$$

where $\tau: \Delta^n \otimes E(j) \rightarrow \Delta^{n \otimes j}$ is an E -coalgebra structure on the complex Δ^n .

Direct verification show that the required relations are satisfied.

Similary define mappings

$$\Delta^n \otimes \overline{E}(j; X) \rightarrow \overline{E}(j; \Delta^n \otimes X)$$

or, that is the same, mappings

$$E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \rightarrow (\Delta^n \otimes X)^{\otimes j}$$

to be the compositions

(16)

$$E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) @>\nabla \otimes 1 \otimes 1>> E(j) \otimes E(j) \otimes \Delta^n \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j})$$

(17)

$$\rightarrow E(j) \otimes \Delta^{n \otimes j} \otimes \text{Hom}_{\Sigma_j}(E(j); X^{\otimes j}) \rightarrow (\Delta^n \otimes X)^{\otimes j}.$$

Direct verification show that the required relations are satisfied.

Our aim is to calculate the functorial homology operations for the functors $\underline{E}_*$, $\overline{E}_*$. It means that for any sequence

$$f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$$

of chain mappings we need to calculate the mappings

$$\underline{E}_*(f^n, \dots, f^1): \underline{E}_*(X^1) \rightarrow \underline{E}_*(X^{n+1}), \quad \overline{E}_*(f^n, \dots, f^1): \overline{E}_*(X^1) \rightarrow \overline{E}_*(X^{n+1}).$$

Consider firstly the functor $\underline{E}(2; -)$ corresponding to a complex X the complex

$$\underline{E}(2; X) = E(2) \otimes_{\Sigma_2} X \otimes X,$$

where $E(2) - \Sigma_2$ -free and acyclic complex with generators e_i of dimensions i . A differential is defined by the formula

$$d(e_i) = e_{i-1} + e_{i-1}T, \quad T \in \Sigma_2.$$

The homology $\underline{E}_*(2; -)$ of this functor, as it was poined out above, is not only a functor but an A_∞ -functor. It means that for any sequence of chain mappings

$$f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$$

there is the operation

$$\underline{E}_*(2; f^n, \dots, f^1): \underline{E}_*(2; X^1) \rightarrow \underline{E}_*(2; X^{n+1}).$$

Let us calculate these operations.

Note that for a chain complex X there is an isomorphism

$$\underline{E}_*(2, X) \cong \underline{E}_*(2, X_*).$$

If X_* is a graded module then $\underline{E}_*(2; X_*)$ is the sum of two factors. The first factor is the quotient module $X_* \cdot X_*$ of the tensor product $X_* \otimes X_*$ up to permutation of factors. The second factor is the module generated by the elements of the form $e_i \times y_n$, $i \geq 1$ of dimensions $i + 2n$. The elements $y_n \cdot y_n \in X_* \cdot X_*$ will be also denoted as $e_0 \times y_n$.

Let $\xi: X_* \rightarrow X$, $\eta: X \rightarrow X_*$, $h: X \rightarrow X$ are mappings giving a chain equivalence between X and X_* . Denote by

$$E(\xi): E(2, X_*) \rightarrow E(2, X), \quad E(\eta): E(2, X) \rightarrow E(2, X_*), \quad E(h): E(2, X) \rightarrow E(2, X)$$

the mappings defined by the formulas

(18)

$$E(\xi)(e_i \otimes y_1 \otimes y_2) = e_i \otimes \xi(y_1) \otimes \xi(y_2), \quad E(\eta)(e_i \otimes x_1 \otimes x_2) = e_i \otimes \eta(x_1) \otimes \eta(x_2),$$

(19)

$$E(h)(e_i \otimes x_1 \otimes x_2) = e_i \otimes (x_1 \otimes h(x_2) + h(x_1) \otimes \xi\eta(x_2)) + e_{i-1} \otimes h(x_1) \otimes h(x_2).$$

It is clear they give a chain equivalence $\underline{E}(2, X) \simeq \underline{E}(2, X_*)$.

