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Higher Algebra and Stable Homotopy Groups

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Abstract. A fundamental problem in homotopy theory is to understand the higher homotopy groups of spheres, $\pi_n(S^k)$. Freudenthal showed that for large k, these groups depend only on the difference $n-k$. The corresponding group for $m = n-k$, denoted π_m^S, is finite for $m > 0$, and the sequence of finite abelian groups π_m^S exhibits fascinating patterns. We shall survey how an "∞-categorical" form of algebra, called higher algebra, places these patterns into an algebraic framework, and how these ideas allow us to give asymptotic bounds on the sizes of π_m^S.

Key words. chromatic homotopy, higher algebra, homotopy Groups, stable homotopy

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1 Motivation: Stable homotopy groups. A fundamental problem in homotopy theory is to understand the higher homotopy groups of spheres, $\pi_{n+d}(S^d)$. Ever since Hopf's 1931 theorem [Hop31] established that $\pi_3(S^2) \simeq \mathbb{Z}$, this problem has served as a key motivation in the field.

Freudenthal suspension theorem [Fre38] says that for any pointed CW-complex X, the suspension map

$$\Sigma\colon \pi_{n+d}(S^d \wedge X) \to \pi_{n+d+1}(S^{d+1} \wedge X)$$

is an isomorphism when $d > n+1$. Thus, these groups stabilize as $d \to \infty$, and we denote this stable group by $\pi_n^S(X)$. For $X = S^0$, we write $\pi_n^S := \pi_n^S(S^0)$.

It follows from the work of Serre that if X is a CW-complex with finitely many cells in each dimension, then $\pi_n^S(X)$ is a finitely generated abelian group for every n. Moreover, $\pi_n^S(X) \otimes \mathbb{Q} \simeq \tilde{H}_n(X;\mathbb{Q})$. Consequently, for $n > 0$, π_n^S is a finite abelian group, while $\pi_0^S \simeq \mathbb{Z}$, where the isomorphism sends a map to its degree.

Furthermore, the smash product construction endows the graded abelian group π_*^S with the structure of a skew-commutative graded ring, via the pairing

$$\pi_{n_1+d_1}(S^{d_1}) \times \pi_{n_2+d_2}(S^{d_2}) \to \pi_{n_1+d_1+n_2+d_2}(S^{d_1} \wedge S^{d_2}) \simeq \pi_{n_1+n_2+d_1+d_2}(S^{d_1+d_2}).$$

This infinite sequence of abelian groups is a fundamental object in homotopy theory, relating in remarkable and potent ways to other mathematical subjects such as differential topology, algebraic K-theory, mathematical physics, p-adic geometry, and more. The goal of this talk is to describe a systemic approach to studying the patterns in this sequence and to describe recent novel results that exhibit its efficacy.

To start, Table 1 provides a list of these groups for low values of n.

The groups are presented in a way that emphasizes their primary decomposition.

Indeed, one key insight is that the relevant patterns become clearer when one localizes at a given prime p (that is, inverting all other primes). For example, for $p = 3$, the corresponding p-localized stable homotopy groups, $\pi_{(3),n}^S$, are given in Table 2.

A pattern already seems to emerge: the group does not vanish whenever $n \equiv -1 \pmod 4$. Indeed, Adams's work on the image of J provided an infinite family of elements in the homotopy groups of spheres [Ada66]. For odd primes, this family consists of a copy of $\mathbb{Z}/p^{k+1}$ in each degree n which is congruent to $-1 \mod 2p-2$ where $p^k \| n+1$.

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Table 1 Stable homotopy groups of spheres

n	0	1	2	3	4
π_n^S	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8\oplus\mathbb{Z}/3$	0
π_{n+5}^S	0	$\mathbb{Z}/2$	$\mathbb{Z}/16\oplus\mathbb{Z}/3\oplus\mathbb{Z}/5$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$
π_{n+10}^S	$\mathbb{Z}/2\oplus\mathbb{Z}/3$	$\mathbb{Z}/8\oplus\mathbb{Z}/9\oplus\mathbb{Z}/7$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$
π_{n+15}^S	$\mathbb{Z}/32\oplus\mathbb{Z}/2$ $\oplus\mathbb{Z}/3\oplus\mathbb{Z}/5$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$ $\oplus\mathbb{Z}/3\oplus\mathbb{Z}/11$
π_{n+20}^S	$\mathbb{Z}/8\oplus\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/16\oplus\mathbb{Z}/8\oplus\mathbb{Z}/2$ $\oplus\mathbb{Z}/9\oplus\mathbb{Z}/3$ $\oplus\mathbb{Z}/5\oplus\mathbb{Z}/7\oplus\mathbb{Z}/13$	$(\mathbb{Z}/2)^2$

Table 2 stable homotopy groups of spheres localized at the prime 2

n	0	1	2	3	4
$\pi_{(3),n}^S$	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/3$	0
$\pi_{(3),n+5}^S$	0	0	$\mathbb{Z}/3$	0	0
$\pi_{(3),n+10}^S$	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0	$\mathbb{Z}/3$	0
$\pi_{(3),n+15}^S$	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$
$\pi_{(3),n+20}^S$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9\oplus\mathbb{Z}/3$	0

The main tool to compute specific values of $\pi_{(p),n}^S$ is the Adams [Alg58] and Adams–Novikov [Nov67] spectral sequence. At odd primes, the state of the art is computed by Ravenel in [Rav03]. As the prime grows, the range of known stems grows. For example, for $p=3$ and $p=5$, we have complete knowledge up to around 100 and 1000 stems, respectively [Rav03]. These ranges are both approximately equal to $p^3(2p-2)$. For $p=2$ the classical Adams spectral sequence has been more efficient in the computation of $\pi_{(2),n}^S$. These groups were computed up to $n=90$ by Isaksen, Wang, and Xu [IWX20]. Moreover in the case $p=2$, recent breakthrough work by Lin, Wang, and Xu has used computer-assisted methods to obtain results for $\pi_{(2),n}^S$ in the range of $n\approx 126$ [LWX24]. The focus of this talk, however, will be on different methods that concentrate not on specific values, but rather deal with the asymptotic and average behavior of these groups (and, more generally, $\pi_{(p),n}^S(X)$ for a finite CW-complex X) as n grows as $n\to\infty$.

Based on their computations, Isaksen, Wang, and Xu proposed the following conjecture.

Conjecture 1.1. *There exists a constant $C_2>0$ such that*

$$\lim_{k\to\infty}\frac{1}{k^2}\sum_{n=1}^{k}\log_2(|\pi_{(2),n}|)=C_2.$$

Although this conjecture seems still far from reach we shall discuss how insights from arithmetic geometry, higher category theory, and algebraic K-theory give new results on the asymptotic behavior of $\pi_{(p),n}^S$ as $n\to\infty$.

2 Spectra: The stable world.

The key mathematical object that binds the groups π_n^S together, and the notion we shall use to study them, is that of a *spectrum.*

Definition 2.1. *A spectrum X is a sequence $\{(\mathsf{X}_d,\omega_d)\}_{d\in\mathbb{Z}}$, where each X_d is a pointed CW-space and each $\omega_d\colon \mathsf{X}_d\to\Omega\mathsf{X}_{d+1}$ is a homotopy equivalence.*

Given a spectrum $\mathsf{X}\in\mathrm{Sp}$, we have a sequence of isomorphisms

$$\pi_{n+d}(\mathsf{X}_d)\simeq\pi_{n+d}(\Omega\mathsf{X}_{d+1})\simeq\pi_{n+d+1}(\mathsf{X}_{d+1}).$$

We can thus define, for $n\in\mathbb{Z}$, the nth homotopy group of X to be

$$\pi_n(\mathsf{X}):=\pi_{n+d}(\mathsf{X}_d)\quad\in\quad\mathrm{Ab},$$

which is independent of d for sufficiently large d. Note that since we can choose d such that $n+d>1$, $\pi_n(\mathsf{X})$ is defined for all $n\in\mathbb{Z}$ and is always an abelian group.