Define mappings

$$(20) \quad \xi(E): \underline{E}_*(2; X_*) \rightarrow \underline{E}(2; X_*), \quad \eta(E): \underline{E}(2; X_*) \rightarrow \underline{E}_*(2; X_*),$$

(21)

$$h(E): \underline{E}(2; X_*) \rightarrow \underline{E}(2; X_*).$$

To do it firstly we choose an ordering basis $\{y\}$ in X_* . Then define the mapping $\xi(E)$ putting

$$\xi(E)(e_i \otimes y) = e_i \otimes y \otimes y; \quad \xi(E)(y_1 \cdot y_2) = e_0 \otimes (y_1 \otimes y_2), \quad y_1 \leq y_2.$$

Define the mapping $\eta(E)$ putting

$$\eta(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} e_i \otimes y_1, & y_1 = y_2 \\ y_1 \cdot y_2, & y_1 < y_2, i = 0 \\ 0, & \text{in other cases} \end{cases}$$

Define the mapping $h(E)$ putting

$$h(E)(e_i \otimes y_1 \otimes y_2) = \begin{cases} e_{i+1} \otimes y_2 \otimes y_1, & y_1 > y_2, \\ 0, & \text{in other cases} \end{cases}$$

Direct calculations show that the required relations are satisfied.

The mappings

$$(22) \quad E(\xi) \circ \xi(E): E_*(2, X_*) \rightarrow E(2, X), \quad \eta(E) \circ E(\eta): E(2, X) \rightarrow E_*(2, X_*),$$

(23)

$$E(\xi) \circ h(E) \circ E(\eta) + E(h): E(2, X) \rightarrow E(2, X)$$

give us a chain equivalence between $E(2, X)$ and $E_*(2, X_*)$.

From the general formula of functional homology operations for a chain functor it follows that for the functor $\underline{E}(2, -)$ the following formula is taken place

$$\underline{E}_*(2, f^n, \dots, f^1) = \sum \underline{E}_*(2, H_*(f^n, \dots, f^{n_m+1}), \dots, H_*(f^{n_1}, \dots, f^1)),$$

where the sum is taken over all m and $n_1, \dots, n_m$ such that $1 \leq n_1 < \dots < n_m < n$.

For a graded module X with fixed ordering basis $\{x_i\}$ define mappings $p: X \otimes X \rightarrow X$, $q: X \rightarrow X \otimes X$, $r: X \otimes X \rightarrow X \otimes X$ putting $q(x_i) = x_i \otimes x_i$ and

$$p(x_i \otimes x_j) = \begin{cases} x_i, & i = j, \\ 0, & i \neq j, \end{cases} \quad r(x_i \otimes x_j) = \begin{cases} x_j \otimes x_i, & i > j, \\ 0, & i \leq j. \end{cases}$$

For a sequence of mappings $f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$ of graded modules with ordering bases define the mapping $(f^n, \dots, f^1): X^1 \rightarrow X^{n+1}$ putting

$$(f^n, \dots, f^1) = p \circ (f^n)^{\otimes 2} \circ r \circ (f^{n-1})^{\otimes 2} \circ \dots \circ r \circ (f^1)^{\otimes 2} \circ q.$$

Directly from the definition of the homology operations it follows

Theorem 13. *For a sequence of mappings $f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$ of graded modules the following formula is taken place*

$$\underline{E}_*(2; f^n, \dots, f^1)(e_i \times x) = e_{i+n-1} \times (f^n, \dots, f^1)(x).$$

To obtain the correpondig formula for the functor $\underline{E}_*$ it needs to use a monad structure $\gamma_*: \underline{E}_* \circ \underline{E}_* \rightarrow \underline{E}_*$ and the formula

$$\underline{E}_*(f^n, \dots, f^1) \circ \gamma_* = \sum \gamma_* \circ \underline{E}_*(\underline{E}_*(f^n, \dots, f^{n_m+1}), \dots, \underline{E}_*(f^{n_1}, \dots, f^1)),$$

where the sum is taken over all m and $n_1, \dots, n_m$ such that $1 \leq n_1 < \dots < n_m < n$.