To a pointed space $X \in \mathcal{S}_*$, one can associate its *suspension spectrum* $\overline{\mathbb{S}}\langle X\rangle$,[1] whose dth space is

$$\overline{\mathbb{S}}\langle X\rangle_d = \operatorname*{colim}_{k\to\infty}(\Omega^{k-d}\Sigma^k X).$$

Stable homotopy groups are then related to spectra via a natural isomorphism,

$$\pi_n\left(\overline{\mathbb{S}}\langle X\rangle\right) \simeq \pi_n^S(X).$$

We denote the sphere spectrum by $\mathbb{S} := \overline{\mathbb{S}}\langle S^0\rangle$. More generally, if A is a torsion-free abelian group, we can construct spectra $\overline{\mathbb{S}}_A\langle X\rangle$ such that

$$\pi_n\left(\overline{\mathbb{S}}_A\langle X\rangle\right) \simeq \pi_n^S(X)\otimes A.$$

We abuse notation by writing $\overline{\mathbb{S}}_{(p)}\langle X\rangle$ for $\overline{\mathbb{S}}_{\mathbb{Z}_{(p)}}\langle X\rangle$, and $\mathbb{S}_{(p)}$ for $\overline{\mathbb{S}}_{\mathbb{Z}_{(p)}}\langle S^0\rangle$.

Many other naturally occurring groups appear as homotopy groups of spectra. Classical examples include algebraic K-theory groups and cobordism groups.

Given a ring R, its algebraic K-theory groups $K_n(R)$ are the homotopy groups of its algebraic K-theory spectrum $\mathsf{K}(R)$. These groups are known to have fundamental connections to differential topology, number theory, and algebraic geometry. For instance, K_1 of group rings relates to the classification h-cobordism classes, the vanishing of $K_{4k}(\mathbb{Z})$ is equivalent to the Kummer–Vandiver conjecture on the class number of real cyclotomic fields, and Bloch's formula connects algebraic K-theory to algebraic cycles on varieties. Algebraic K-theory has seen many recent breakthroughs, based on the study of $\mathsf{K}(R)$ as a spectrum [AKN24, CMM21, CM21, BCM20].

Another key classical example is cobordism theory. Fix a tangential structure, that is, a map $X \to BO$. A closed *X-framed m-manifold* is a smooth manifold M of dimension m together with a lift of the map $M \xrightarrow{-TM} BO$ along $X \to BO$. If W is an X-framed $(m+1)$-manifold and $M \subset \partial W$, one can induce two different X-framed structures on M by using either the inward or outward normal vector for the identification $TM\oplus\mathbb{R} \simeq TW|_M$. If W is a compact X-framed $(m+1)$-manifold with boundary $\partial W \simeq M_0 \coprod M_1$ as closed X-framed m-manifolds (where we use the inward identification on M_0 and the outward identification on M_1), we say that W is a cobordism between M_0 and M_1. If such a W exists, we say that M_0 and M_1 are *cobordant.* The collection of these cobordism classes forms an abelian group under disjoint union, denoted by Ω_m^X. These groups are related to spectra: there exists a spectrum $\mathsf{M}(X)$, called the Thom spectrum, and an isomorphism proved by Pontryagin and Thom:

$$\Omega_m^X \simeq \pi_m(\mathsf{M}(X)). \tag{2.1}$$

A key example is the case where X is a point, $X = *$. Manifolds with a $*$-framing are called *stably framed* manifolds, and in this case $\mathsf{M}(*) \simeq \mathbb{S}$. Thus, the stable homotopy groups of spheres can also be interpreted as cobordism classes of stably framed manifolds.

When $X = BO \to BO$ is the identity map, a closed BO-framed manifold is just a closed smooth manifold. And Ω_*^{BO} is simply the graded cobordism ring of closed manifolds. Using the isomorphism $\pi_*(\mathsf{M}(BO)) \simeq \Omega_*^{BO}$ in [Tho54] Thom computed this ring to be

$$\Omega_*^{BO} \simeq \mathbb{F}_2[x_i | 1 \le i \le \infty, i+1 \ne 2^j],$$

where $x_i \in \Omega_i^{BO}$

Similarly one can consider $X = BSO, BSpin, BString$ for oriented, spin and string cobordism rings, respectively.

3 Spectra and abelian Groups. The ubiquity of spectra might seem surprising given the technical definition above, but we suggest it can be understood through the following analogy:

$$\text{Spectra} : \text{Homotopy types} \quad :: \quad \text{Abelian groups} : \text{Sets}.$$

This analogy will be a central guide for us going forward. Let us put the right-hand side of this analogy in the proper context. In Figure 1, all arrows going from left to right are symmetric monoidal left adjoints. Let us analyze these more carefully:

[1] $\overline{\mathbb{S}}\langle X\rangle$ is denoted by $\Sigma^\infty X$ in many classical texts.

$$\begin{tikzcd}
(\mathrm{Set},\times,*) \arrow[r, shift left, "(-)_+"] \arrow[rrr, bend left=30, "\mathbb{Z}\langle - \rangle"] & (\mathrm{Set}_*,\wedge,S^0) \arrow[l, shift left, "\underline{(-)}"] \arrow[r, shift left, "\overline{\mathbb{N}}\langle - \rangle"] \arrow[rr, bend left=25, "\overline{\mathbb{Z}}\langle - \rangle"] & (\mathrm{CMon}(\mathrm{Set}),\otimes_{\mathbb{N}},\mathbb{N}) \arrow[l, shift left, "\underline{(-)}"] \arrow[r, shift left, "(-)^{\mathrm{gp}}"] & (\mathrm{Ab},\otimes,\mathbb{Z}) \arrow[l, shift left, "\underline{(-)}"]
\end{tikzcd}$$

Fig 1: *The path from sets to abelian groups.*

$$\begin{tikzcd}
(\mathcal{S},\times,*) \arrow[r, shift left, "(-)_+"] \arrow[rrr, bend left=30, "\mathbb{S}\langle - \rangle"] & (\mathcal{S}_*,\wedge,S^0) \arrow[l, shift left, "\underline{(-)}"] \arrow[r, shift left, "\overline{\mathbb{M}}\langle - \rangle"] \arrow[rr, bend left=25, "\overline{\mathbb{S}}\langle - \rangle"] & (\mathrm{CMon}(\mathcal{S}),\otimes_{\mathbb{M}},\mathbb{M}) \arrow[l, shift left, "\underline{(-)}"] \arrow[r, shift left, "(-)^{\mathrm{gp}}"] & (\mathrm{Sp},\otimes,\mathbb{S}) \arrow[l, shift left, "\underline{(-)}"]
\end{tikzcd}$$

Fig 2: *The analogous path from spaces to spectra.*

- The functor $(-)_+\colon \mathrm{Set}\to\mathrm{Set}_*$ is defined by adding a disjoint base point.
- $\wedge$ is the smash product of pointed sets $(X,x_0)\wedge(Y,y_0)=(X\times Y/(X\vee Y),(X\vee Y)/(X\vee Y))$.
- The functor $\overline{\mathbb{N}}\langle - \rangle\colon \mathrm{Set}_*\to\mathrm{CMon}(\mathrm{Set})$ sends a pointed set (X,x_0) to the free commutative monoid on X with the relation $x_0=0$.
- The functor $(-)^{\mathrm{gp}}\colon \mathrm{CMon}(\mathrm{Set})\to\mathrm{Ab}$ is the group-completion functor, which is given a commutative monoid M it freely adds inverses to make M into an abelian group (for instance, $\mathbb{N}^{\mathrm{gp}}\simeq\mathbb{Z}$).
- The functor $\overline{\mathbb{Z}}\langle - \rangle\colon \mathrm{Set}_*\to\mathrm{Ab}$ sends a pointed set (X,x_0) to the free abelian group on X with the relation $x_0=0$.
- The functor $\mathbb{Z}\langle - \rangle\colon \mathrm{Set}\to\mathrm{Ab}$ sends a pointed set (X,x_0) to the free abelian group on X.

Their corresponding right adjoints are all abusively denoted by $\underline{(-)}$ since they are given by taking an "underlying structure." For example, the functor $\underline{(-)}\colon \mathrm{Ab}\to\mathrm{Set}$ sends an abelian group to the underlying set of elements.

This sequence of constructions has an analogue in the homotopical realm; see Figure 2.

Again, in this diagram, all arrows are symmetric monoidal left adjoints, and their right adjoints are denoted abusively by $\underline{(-)}$. This time, the relevant adjunctions are between ∞-categories. That is, for any two objects $x,y\in\mathcal{C}$, the mapping object is not a set but rather a homotopy type, i.e., a space $\mathrm{Map}_{\mathcal{C}}(x,y)\in\mathcal{S}$, well-defined up to weak homotopy equivalence. From this perspective, a spectrum is a homotopy type endowed with a form of commutative addition.