Passing to the Dyer-Lashof algebra $\mathcal{R}$ we obtain the operations

$$\mathcal{R}(f^n, \dots, f^1): \mathcal{R} \times X^1 \rightarrow \mathcal{R} \times X^{n+1},$$

which on the generators Q^i are expressed by the formulas

$$\mathcal{R}(f^n, \dots, f^1)(Q^i \otimes x) = Q^{i+n-1} \otimes (f^n, \dots, f^1)(x).$$

Dually for the functor $\overline{E}_*(2; -)$ there is

Theorem 14. *For a sequence of mappings $f^1: X^1 \rightarrow X^2, \dots, f^n: X^n \rightarrow X^{n+1}$ of graded modules the following formula is taken place*

$$\overline{E}_*(2; f^n, \dots, f^1)(\bar{e}_i \times x) = \begin{cases} \bar{e}_{i-n+1} \times (f^n, \dots, f^1)(x), & i \geq n-1 \\ 0, & \text{in other cases} \end{cases}$$

To obtain the corresponding formula for the functor $\overline{E}_*$ it needs to use a comonad structure $\bar{\gamma}_*: \overline{E}_* \rightarrow \overline{E}_* \circ \overline{E}_*$ and the formula

$$\bar{\gamma}_* \circ \overline{E}_*(f^n, \dots, f^1) = \sum \overline{E}_*(\overline{E}_*(f^n, \dots, f^{n_m+1}), \dots, \overline{E}_*(f^{n_1}, \dots, f^1)) \circ \bar{\gamma}_*,$$

where the sun is taken over all m and $n_1, \dots, n_m$ such that $1 \leq n_1 < \dots < n_m < n$.

Passing to the Milnor coalgebra $\mathcal{K}$ we obtain the operations

$$\mathcal{K}(f^n, \dots, f^1): \mathcal{K} \times X^1 \rightarrow \mathcal{K} \times X^{n+1},$$

which are expressed by the formulas

$$\mathcal{K}(f^n, \dots, f^1)(y \otimes x) = y \cdot \xi_1^{n-1} \otimes (f^n, \dots, f^1)(x).$$

Consider operations associated with a comultiplication ∇ of the Milnor coalgebra $\mathcal{K}$. Denote

$$\nabla(n) = \nabla \otimes 1 \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \nabla: \mathcal{K}^{\times n} \rightarrow \mathcal{K}^{\times n+1}.$$

Direct calculations show that the operations

$$(\nabla(n), \dots, \nabla): \mathcal{K} \rightarrow \mathcal{K}^{\times n+1}, \quad n \geq 2,$$

are trivial on the elements $\xi_i^{2^k}$. However on the other elements these operations in general are not trivial. For example there is the formula

$$(\nabla(2), \nabla)(\xi_i \xi_j) = \xi_{j-i}^{2^i} \xi_0^{2^i} \otimes \xi_i \xi_0^{2^i} \otimes \xi_i, \quad i < j.$$

Denote by $\tilde{\nabla}$ a comultiplication in the tensor product $\mathcal{K} \otimes \mathcal{K}$,

$$\tilde{\nabla} = (1 \otimes T \otimes 1)(\nabla \otimes \nabla).$$

Put

$$\tilde{\nabla}(n) = \tilde{\nabla} \otimes 1 \cdots \otimes 1 - \cdots + (-1)^{n-1} 1 \otimes \cdots \otimes 1 \otimes \tilde{\nabla}: (\mathcal{K} \otimes \mathcal{K})^{\times n} \rightarrow (\mathcal{K} \otimes \mathcal{K})^{\times n+1}.$$

Consider the operations

$$(\pi^{\times n+1}, \tilde{\nabla}(n), \dots, \tilde{\nabla}): \mathcal{K}^{\otimes 2} \rightarrow \mathcal{K}^{\times n+1}.$$

Its restriction on the elements $x \otimes x \in \mathcal{K} \otimes \mathcal{K}$ we denote by

$$\Psi^n: \mathcal{K} \rightarrow \mathcal{K}^{\times n+1}.$$

From the formula of a comultiplication in the Milnor coalgebra directly follows the formula

$$\Psi^1(\xi_n) = \sum_{i < j} \xi_{n-i}^{2^i} \xi_{n-j}^{2^j} \otimes \xi_i \xi_j,$$

or in more general case

$$\Psi^1(\xi_n^{2^m}) = \sum_{i < j} \xi_{n-i}^{2^{i+m}} \xi_{n-j}^{2^{j+m}} \otimes \xi_i^{2^m} \xi_j^{2^m}.$$

In particular for the primitive elements $\xi_1^{2^m} \in \mathcal{K}$ we have the formula

$$\Psi^1(\xi_1^{2^m}) = \xi_1^{2^m} \xi_0^{2^m} \otimes \xi_1^{2^m}.$$

Similary for the operation Ψ^2 we have the formula

$$\Psi^2(\xi_n^{2^m}) = \sum_{i < j, k > l} \xi_{n-i}^{2^{i+m}} \xi_{n-j}^{2^{j+m}} \otimes \xi_{i-k}^{2^{k+m}} \xi_{j-l}^{2^{l+m}} \otimes \xi_k^{2^m} \xi_l^{2^m}.$$

In particular for the primitive elements $\xi_1^{2^m} \in \mathcal{K}$ we have the formula

$$\Psi^2(\xi_1^{2^m}) = 0.$$

And so on.