4 The ∞-categorical framework. The use of the ∞-categorical framework for the study of spectra has proven to be extremely beneficial. A key feature of this theory is that an ∞-category in which every morphism is an isomorphism (that is, an ∞-groupoid) is the same as a homotopy type in $\mathcal{S}$. This has several implications.

Given a (small) infinity category $\mathcal{C}$, we can consider the full ∞-subcategory $\mathcal{C}^{\simeq}\subset\mathcal{C}$ supported on all isomorphisms. A symmetric monoidal structure on $\mathcal{C}$ induces in turn a coherent commutative monoid structure on $\mathcal{C}^{\simeq}$. For example, if R is a ring, the category of projective finitely generated R-modules $\mathrm{Proj}(R)^{fg}$ admits a symmetric monoidal structure given by $\oplus$. We then get

$$\left(\mathrm{Proj}(R)^{fg}\right)^{\simeq}\in\mathrm{CMon}(\mathcal{S}).$$

Taking group completion we get the algebraic K-theory spectrum

$$\mathsf{K}(R)\colon=\left(\left(\mathrm{Proj}(R)^{fg}\right)^{\simeq}\right)^{gp}\in\mathrm{Sp}.$$

The equivalence of homotopy types of spaces and ∞-groupoids also allows us to consider diagrams in an ∞-category $\mathcal{C}$ with indexed on a space $A\in\mathcal{S}$. Such a functor $F\colon A\to\mathcal{C}$ can be thought of as a local system of objects in $\mathcal{C}$ on A. When $\mathcal{C}$ has enough limits or colimits we can then consider $\lim_A F\in\mathcal{C}$ and $\mathrm{colim}_A F\in\mathcal{C}$, respectively.

The former can be thought of as derived global sections of the local system F (and the latter as derived global cosections). When $A = BG$ is the classifying space of a group G, we consider $F\colon BG \to \mathcal{C}$ as an object $X = F(*) \in \mathcal{C}$ with a G-action. We then denote

$$X^{hG} := \lim_{BG} F, \quad X_{hG} := \operatorname*{colim}_{BG} F$$

and refer to these as G-fixed points and G-orbits, respectively. For any $A \in \mathcal{S}$ and $X \in \mathcal{C}$ one can consider the composition $A \to * \xrightarrow{X} \mathcal{C}$ as the constant A-shaped diagram on X, $\mathrm{cns}_{\mathsf{X}}$. We denote

$$X^A := \lim_A \mathrm{cns}_X, \ \ X\langle A\rangle := \operatorname*{colim}_A \mathrm{cns}_X.$$

Given a spectrum $\mathsf{E} \in \mathrm{Sp}$, we denote $\mathsf{E}_*(A) := \pi_*(\mathsf{E}\langle A\rangle)$ and $\mathsf{E}^*(A) := \pi_{-*}(\mathsf{E}^A)$. This collection of groups satisfies the Eilenberg–Steenrod axioms [ES45] for an extraordinary homology theory (resp., extraordinary cohomology theory). It is a celebrated theorem of Brown [Bro62] that every extraordinary homology or cohomology theory arises in this way from some spectrum E.

5 Homotopical measures. Going back to the fundamental analogy between spectra and abelian groups, we suggest the following picture: Letting $\mathsf{X} \in \mathrm{Sp}$ be a spectrum, its commutative addition property implies that, given a map $f\colon A \to \underline{\mathsf{X}}$ from a finite set A, we can obtain a summed map $* \xrightarrow{\sum_A f} \underline{\mathsf{X}}$. Using the $\mathbb{S}\langle -\rangle \dashv \underline{\bullet}$ adjunction, we can say this differently. For every map $\mathbb{S}\langle A\rangle \xrightarrow{f} \mathsf{X}$, we should obtain a natural summed map $\mathbb{S} \xrightarrow{\sum_A f} \mathsf{X}$. By the Yoneda lemma such a map must come from a precomposition with some map $\mu_A\colon \mathbb{S} \to \mathbb{S}\langle A\rangle$, that is, $\mu_A \in \pi_0(\mathbb{S}\langle A\rangle)$.

Consider the analogous discrete case, in which we are looking at an element $\sum_{a\in A} c_a\langle a\rangle \in \mathbb{Z}\langle A\rangle$. $\sum_{a\in A} c_a\langle a\rangle$ can be thought of as a "measure" on A, where a has weight c_a. Indeed such a measure allows us to sum maps from the set A. The counting measure is given by the element $\sum_{a\in A}\langle a\rangle \in \mathbb{Z}\langle A\rangle$. Since we have a natural isomorphism $\mathbb{Z}\langle A\rangle \simeq \bigoplus_A \mathbb{Z}$, we can identify the counting measure as corresponding to the diagonal map $\mathbb{Z} \to \prod_A \mathbb{Z}$ (for finite A, $\prod_A \mathbb{Z} \simeq \bigoplus_A \mathbb{Z}$).

Similarly, we have a natural equivalence $\mathbb{S}\langle A\rangle \simeq \bigvee_A \mathbb{S}$. For finite A, this wedge is equivalent to the product, $\prod_A \mathbb{S}$, and the map $\mathbb{S} \to \mathbb{S}\langle A\rangle$ that corresponds to the diagonal $\mathbb{S} \to \prod_A \mathbb{S}$ gives the relevant "homotopical" counting measure $\mu_A \in \pi_0(\mathbb{S}\langle A\rangle)$.

From this perspective, for a space $M \in \mathcal{S}$, any element $\mu \in \pi_0(\mathbb{S}\langle M\rangle)$ can be considered a *homotopical measure* on M. Given such $\mu \in \pi_0(\mathbb{S}\langle M\rangle)$ and a map $M \xrightarrow{f} \underline{\mathsf{X}}$, we can compose

$$\mathbb{S} \xrightarrow{\mu} \mathbb{S}\langle M\rangle \xrightarrow{f} \mathsf{X}$$

and obtain an element $\int_M f\mu \in \pi_0(\mathsf{X})$

Generalizing further, for any $m \in \mathbb{Z}$, we shall consider an element $\mu \in \pi_m(\mathbb{S}\langle M\rangle)$ as an *m-shifted measure*, such that for a map $f\colon M \to \underline{\mathsf{X}}$, we get an element

$$\int_M f \mathrm{d}\mu \in \pi_m(\mathsf{X}).$$

The unit of the adjunction $\mathbb{S}\langle -\rangle \dashv \underline{\bullet}$ gives rise to a map $* \to \underline{\mathbb{S}\langle *\rangle} = \underline{\mathbb{S}}$ that chooses the element $1 \in \mathbb{Z} \simeq \pi_0(\mathbb{S})$. Composing with the unique map $M \to *$ gives rise to the constant unit map $M \xrightarrow{1} \underline{\mathbb{S}}$, and we denote the volume of M according to μ by

$$|M|_\mu = \int_M 1 \mathrm{d}\mu \in \pi_m(\mathbb{S}).$$

For example, for a finite set A as above $|A|_{\mu_A} = \#A \in \mathbb{Z} \simeq \pi_0(\mathbb{S})$ is the cardinality of the finite set A.

One example of an m-shifted measure comes from closed stably framed m-manifolds. Indeed, if M is such a manifold, there exists an embedding $M \hookrightarrow \mathbb{R}^{N+m}$ with a trivialization of its normal bundle. Thus we have an open tubular neighborhood $M \subset U \subset \mathbb{R}^{N+m}$ together with an identification $U \simeq M \times \mathbb{R}^N$ The Pontryagin–Thom construction yields from the open embedding $U \subset \mathbb{R}^{N+m}$ a collapse map of one-point compactifications

$$S^{N+m} \simeq \widehat{\mathbb{R}^{N+m}} \to \widehat{U} \simeq \widehat{M \times \mathbb{R}^N} \simeq \Sigma^N(M_+).$$

This determines a spectrum map $\overline{\mathbb{S}}\langle S^m\rangle \to \overline{\mathbb{S}}\langle M_+\rangle \simeq \mathbb{S}\langle M\rangle$ and thus a measure $\mu_M \in \pi_m(\mathbb{S}\langle M\rangle)$. The Pontryagin–Thom theorem implies that this construction realizes the isomorphism discussed in subsection 2:

$$\Omega_m^{\mathrm{fr}} \simeq \pi_m(\mathbb{S}), \qquad [M] \mapsto |M|_{\mu_M}.$$

6 Higher algebra of spectra. Considering spectra as analogous to abelian groups paves the way for redeveloping algebra with spectra in the role of abelian groups. This endeavor was named "Brave New Algebra" by Waldhausen, and, being quite daring ourselves, it is the main approach we shall take in this talk.

The tensor product $\otimes$ on the category of spectra allows us, as in the case of abelian groups, to consider multilinear maps. From here, it is possible to define a *ring spectrum* as a spectrum R endowed with a bilinear multiplication map $R \otimes R \to R$ and a unit map $\mathbb{S} \to R$.

In the case of abelian groups, the general theory is substantially different when we consider commutative rings versus general associative rings. In the homotopical realm, the situation is even more subtle. Properties like associativity and commutativity have many possible analogues, which differ from each other by a level of "coherence." The key notion for our story is that of an $\mathbb{E}_\infty$-*ring spectrum*, which is the most coherent form of commutativity. Other, weaker forms of structure will make an appearance, and we shall remark on them when relevant.

The universal example is the sphere spectrum $\mathbb{S}$ itself, which is the initial object in the category of $\mathbb{E}_\infty$-ring spectra, in analogy with $\mathbb{Z}$ being the initial commutative ring. As another example, the algebraic K-theory spectrum $\mathsf{K}(R)$ of a commutative ring R is an $\mathbb{E}_\infty$-ring, where the product structure comes from the tensor product of R-modules. A further example comes from cobordisms. For certain kinds of tangential structures, classified by a map $X \to BO$, with notable examples including $X = BO, BSO, BSpin, BString, *$, the product of manifolds gives rise to an $\mathbb{E}_\infty$-ring structure on the Thom spectrum $\mathsf{M}(X)$.

Similarly to the discrete case, given an $\mathbb{E}_\infty$-ring R, we can construct a symmetric monoidal ∞-category of modules $(\mathrm{Mod}_R, \otimes_R, R)$. Using this, one can also define the algebraic K-theory of R, also denoted $\mathsf{K}(R)$, which is again an $\mathbb{E}_\infty$-ring. The category Mod_R is a stable, presentable, symmetric monoidal ∞-category. For any such category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$, there is a free-forgetful adjunction:

$$(\mathcal{S}, \times, *) \underset{\underline{(-)}}{\overset{\mathbb{1}_{\mathcal{C}}\langle - \rangle}{\rightleftarrows}} (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}).$$

We can thus also discuss $\mathcal{C}$-valued (shifted) measures. For example, when the Thom spectrum $\mathsf{M}(X)$ is an $\mathbb{E}_\infty$-ring, every X-framed m-manifold M carries a canonical $\mathcal{C}$-valued measure $\mu_M \in \pi_m(\mathbb{1}_{\mathcal{C}}\langle M\rangle)$, where $\mathcal{C} = \mathrm{Mod}_{\mathsf{M}(X)}$, and the map

$$\Omega_m^X \simeq \pi_m(\mathsf{M}(X)), \qquad [M] \mapsto |M|_{\mu_M}$$

realizes the Thom isomorphism (2.1) above.

7 Lessons from arithmetic geometry. Let us now focus on the attempt to understand $\pi^S_{(p),*} \simeq \pi_*(\mathbb{S}_{(p)})$. The spectrum $\mathbb{S}_{(p)}$ is an $\mathbb{E}_\infty$-ring spectrum, and thus a homotopical analogue of a commutative ring. Many of the great triumphs in understanding commutative rings came from the perspective of arithmetic geometry, which suggests methods for decomposing commutative rings into simpler pieces.

The philosophical idea is as follows:

> First, every commutative ring R can be imagined to be a ring of functions on some space $\mathrm{Spec}(R)$. Second, a "divide and conquer" approach is taken by breaking the space $\mathrm{Spec}(R)$ into smaller pieces.

Specifically, given a "function" $f \in R$, we can consider the zero and nonzero loci of f in $\mathrm{Spec}(R)$, which are given by $\mathrm{Spec}(R/f) \subset \mathrm{Spec}(R)$ and $\mathrm{Spec}(R[f^{-1}]) \subset \mathrm{Spec}(R)$, respectively. Whenever f is neither invertible nor nilpotent, these inclusions are strict. Furthermore, the subsets $\mathrm{Spec}(R[f^{-1}])$ and $\mathrm{Spec}(R/f)$ depend only on the radical ideal generated by f. In particular, they are unchanged when f is replaced by f^m (for $m \geq 1$) or by uf (for a unit $u \in R^\times$). When R has no nonnilpotent, noninvertible elements, we say it is "hyperlocal."

We shall attempt to run a similar approach for a ring spectrum R, such as $\mathsf{R} = \mathbb{S}_{(p)}$, using an element $f \in \pi_*(\mathsf{R})$. First, given such an element $f \in \pi_d(\mathsf{R})$ for some $d \in \mathbb{Z}$, if f is central (which is automatic if R is commutative), it is indeed possible to define a new ring spectrum $\mathsf{R}[f^{-1}]$ with

$$\pi_*(\mathsf{R}[f^{-1}]) \simeq \pi_*(\mathsf{R})[f^{-1}] \simeq \operatorname{colim}\left[\pi_*(\mathsf{R}) \xrightarrow{\times f} \pi_{*+d}(\mathsf{R}) \xrightarrow{\times f} \pi_{*+2d}(\mathsf{R}) \xrightarrow{\times f} \cdots\right]. \tag{7.1}$$

Similarly, one can consider the quotient R/f. Here, however, two subtleties arise that are not present in the discrete case.

First, instead of $\pi_*(\mathsf{R}/f)$ simply being $\pi_*(\mathsf{R})/f$, we have a long exact sequence in homotopy arising from the cofiber sequence $\mathsf{R} \xrightarrow{\times f} \mathsf{R} \to \mathsf{R}/f$:

$$\cdots \to \pi_{*-d}(\mathsf{R}) \xrightarrow{\times f} \pi_*(\mathsf{R}) \to \pi_*(\mathsf{R}/f) \to \pi_{*-1-d}(\mathsf{R}) \xrightarrow{\times f} \pi_{*-1}(\mathsf{R}) \to \cdots$$

This gives rise to a short exact sequence:

$$0 \to \pi_*(\mathsf{R})/f \to \pi_*(\mathsf{R}/f) \to \pi_{*-1-d}(\mathsf{R})[f] \to 0, \tag{7.2}$$

where $(-)/f$ denotes the cokernel of multiplication by f and $(-)[f]$ stands for the f-torsion subgroup.

A second subtlety is that the level of coherence of commutativity and associativity tends to decrease when passing from R to R/f.[2] Although this is a very interesting phenomenon, in the context of our story it appears as a mild technicality, as we can always obtain a suitable amount of coherent structure by replacing f with some power of f (a particular strong version of this principle has been proven by Burklund in [Bur22]). We will suppress this subtlety for the remainder of the paper.