6. $\cup_\infty - A_\infty$ -Hopf algebras

To calculate higher differentials of the Adams spectral sequence we need to use not only the action of the Dyer-Lashof algebra, but the E_∞ -structure. However this structure is too complicate. Some of the calculations were made in [6]. Here we'll use only a part of the E_∞ -structure consisting of $\cup_i$ -products.

A chain complex A will be called a $\cup_\infty$ -algebra if there are given operations $\cup_i: A \otimes A \rightarrow A$, $i \geq 0$, called $\cup_i$ -products, increasing dimensions by i and satisfying the relation

$$d(x \cup_i y) = d(x) \cup_i y + x \cup_i d(y) + x \cup_{i-1} y + y \cup_{i-1} x.$$

A differential coalgebra K will be called a $\cup_\infty$ -Hopf algebra if there are given $\cup_i$ -products $\cup_i: K \otimes K \rightarrow K$ satisfying the distributivity relation

$$\nabla(x \cup_i y) = \sum_k (x' \cup_{i-k} T^k y') \otimes (x'' \cup_k y''),$$

where $\nabla(x) = \sum x' \otimes x''$, $\nabla(y) = \sum y' \otimes y''$, $T: K \otimes K \rightarrow K \otimes K$ is the permutation mapping, T^k it's k -th iteration.

Theorem 15. *The cobar construction FK over a $\cup_\infty$ -Hopf algebra K is a $\cup_\infty$ -algebra. Moreover $\cup_i$ -products $\cup_i: FK \otimes FK \rightarrow FK$ uniquely determined by the formula*

$$[x] \cup_i [y] = \begin{cases} [x \cup_{i-1} y], & i \geq 1, \\ [x, y], & i = 0. \end{cases}$$

and the relations

$$(24) \quad (x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),$$

$$(25) \quad (x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1) (x_2 \cup_k y_2) +$$

$$(26) \quad + (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) +$$

$$(27) \quad + ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) +$$

$$(28) \quad + x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,$$

where $x_1, x_2, y_1, y_2 \in FK$, $y \in K$, $i \geq 1$.

Indeed, the products $[x_1, \dots, x_n] \cup_i [y]$ are determined by the first relation

$$[x_1, \dots, x_n] \cup_i [y] = \sum_{k=1}^n [x_1, \dots, x_k \cup_i y, \dots, x_n]$$

From the second relation it follows that to define $\cup_i$ -products in general case, i.e.

$$[x_1, \dots, x_n] \cup_i [y_1, \dots, y_m]$$

it is sufficient to define only $\cup_i$ -products $[x] \cup_i [y_1, \dots, y_m]$.

We have

$$(29) \quad d([y_1, \dots, y_m] \cup_{i+1} [x]) = \sum_{k=1}^m [y_1, \dots, d(y_k), \dots, y_m] \cup_{i+1} [x] +$$

$$(30) \quad + \sum_{k=1}^m [y_1, \dots, y'_k, y''_k, \dots, y_m] \cup_{i+1} [x] +$$

$$(31) \quad + [y_1, \dots, y_m] \cup_{i+1} ([d(x)] + [x', x'']) +$$

$$(32) \quad + [y_1, \dots, y_m] \cup_i [x] + [x] \cup_i [y_1, \dots, y_m].$$

Thus the product $[x] \cup_i [y_1, \dots, y_m]$ is expressed through already defined products and products of the elements lesser dimensions. Hence $\cup_i$ -products are determined by induction.

So this theorem gives us the formulas for $\cup_i$ -product in the cobar construction. However they are inductive and not so simple even in the case when higher $\cup_i$ -products ($i \geq 1$) on K are trivial, i.e. when K is a commutative Hopf algebra.