8 Decomposing the sphere. We shall now try to apply this algebraic decomposition to the commutative ring spectrum $\mathbb{S}_{(p)}$. It is a theorem of Nishida that all elements in $\pi_n(\mathbb{S}_{(p)})$ for $n > 0$ are nilpotent. The only nonnilpotent element to consider (up to units and powers) is therefore the prime p itself, viewed as an element $p \in \pi_0(\mathbb{S}_{(p)})$. We thus consider the decomposition with respect to p:

$$T(0) := \mathbb{S}_{(p)}[p^{-1}] \quad \text{and} \quad W(0) := \mathbb{S}_{(p)}/p.$$

By the analysis above, the homotopy groups of the former are

$$\pi_*(T(0)) = \pi_*(\mathbb{S}_{(p)})[p^{-1}] = \begin{cases} \mathbb{Q} & \text{if } * = 0, \\ 0 & \text{if } * \neq 0, \end{cases}$$

and by (7.2), there is a short exact sequence

$$0 \to \pi_*(\mathbb{S}_{(p)})/p \to \pi_*(\mathbb{S}_{(p)}/p) \to \pi_{*-1}(\mathbb{S}_{(p)})[p] \to 0.$$

Thus, $\pi_*(T(0))$ is hyperlocal. Understanding $\pi_*(\mathbb{S}_{(p)}/p)$ is closely related to understanding the p-rank of π_*^S. More precisely, when $\mathbb{S}_{(p)}/p$ is a ring (which requires $p > 2$), $\pi_m(\mathbb{S}_{(p)}/p)$ is an $\mathbb{F}_p$-vector space, and for $m > 1$, by (7.2) the dimension is given by

$$\dim_{\mathbb{F}_p} \pi_m(\mathbb{S}_{(p)}/p) = \dim_{\mathbb{F}_p}(\pi_m^S/p) + \dim_{\mathbb{F}_p}(\pi_{m-1}^S/p). \tag{8.1}$$

We now want to apply this decomposition procedure again, this time to the ring spectrum $\mathbb{S}_{(p)}/p$. (When $p = 2$, to get a ring structure we actually need to consider $\mathbb{S}_{(2)}/4$, as mentioned above, regarding the need to replace elements by specific powers to retain structure). For concreteness and simplicity, let us restrict ourselves to an odd prime p, so we can work with $\mathbb{S}_{(p)}/p$.

Adams proved that there exists a nonnilpotent, noninvertible element $v_1 \in \pi_{2p-2}(\mathbb{S}_{(p)}/p)$. We can thus decompose $\mathbb{S}_{(p)}/p$ with respect to v_1, considering $T(1) := (\mathbb{S}_{(p)}/p)[v_1^{-1}]$ and $W(1) := (\mathbb{S}_{(p)}/p)/v_1$. By (7.2), for $m > 2p$ we have

$$\begin{aligned} \dim_{\mathbb{F}_p} \pi_m(W(1)) &= \dim_{\mathbb{F}_p} \pi_m(W(0))/v_1 + \dim_{\mathbb{F}_p} \pi_{m-2p+1}(W(0))[v_1] \\ &\leq \dim_{\mathbb{F}_p} \pi_m(W(0)) + \dim_{\mathbb{F}_p} \pi_{m-2p+1}(W(0)) \\ &= \dim_{\mathbb{F}_p}(\pi_m^S/p) + \dim_{\mathbb{F}_p}(\pi_{m-1}^S/p) + \dim_{\mathbb{F}_p}(\pi_{m-2p+1}^S/p) + \dim_{\mathbb{F}_p}(\pi_{m-2p}^S/p), \end{aligned} \tag{8.2}$$

where we recall that $\pi_{m-2p+1}(W(0))[v_1]$ denotes the v_1-torsion subgroup of $\pi_{m-2p+1}(W(0))$.

In [Rav84] Ravenel made an amazing list of conjectures that, in particular, implies that this pattern continues indefinitely. That is, denoting $W(-1) = \mathbb{S}_{(p)}$ as the base of our induction, for every $h \geq 0$ there exists a

[2] In fact, it is even possible for R to be an $\mathbb{E}_\infty$-ring, while R/f does not even have a unital product structure.

nonnilpotent and noninvertible element $v_h \in \pi_*(W(h-1))$. We can then define the next layers of the tower as $T(h) := W(h-1)[v_h^{-1}]$ and $W(h) := W(h-1)/v_h$. Indeed the continuation of this pattern together with all of Ravenel's conjectures but one were proven in a pair of landmark papers by Devinatz, Hopkins, and Smith [DHS88, HS98], which also implies that $\pi_*(T(h))$ is hyperlocal for all $h \geq 0$. Repeating a similar argument to that in (8.2) gives for any h that there exists some constant $C_{W(h)}$ such that for $m \gg 0$

$$\dim_{\mathbb{F}_p} \pi_m(W(h)) \leq \sum_{i=m-C_{W(h)}}^{m} \dim_{\mathbb{F}_p}(\pi_i^S/p). \tag{8.3}$$

Thus we can get lower bounds on the asymptotic of $\dim_{\mathbb{F}_p}(\pi_i^S/p)$ from the asymptotic of $\dim_{\mathbb{F}_p} \pi_m(W(h))$. On the other hand, (7.1) implies that

$$\dim_{\mathbb{F}_p} \pi_m(T(h)) \leq \liminf_{d\to\infty} \dim_{\mathbb{F}_p}(\pi_{m+|v|d} W(h)), \tag{8.4}$$

thus relating the homotopy groups of $T(h)$ to the asymptotic behavior of $\dim_{\mathbb{F}_p}(\pi_i^S/p)$. In particular we shall get lower bounds by showing that certain homotopy groups of $T(h)$ are infinite. Note that we have made many choices when constructing the different $T(h)$'s. However, one can show that, given two such choices, say $T(h)_1$ and $T(h)_2$, the spectrum $T(h)_2$ can be constructed from $T(h)_1$ via limits of $T(h)_1$-modules. We are thus motivated to consider the subcategory

$$\mathrm{Sp}_{T(h)} \subset \mathrm{Sp},$$

which is the smallest subcategory closed under all limits and containing all $T(h)$-modules. This category does not depend on the choice of $T(h)$.

Furthermore, the inclusion $\mathrm{Sp}_{T(h)} \subset \mathrm{Sp}$ admits a left adjoint $L_{T(h)}\colon \mathrm{Sp} \to \mathrm{Sp}_{T(h)}$. Moreover, the category $\mathrm{Sp}_{T(h)}$ is itself a symmetric monoidal stable ∞-category with tensor product and unit given by

$$X \widehat{\otimes} Y := L_{T(h)}(X \otimes Y), \qquad \mathbb{S}_{T(h)} := L_{T(h)}\mathbb{S}.$$

The symmetric monoidal localization functor $L_{T(h)}$ is an instance of a Bousfield localization [Bou79]. Bousfield showed that for spectrum $\mathsf{E} \in \mathrm{Sp}$ there is a symmetric monoidal functor $L_{\mathsf{E}}\colon \mathrm{Sp} \to \mathrm{Sp}_{\mathsf{E}}$ with a fully faithful right adjoint $\mathrm{Sp}_{\mathsf{E}} \subset \mathrm{Sp}$ such that

$$L_{\mathsf{E}}(\mathsf{X}) = 0 \iff X \otimes \mathsf{E} = 0.$$

We say that E_1 and E_2 are in the same "Bousfield class" if we can identify $(\mathrm{Sp}_{\mathsf{E}_1}, L_{\mathsf{E}_1})$ with $(\mathrm{Sp}_{\mathsf{E}_2}, L_{\mathsf{E}_2})$. As mentioned above the Bousfield class of $T(h)$ is independent of the choices made to construct $T(h)$[3] and is determined by the pair (p, h).

9 Chromatic support. The localization functor

$$L_{T(h)}\colon \mathrm{Sp} \to \mathrm{Sp}_{T(h)}$$

can be thought of as a restriction functor to an infinitesimal neighborhood of a point $(p, h) \in$ "Spec($\mathbb{S}$)". Given an $\mathbb{E}_\infty$-ring R, Hahn [Hah16] proved the following theorem.

THEOREM 9.1 (Hahn). *For* R *an* $\mathbb{E}_\infty$*-ring and* $h \geq 0$,

$$L_{T(h+1)}\mathsf{R} \neq 0 \implies L_{T(h)}\mathsf{R} \neq 0.$$

Thus, one can define the *p-height* of R as

$$\mathrm{height}_p(\mathsf{R}) := \max\{h \mid L_{T(h)}\mathsf{R} \neq 0\}.$$

In particular, if $0 \neq \mathsf{R} \in \mathrm{Sp}_{T(h)}$ is an $\mathbb{E}_\infty$-ring, then $\mathrm{height}_p(\mathsf{R}) = h$. The support of $\mathbb{E}_\infty$-rings is a key invariant. For example, when working at the prime $p = 2$, we have

$$\mathrm{height}_2(\mathsf{MSO}) = 0, \quad \mathrm{height}_2(\mathsf{MSpin}) = 1, \quad \mathrm{height}_2(\mathsf{MString}) = 2.$$

These facts are closely related to the Hirzebruch, Atiyah–Bott–Shapiro, and Witten genera, respectively. Algebraic K-theory has a particularly interesting interaction with chromatic support. The following result (sometimes referred to as *redshift*) was conjectured by Ausoni and Rognes [AR02, AR08] and proved in [Yua24, BSY22, CMNN20, LMMT20].