An $\cup_\infty$ -Hopf algebra K will be called commutative if the coproduct $\nabla: K \rightarrow K \otimes K$ is commutative.

Theorem 16. *The cobar construction FK over a commutative $\cup_\infty$ -Hopf algebra is a commutative $\cup_\infty$ -Hopf algebra. So the cobar construction over a commutative $\cup_\infty$ -Hopf algebra may be iterated.*

Proof. Define the coproduct $\nabla: FK \rightarrow FK \otimes FK$ putting

$$\nabla[x_1, \dots, x_n] = \sum [x_{i_1}, \dots, x_{i_p}] \otimes [x_{j_1}, \dots, x_{j_q}],$$

where the sum is taken over all (p, q) -shuffles of $1, 2, \dots, n$. Direct calculations show that the required relations are satisfied.

Consider the question about the structure on the homology of a $\cup_\infty$ -Hopf algebra.

Consider the question about the structure on the homology of a $\cup_\infty$ -Hopf algebra. It is clear that on the homology K_* of a $\cup_\infty$ -Hopf algebra K there are $\cup_\infty$ -algebra structure, consisting of the operations

$$\cup_i: K_* \otimes K_* \rightarrow K_*$$

and A_∞ -coalgebra structure, consisting of the operations

$$\nabla_n: K_* \rightarrow K_*^{\otimes n+2}.$$

But besides that there are another operations of the form

$$\Psi_{i,n}: K_* \otimes K_* \rightarrow K_*^{\otimes n+2}.$$

To describe these operations and relations between them we introduce the notion of a $\cup_\infty - A_\infty$ -Hopf algebra.

An A_∞ -coalgebra K will be called an $\cup_\infty - A_\infty$ -Hopf algebra if on the cobar construction $\tilde{F}K$ there is given $\cup_\infty$ -algebra structure satisfying the relations

$$(33) \quad (x_1 x_2) \cup_i [y] = (x_1 \cup_i [y]) x_2 + x_1 (x_2 \cup_i [y]),$$

$$(34) \quad (x_1 x_2) \cup_i (y_1 y_2) = \sum_k (x_1 \cup_{i-k} T^k y_1) (x_2 \cup_k y_2) +$$

$$(35) \quad + (x_1 \cup_i (y_1 y_2)) x_2 + x_1 (x_2 \cup_i (y_1 y_2)) +$$

$$(36) \quad + ((x_1 x_2) \cup_i y_1) y_2 + y_1 ((x_1 x_2) \cup_i y_2) +$$

$$(37) \quad + x_1 (x_2 \cup_i y_1) y_2 + y_1 (x_1 \cup_i y_2) x_2,$$

where $x_1, x_2, y_1, y_2 \in FK$, $y \in K$, $i \geq 1$.

Theorem 17. *If K is a $\cup_\infty$ -Hopf algebra then its homology $K_* = H_*(K)$ is $\cup_\infty - A_\infty$ -Hopf algebra and there is an equivalence of $\cup_\infty$ -algebras $\tilde{F}K_* \simeq FK$.*

Proof. It is known [16] that the homology K_* of a differential coalgebra K is A_∞ -coalgebra and there are algebra mappings $\xi: \tilde{F}K_* \rightarrow FK$, $\eta: FK \rightarrow \tilde{F}K_*$ and an algebra chain homotopy $h: FK \rightarrow FK$ such that $\eta \circ \xi = Id$, $d(h) = \xi \circ \eta - Id$. So we need to define on $\tilde{F}K_*$ the $\cup_i$ -products. Put on generators

$$[x] \cup_i [y] = \eta(\xi[x] \cup_i \xi[y]).$$

On the other elements the $\cup_i$ -products determines by the relations.

Applying this theorem to the Milnor coalgebra we obtain

Theorem 18. *The Milnor coalgebra $\mathcal{K}$ possesses $\cup_\infty - A_\infty$ -Hopf algebra structure. The homology of the corresponding cobar construction $\tilde{F}\mathcal{K}$ is isomorphic to the E^∞ term of the Adams spectral sequence.*

7. Differentials of the Adams spectral sequence and the Kervaire invariant problem

Apply developed methods to calculate higher differentials of the Adams spectral sequence or, that is the same to calculate the differential in $\tilde{F}\mathcal{K}$.

Since any element of $\mathcal{K}$ may be obtained from ξ_0 by applying $\cup_i$ -products, we have

Theorem 19. *The formulas for $\cup_i$ -products in the Milnor coalgebra and the relations for $\cup_\infty$ -algebra structure in the cobar construction $\tilde{F}\mathcal{K}$ completely determine the differential in $\tilde{F}\mathcal{K}$.*

However the formulas for the differential are inductive and very complicated. So the next step in the calculation of the differential is to replace the Milnor coalgebra $\mathcal{K}$ and the cobar construction $\tilde{F}\mathcal{K}$ by more simply objects.

To do it consider the filtration of $\tilde{F}\mathcal{K}$ putting the filtration of the elements $\xi_{i_1} \cdots \xi_{i_n} \in K$ to be equal n . Then the first term of the corresponding spectral sequence will be isomorphic to the polynomial algebra $PS^{-1}X$ over the module X generated by the elements $\xi_i^{2^k}$. We'll denote elements of $PS^{-1}X$ as elements of the cobar construction, i.e. $[x_1, \dots, x_n]$, $x_i \in X$.

Note that there is an algebra mapping $\eta: FK \rightarrow PS^{-1}X$, given by the formula

$$\eta[x] = \begin{cases} [\xi_i^{2^k}], & x = \xi_i^{2^k}, \\ 0, & \text{in other cases.} \end{cases}$$

The inverse mapping $\xi: PS^{-1}X \rightarrow FK$ may be given by the formula

$$\xi([x_1, \dots, x_n]) = [x_1, \dots, x_n], \quad x_1 \leq \dots \leq x_n.$$

It is not an algebra mapping, but if $x \leq y$ then $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$.

From here and from the Perturbation Theory it follows

Theorem 20. *The polynomial algebra $PS^{-1}X$ possesses a $\cup_\infty$ -algebra structure. The homology of the corresponding complex $\tilde{P}S^{-1}X$ is isomorphic to the homology of $\tilde{F}\mathcal{K}$ and hence to the E^∞ term of the Adams spectral sequence.*

This theorem gives us the inductive formulas for the differential and $\cup_i$ -products in $\tilde{P}S^{-1}X$.

Since any element of $\tilde{P}S^{-1}X$ may be obtained from $[\xi_0]$ by applying $\cup_i$ -products, we have

Theorem 21. *The formulas for $\cup_i$ -products in the module X and the relations for $\cup_i$ -products in $\tilde{P}S^{-1}X$ completely determine the differential in $\tilde{P}S^{-1}X$.*

Following the Adams notations we put $h_n = [\xi_1^{2^n}]$. Using the formula $h_n \cup_1 h_n = h_{n+1}$ we prove the following theorem

Theorem 22. *For the differential in $\tilde{P}S^{-1}X$ on the elements h_n there is the next formula*

$$d(h_n) = \sum_{i=0}^{n-1} \xi_i h_{n-i}^{2^i}.$$

Proof. It is clear that $d(h_1) = h_{-1}h_1$. For the element h_2 we have

$$(38) \quad d(h_2) = d(h_1 \cup_1 h_1) = (h_{-1}h_1 \cup_1 h_1 + h_1 \cup_1 (h_{-1}h_1) =$$

$$(39) \quad = (h_{-1}h_1) \cup_2 (h_{-1}h_1) = h_{-1}h_2 + h_0h_1^2.$$

Suppose the required formula is true for n and prove it is true for $n + 1$. We have

$$(40) \quad d(h_{n+1}) = d(h_n \cup_1 h_n) = d(h_n) \cup_1 h_n + h_n \cup_1 d(h_n) =$$

$$(41) \quad = \left(\sum_{i=0}^{n-1} [\xi_i] h_{n-i}^{2^i} \right) \cup_2 \left(\sum_{i=0}^{n-1} [\xi_i] h_{n-i}^{2^i} \right) =$$

$$(42) \quad = \sum_{i=0}^n [\xi_i] h_{n+1-i}^{2^i}.$$

Apply this formula to solve the Kervaire invariant problem.