[3] We follow here a standard convection of suppressing the dependence of $T(h)$ in p from the notation.

THEOREM 9.2 (redshift). *If* R *is an* $\mathbb{E}_\infty$*-ring with* $\mathrm{height}_p(\mathsf{R}) \geq 0$*, then*

$$\mathrm{height}_p(\mathsf{K}(\mathsf{R})) = \mathrm{height}_p(\mathsf{R}) + 1.$$

In fact, these height-shifting phenomena in algebraic K-theory are expected to hold in far greater generality than merely for $\mathbb{E}_\infty$-rings, and they remain only partially understood. A major recent advance is the work of Hahn and Wilson [HW22], which establishes a more quantitative form of redshift—one that is in fact closer to the original perspective of Ausoni and Rognes.

We say that a spectrum R is *h-cropped* if $\mathsf{R} \otimes W(h)$ is bounded above but $\mathsf{R} \otimes W(h-1)$ is not. (This notion is closely related to the fp-spectra studied by Mahowald and Rezk [MR99].) Since the v_h- and v_{h+1}-maps have nonnegative degree, an h-cropped spectrum necessarily satisfies

$$\mathrm{height}_p(\mathsf{R}) = h.$$

Another manifestation of redshift—verified in many important cases—is the tendency of algebraic K-theory to send h-cropped ring spectra to $(h+1)$-cropped spectra. The full scope of this phenomenon is still unknown, but evidence points to it being very general. For suitable classical rings R the claim that $\mathsf{K}(R)$ is 1-cropped had been shown by Waldhausen [Wal84] (based on work of Thomason [Tho82]) to imply the celebrated Lichtenbaum–Quillen conjecture [Qui75].

Now assume that R is h-cropped and that $\pi_*(R \otimes W(h)) = 0$ for $* \geq b$. Then the natural map

$$\mathsf{R} \otimes W(h-1) \longrightarrow \mathsf{R} \otimes W(h-1)[v_h^{-1}] \simeq R \otimes T(h)$$

is an isomorphism on homotopy groups in degrees $* > b - |v_h|$. In particular, $\pi_*(R \otimes T(h))$ can be understood by analyzing $\pi_*(R)$ in a finite range of degrees. This stands in sharp contrast to the case $\mathsf{R} = \mathbb{S}$ (or any finite complex), for which $\pi_*(T(h))$ depends on homotopy groups in arbitrarily high degrees.

Our strategy for proving that $\pi_m(T(h))$ is infinite-dimensional for suitable pairs (m, h) will be as follows. We construct infinitely many classes in $\pi_m(T(h))$ and then establish their linear independence using the unit map

$$T(h) \longrightarrow \mathsf{R} \otimes T(h)$$

for a carefully chosen h-cropped spectrum R. Leveraging redshift, we take $\mathsf{R} \simeq \mathsf{K}(\mathsf{R}')$ for some $(h-1)$-cropped spectrum R'. We now introduce the main tool that enables the construction of this abundance of classes.

10 Higher semiadditivity. A key property of $\mathrm{Sp}_{T(h)}$ is the existence of a natural isomorphism between finite coproducts and finite products, as in any stable ∞-category. This can be restated as a natural isomorphism between colimits and limits over a diagram A when A is a finite set. For $\mathrm{Sp}_{T(h)}$ this phenomena can be generalized to a much larger class of diagrams than the class of finite sets. Namely, generalizing the work of Hopkins and Lurie in [HL13], in [CSY22] we prove the following.

THEOREM 10.1 (Carmeli–Schlank–Yanovski). *Let* $A \in \mathcal{S}$ *be a space such that*
1. $|\pi_0(A)| < \infty$*;*
2. *for all* $a \in A$*, the product* $\prod_i \pi_i(A, a)$ *is finite.*

Then for every functor $F\colon A \to \mathrm{Sp}_{T(h)}$*, there exists a natural isomorphism*

$$\mathrm{Nm}_A\colon \operatorname*{colim}_A F \xrightarrow{\sim} \lim_A F.$$

A space A as in the statement of this theorem is called *π-finite*. The existence of such isomorphisms Nm_A for special subcategories of Sp has been previously considered by Greenlees, Hovey, and Sadofsky [HS96, GS96], Kuhn [Kuh04] and Hopkins and Lurie [HL13]. This special property of $\mathrm{Sp}_{T(h)}$ allows the definition of "uniform" counting $\mathrm{Sp}_{T(h)}$-measures on π-finite spaces. Indeed, we get an isomorphism $\mathbb{S}_{T(h)}\langle A\rangle \simeq \mathbb{S}_{T(h)}^A$ for π-finite A. We call the measure $\mu_A \in \pi_0(\mathbb{S}_{T(h)}\langle A\rangle) \simeq \pi_0(\mathbb{S}_{T(h)}^A)$ corresponding to the map $A \to *$ the *counting measure* on A. We denote $|A|_h := |A|_{\mu_A} \in \pi_0(\mathbb{S}_{T(h)})$. Furthermore we show in [CSY21a] that if A is $(h-1)$-connected with an abelian fundamental group, then $|A|_h \in \pi_0(\mathbb{S}_{T(h)})^\times$ is a unit.

Given an m-dimensional stably framed manifold M, the image of its canonical measure $\mu_M \in \pi_m(\mathbb{S})$ under the map $\pi_m(\mathbb{S}) \to \pi_m(\mathbb{S}_{T(h)})$ gives an m-shifted $\mathrm{Sp}_{T(h)}$-measure on M. In fact, one can construct $\mathrm{Sp}_{T(h)}$-measures on spaces which are "mixed" from π-finite spaces and stably framed manifolds. That is, let X be a space endowed with a map $X \to M$ to a stably framed m-manifold, such that the fibers of the map $X \to M$ are π-finite. Then X carries an m-shifted $\mathrm{Sp}_{T(h)}$-measure $\mu_X \in \pi_m(\mathbb{S}\langle X\rangle)$, informally given by first summing with the counting measure along the fibers and then integrating with μ_M.

As an example, consider an automorphism $g\colon A \to A$ of a π-finite space A. The mapping torus

$$A//g := A \times [0,1]/\{(a,0) \sim (g(a),1)\}$$

is a fiber bundle over S^1 (a stably framed 1-manifold) with fibers homotopic to A. In this case, we get a 1-shifted $\mathrm{Sp}_{T(h)}$-measure $\mu_{A//g} \in \pi_1(\mathbb{S}_{T(h)}\langle A//g\rangle)$. An example that will be particularly relevant for us is the case where $A = B^{h-1}(\mathbb{Z}/p^k)$ is the Eilenberg–MacLane space with $\pi_{h-1}A = \mathbb{Z}/p^k$ and g is the map induced by multiplication by $(p+1)$ on $\mathbb{Z}/p^k$. We denote the resulting volume by

$$\tau_k := |A//g|_{\mu_{A//g}} \quad \in \quad \pi_1(\mathbb{S}_{T(h)}).$$

These kinds of generalized volumes will allow us to construct many classes in the homotopy groups of ring spectra in $\mathrm{Sp}_{T(h)}$, such as $T(h)$.

11 Galois theory. We shall now take another hint from arithmetic geometry. The decomposition we used so far is analogous to the use of the Zariski topology, but an even finer geometry—the étale topology—is useful for simplifying things further. For example, the theory of fields is generally simpler than that of rings, and separably closed fields are simpler still. Put differently, we can try to use Galois theory to understand $T(h)$ (or $L_{T(h)}\mathsf{X}$ for other $\mathsf{X} \in \mathrm{Sp}$). The theoretical framework of Galois extensions is based on Rognes's fundamental work [Rog08] that was expanded by Mathew in [Mat16a].