Theorem 23. *For the elements h_n^2 , $n \geq 4$, of the Adams spectral sequence there is the following formula*

$$d_5(h_n^2) = h_1^2 g_{n-4} h_{n-1}.$$

Proof. The differential on the elements h_n^2 is given by the formula

$$(43) \quad d(h_n^2) = d(h_n) h_n + h_n d(h_n) = d(h_n) \cup_1 d(h_n) =$$

$$(44) \quad = (h_{-1} h_n + h_0 h_{n-1}^2 + \dots) \cup_1 (h_{-1} h_n + h_0 h_{n-1}^2 + \dots) =$$

$$(45) \quad = h_{-1} h_n^2 + h_1 h_{n-1}^4 + [\xi_2^2] h_{n-2}^8 + \dots$$

After the factorization over h_{-1} we obtain the formula

$$d(h_n^2) = h_1 h_{n-1}^4 + [\xi_2^2] h_{n-2}^8 + \dots$$

Write the differential d as the sum $d = d_1 + d_2 + \dots$, where d_n increase the filtration by n . Note that the complex $\tilde{P}S^{-1}X$ with the differential d_1 is isomorphic to the E^1 term of the Adams spectral sequence. The corresponding homology is isomorphic to the E^2 term of this spectral sequence.

In the E^1 term there is the formula

$$d_1(h_{n-1}[\xi_2^{2^{n-2}}]^2 + h_{n-2}^2[\xi_2^{2^{n-1}}]) = h_{n-1}^4. \quad (*)$$

From this formula it follows that in the E_2 term the differential d_3 on the elements h_n^2 is equal to zero. Improve the elements h_n^2 and consider the elements

$$\tilde{h}_n^2 = h_n^2 + h_1(h_{n-1}[\xi_2^{2^{n-2}}]^2 + h_{n-2}^2[\xi_2^{2^{n-1}}]).$$

The differential on the elements $[\xi_2^{2^{n-1}}]$, $[\xi_2^{2^{n-2}}]^2$ is given by the formula

$$(46) \quad d[\xi_2^{2^{n-1}}] = h_{n-1} h_n + h_0[\xi_2^{2^{n-2}}]^2 + h_1 h_{n-2} h_{n-1}[\xi_2^{2^{n-2}}] + \dots,$$

$$(47) \quad d[\xi_2^{2^{n-2}}]^2 = h_{n-2}^2 h_n + h_{n-1}^3 + h_1[\xi_2^{2^{n-3}}]^4 + \dots$$

Hence

$$d(\tilde{h}_n^2) = h_1^2(h_{n-1}[\xi_2^{2^{n-3}}]^4 + h_{n-3}^4[\xi_2^{2^{n-1}}] + h_{n-2}^3 h_{n-1}[\xi_2^{2^{n-2}}]) + \dots$$

Thus $d_i(\tilde{h}_n^2) = 0$ if $i \leq 4$ and

$$d_5(\tilde{h}_n^2) = h_1^2(h_{n-1}[\xi_2^{2^{n-3}}]^4 + h_{n-3}^4[\xi_2^{2^{n-1}}] + h_{n-2}^3 h_{n-1}[\xi_2^{2^{n-2}}]).$$

From the formula $(*)$ it follows that in the E_1 term the element $h_{n-3}^4[\xi_2^{2^{n-1}}]$ is homological to the element $(h_{n-3}[\xi_2^{2^{n-4}}]^2 + h_{n-4}^2[\xi_2^{2^{n-3}}])h_{n-1}h_n$. Hence the element $d_5(\tilde{h}_n^2)$ is homological to the element

$$h_1^2([\xi_2^{2^{n-3}}]^4 + h_{n-3}[\xi_2^{2^{n-4}}]^2 h_n + h_{n-4}^2[\xi_2^{2^{n-3}}] h_n + h_{n-2}^3[\xi_2^{2^{n-2}}])h_{n-1}.$$

Direct calculations show that the expression in the parentheses is a cycle in E_1 of the filtration 4. The corresponding homology class in the E^2 term of the Adams spectral sequence is denoted by g_{n-4} .

So we have the formula

$$d_5(h_n^2) = h_1^2 g_{n-4} h_{n-1}.$$

Corollary. *The elements h_n^2 survive till the E_6 term of the Adams spectral sequence.*

Indeed it is known that the element $g_0 h_3$ is equal to zero in the E_2 term of the Adams spectral sequence. Hence the elements $g_{n-4} h_{n-1}$ also are equal to zero. Thus $d_5(h_n^2) = 0$ and the elements h_n^2 survive till the E_6 term.

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