To build Galois extensions of $\mathbb{S}_{T(h)}$, we shall mimic the construction of cyclotomic extensions. The idea is to observe that the group ring $\mathbb{Q}\langle\mathbb{Z}/p^k\rangle$ splits as a product $\mathbb{Q}\langle\mathbb{Z}/p^k\rangle \simeq \mathbb{Q}(\omega_{p^k}) \times \mathbb{Q}\langle\mathbb{Z}/p^{k-1}\rangle$. This splitting is induced by an idempotent element $e_k \in \mathbb{Q}\langle\mathbb{Z}/p^k\rangle$. More precisely, we have the idempotent element

$$e_k := \frac{1}{|\mathbb{Z}/p|}\sum_{x\in\mathbb{Z}/p}\phi(x) \quad \in \quad \mathbb{Q}\langle\mathbb{Z}/p^k\rangle,$$

where ϕ is the composition $\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^k \to \mathbb{Q}\langle\mathbb{Z}/p^k\rangle$, satisfying that

$$\mathbb{Q}\langle\mathbb{Z}/p^k\rangle[(1-e_k)^{-1}] \simeq \mathbb{Q}(\omega_{p^k}).$$

Furthermore, the element e_k is fixed by the action of $(\mathbb{Z}/p^k)^\times$ on $\mathbb{Q}\langle\mathbb{Z}/p^k\rangle$, hence the action restricts to an action on $\mathbb{Q}(\omega_{p^k})$, which is the action of its Galois group. Moreover, taking

$$\mathbb{Z}/p^\infty := \mathbb{Q}_p/\mathbb{Z}_p \simeq \operatorname*{colim}_{k\to\infty}\mathbb{Z}/p^k,$$

we similarly get

$$\mathbb{Q}\langle\mathbb{Z}/p^\infty\rangle[(1-e_\infty)^{-1}] \simeq \mathbb{Q}(\omega_{p^\infty}),$$

which is a $\mathbb{Z}_p^\times$-pro-Galois extension.

In [CSY21b], we show that this story has an analogue in $\mathrm{Sp}_{T(h)}$ which uses the existence of the counting $\mathrm{Sp}_{T(h)}$-measures μ_A and the fact that for an $(h-1)$-connected space A, the volume $|A|_h$ is a unit. Generalizing the construction above, for $1 \le k \le \infty$ we can define an idempotent element

$$e_k := \frac{1}{|B^h(\mathbb{Z}/p)|_h}\int_{B^h(\mathbb{Z}/p)}\phi \,\mathrm{d}\mu_{B^h(\mathbb{Z}/p)} \quad \in \quad \pi_0(\mathbb{S}_{T(h)}\langle B^h(\mathbb{Z}/p^k)\rangle),$$

where ϕ is the composition

$$B^h(\mathbb{Z}/p) \to B^h(\mathbb{Z}/p^k) \to \underline{\mathbb{S}_{T(h)}\langle B^h(\mathbb{Z}/p^k)\rangle}.$$

This element is an idempotent and gives a splitting

$$\mathbb{S}_{T(h)}\langle B^h(\mathbb{Z}/p^k)\rangle \simeq \mathbb{S}_{T(h)}(\omega_{p^k}^{(h)}) \times \mathbb{S}_{T(h)}\langle B^h(\mathbb{Z}/p^{k-1})\rangle,$$

such that

$$\mathbb{S}_{T(h)}\langle B^h(\mathbb{Z}/p^k)\rangle[(1-e_k)^{-1}] \simeq \mathbb{S}_{T(h)}(\omega_{p^k}^{(h)}).$$

The action of $(\mathbb{Z}/p^k)^\times$ on $B^h(\mathbb{Z}/p^k)$ fixes e_k, and moreover the map $\mathbb{S}_{T(h)} \to \mathbb{S}_{T(h)}(\omega_{p^k}^{(h)})$ is a $(\mathbb{Z}/p^k)^\times$-Galois extension.

As in the classical setting, we have maps $\mathbb{S}_{T(h)}(\omega_p^{(h)}) \to \mathbb{S}_{T(h)}(\omega_{p^2}^{(h)}) \to \cdots$. Taking the colimit, we obtain the $\mathbb{Z}_p^\times$-extension $\mathbb{S}_{T(h)}(\omega_{p^\infty}^{(h)})$. The behavior of these extensions mirrors in many ways the classical theory of cyclotomic extensions—for instance, a natural generalization of the classical discrete Fourier transform.

Indeed the discrete Fourier transform can be described in the following way.

THEOREM 11.1 (classical discrete Fourier transform). *Let R be a commutative ring that admits a map $\mathbb{Q}[\omega_{p^k}] \to R$. Then for every p^k-torsion finite abelian group A, there exists a natural isomorphism*

$$R[A] \simeq R^{A^*},$$

where $A^* = \hom(A, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual.

Using the higher cyclotomic extensions in [BCSY24] we construct a higher chromatic analogue.

THEOREM 11.2 (Barthel–Carmeli–Schlank–Yanovski). *Let R be a $T(h)$-local $\mathbb{E}_\infty$-ring spectrum that admits a map $\mathbb{S}_{T(h)}(\omega_{p^k}^{(h)}) \to \mathsf{R}$. Then for every p^k-torsion finite abelian group A and for any $0 \le j \le h$, there exists a natural equivalence*

$$\mathsf{R}[B^{h-j}A] \simeq \mathsf{R}^{B^j A^*}.$$

12 Cyclotomic redshift. As we mentioned before, when R is a nonzero $T(h)$-local $\mathbb{E}_\infty$-ring spectrum, the $T(h+1)$-localization of its algebraic K-theory $L_{T(h+1)}\mathsf{K}(\mathsf{R})$ is also nonzero. This redshift phenomenon interacts well with cyclotomic extensions. Indeed, in [BMCSY25] we prove the following theorem.

THEOREM 12.1 (Ben-Moshe–Carmeli–Schlank–Yanovski). *Let R be a $T(h)$-local ring spectrum. Then there is a $\mathbb{Z}_p^\times$-equivariant equivalence*

$$L_{T(h+1)}\mathsf{K}(\mathsf{R}(\omega_{p^\infty}^{(h)})) \simeq (L_{T(h+1)}\mathsf{K}(\mathsf{R}))(\omega_{p^\infty}^{(h+1)}).$$

Theorem 12.1 *generalizes the case of $h=0$ proven by Bhatt, Clausen, and Mathew in* [BCM20].

13 Galois descent. The existence of Galois extensions suggests understanding objects in $\mathrm{Sp}_{T(h)}$ using Galois descent. By analogy, consider that every $\mathbb{R}$-vector space V can be written as $V \simeq (V \otimes_{\mathbb{R}} \mathbb{C})^{hC_2}$. Similarly, given $X \in \mathrm{Sp}_{T(h)}$, we have

$$X \simeq (X(\omega_{p^k}^{(h)}))^{h(\mathbb{Z}/p^k)^\times},$$

where $X(\omega_{p^k}^{(h)})$ stands for $X \widehat{\otimes} \mathbb{S}_{T(h)}(\omega_{p^k}^{(h)})$.

Here, however, a few subtleties arise. First, since $\mathbb{Z}_p^\times$ is a profinite group, the notion of homotopy fixed points is subtle. For now, let us assume that p is odd, in which case $\mathbb{Z}_p^\times$ is procyclic and generated by some element $g_p \in \mathbb{Z}_p^\times$. We can then take homotopy fixed points via the $\mathbb{Z}$-action through g_p. This gives a comparison map:

$$X \to (X(\omega_{p^\infty}^{(h)}))^{h\mathbb{Z}_p^\times} \simeq \mathrm{fiber}\left(X(\omega_{p^\infty}^{(h)}) \xrightarrow{\mathrm{id}-g_p} X(\omega_{p^\infty}^{(h)})\right).$$

There is, however, another subtle issue: the existence of Galois descent for all finite cyclotomic extensions does not imply the same claim for the infinite one.[4] Nevertheless, by work of Mahowald [Mah81] and Miller [Mil81] this descent map is an isomorphism for $h=1$ (the case $h=0$ is also true and essentially classical), and we shall now discuss applications in this case and explore the case $h>1$, where such descent generally fails, in the next sections.

[4]This is a purely ∞-categorical phenomenon and does not appear in the discrete analogue.

It follows from a theorem of Snaith that $\mathbb{S}_{T(1)}(\omega^{(1)}_{p^\infty})$ can be identified with a p-completed form of topological K-theory, KU. As a consequence, for $T(1) = (\mathbb{S}_{(p)}/p)[v_1^{-1}]$, we also have $T(1)(\omega^{(1)}_{p^\infty}) \simeq KU/p$, and

$$\pi_*(T(1)(\omega^{(1)}_{p^\infty})) \simeq \mathbb{F}_p[\beta^{\pm 1}],$$

where the degree is $|\beta| = 2$. Furthermore, the Galois action is given in terms of the progenerator by $g_p \cdot \beta = k\beta$, where k is the image of g_p under the quotient map $\mathbb{Z}_p^\times \to (\mathbb{Z}/p)^\times$. Calculating the homotopy fixed points, we get

$$\pi_*(T(1)) = \mathbb{F}_p[v_1^{\pm 1}] \otimes \Lambda(\epsilon),$$

where $|\epsilon| = -1$ and $|v_1| = 2p-2$.

Consider now the maps $\pi_*(\mathbb{S}_{(p)}) \to \pi_*(\mathbb{S}_{(p)}/p) \to \pi_*(T(1))$. Since $v_1 \in \pi_{2p-2}(\mathbb{S}_{(p)}/p)$ is not nilpotent, it cannot be in the image of $\pi_*(\mathbb{S}_{(p)})$, which consists of nilpotent elements. Therefore, by the short exact sequence associated with the quotient by p,

$$0 \to \pi_*(\mathbb{S}_{(p)})/p \to \pi_*(\mathbb{S}_{(p)}/p) \to \pi_{*-1}(\mathbb{S}_{(p)})[p] \to 0,$$

the element v_1^k must correspond to a nontrivial p-torsion class in $\pi_{k(2p-2)-1}(\mathbb{S})$. The existence of such a torsion class gives p-torsion in both $\pi_{k(2p-2)-1}(\mathbb{S})/p$ and $\pi_{k(2p-2)-1}(\mathbb{S})[p]$, which accounts for the elements corresponding to v_1^k and $v_1^k\epsilon$ in $\pi_*(T(1))$. The height-one part of the chromatic picture thus implies the existence of nontrivial p-torsion in degrees $k(2p-2)-1$. This is the image of J originally defined by Adams [Ada66].

14 Cyclotomic descent and the telescope conjecture. In height 1, we used the descent isomorphism $X \simeq (X(\omega^{(1)}_{p^\infty}))^{h\mathbb{Z}_p^\times}$. When $h > 1$, the situation is more subtle. We denote by $\mathrm{Sp}^{\mathrm{cyc}}_{T(h)} \subset \mathrm{Sp}_{T(h)}$ the subcategory of objects $X \in \mathrm{Sp}_{T(h)}$ such that $X \simeq (X(\omega^{(h)}_{p^\infty}))^{h\mathbb{Z}_p^\times}$. As mentioned above, at height $h = 0, 1$, we have that $\mathrm{Sp}^{\mathrm{cyc}}_{T(h)}$ consists of all objects. On the other hand, in [BHLS23] we show that this fails for higher heights.

Theorem 14.1 (Burklund–Hahn–Levy–Schlank). *$\mathrm{Sp}^{\mathrm{cyc}}_{T(h)} \neq \mathrm{Sp}_{T(h)}$ when $h > 1$.*

In [BHLS23], it is also proven that $\mathrm{Sp}^{\mathrm{cyc}}_{T(h)}$ is closed under limits and colimits. Since $T(h)$ generates $\mathrm{Sp}_{T(h)}$ under limits and colimits, we learn that $T(h) \notin \mathrm{Sp}^{\mathrm{cyc}}_{T(h)}$. In other words, for $h > 1$, the descent map

$$T(h) \to (T(h)(\omega^{(h)}_{p^\infty}))^{h\mathbb{Z}_p^\times}$$

is not an isomorphism. There are now at least two possible approaches to this discrepancy.

First, one can force Galois descent by restricting to a smaller category. For example, forcing Galois descent with respect to the cyclotomic extensions gives us the subcategory $\mathrm{Sp}^{\mathrm{cyc}}_{T(h)}$. We can restrict further by considering more Galois extensions. In fact, Burklund, Clausen, and Levy announced a computation of the absolute Galois group and the "separable closure" of $\mathbb{S}_{T(h)}$. To describe this, we use the following functor defined by Goerss, and Hopkins [GH04] based on work of Morava, and later generalized by Lurie.

Theorem 14.2 (Goerss–Hopkins–Miller). *Let FG_h be the category of pairs (F, Γ), where F is a perfect field and Γ is a 1-dimensional commutative formal group of height h over F. There exists a fully faithful functor*

$$E_h \colon FG_h \to \mathrm{CAlg}(\mathrm{Sp}_{T(h)})$$

such that

1. *$\pi_0(E_h(F,\Gamma)) \simeq LT(F,\Gamma)$ is the Lubin–Tate universal deformation ring of Γ;*
2. *$\pi_{2k+1}(E_h(F,\Gamma)) = 0$ for all k;*
3. *$\pi_{2k}(E_h(F,\Gamma)) \simeq \omega^{\otimes k}$, where ω is the invertible $LT(F,\Gamma)$-module of invariant differentials.*

The result of Burklund, Clausen, and Levy is then the following.

Theorem 14.3 (Burklund–Clausen–Levy). *Let $h \geq 1$ and let Γ_h be the (unique) height h formal group over $\overline{\mathbb{F}}_p$. Then $E_h(\overline{\mathbb{F}}_p, \Gamma_h)$, with its action by the profinite group $\mathbb{G}_h := \mathrm{Aut}(\overline{\mathbb{F}}_p, \Gamma_h)$,[5] is the separable closure of $\mathbb{S}_{T(h)}$.*

[5] E_h is known as Morava E-theory and $\mathbb{G}_h$ is known as the Morava stabilizer group.

One can consider the objects in $\mathrm{Sp}_{T(h)}$ that satisfy Galois descent with respect to this extension, that is, $X \in \mathrm{Sp}_{T(h)}$ such that $X \simeq (X \widehat{\otimes} E_h)^{h\mathbb{G}_h}$. This defines the subcategory

$$\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{T(h)}.$$

It was the last standing from the Ravenel Conjectures [Rav84], called the "Telescope Conjecture," that $\mathrm{Sp}_{K(h)} = \mathrm{Sp}_{T(h)}$. Satisfying descent for the absolute Galois group implies satisfying descent for any intermediate extension, and in particular for the cyclotomic extensions. As a consequence, we learn that we have inclusions

$$\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{T(h)}^{\mathrm{cyc}} \subsetneq \mathrm{Sp}_{T(h)}.$$

Therefore, the failure of cyclotomic descent from Theorem 14.1 also refutes the Telescope Conjecture.

THEOREM 14.4 (Burklund–Hahn–Levy–Schlank). $\mathrm{Sp}_{K(h)} \neq \mathrm{Sp}_{T(h)}$ *when* $h > 1$.

As for $\mathrm{Sp}_{T(h)}$, we have a symmetric monoidal localization of presentably symmetric monoidal ∞-categories

$$L_{K(h)} \colon \mathrm{Sp} \to \mathrm{Sp}_{K(h)}.$$

One can then attempt to understand $\mathbb{S}_{(p)}$ by using E_h-modules in $\mathrm{Sp}_{K(h)}$. Indeed, it is a theorem of Hopkins and Ravenel [Rav92, section 8.6] that $\mathbb{S}_{(p)}$ can be written as a limit of E_h-modules for varying h in a precise way using this descent approach, while the rate of convergence of the $\mathrm{Sp}_{T(h)}$ approach to phenomena in the stable stems is faster, which we will leverage to obtain better bounds on π_*. The advantage of the $\mathrm{Sp}_{K(h)}$ approach is that it allows the use of methods from p-adic geometry and other computational tools. Furthermore, the separable closure E_h is very well behaved and even satisfies a form of Hilbert's Nullstellensatz. The classical Hilbert's Nullstellensatz [Hil70] can be restated as follows.

THEOREM 14.5. *Let R be a nonzero commutative ring. The following are equivalent:*
1. *R is an algebraically closed field.*
2. *For any nonzero compact object A of $\mathrm{Ring}_{R/}$, i.e., a finitely presented R-algebra, there exists a retract of the map $R \to A$.*

In [BSY22] we prove a chromatic analogue to Hilbert's Nullstellensatz in $\mathrm{Sp}_{T(h)}$.

THEOREM 14.6 (Burklund–Schlank–Yuan). *Let R be a nonzero $T(h)$-local $\mathbb{E}_\infty$-ring spectrum. The following are equivalent:*
1. *$\mathsf{R} \simeq E_h(F, \Gamma)$ for an algebraically closed field F.*
2. *For any nonzero compact object A in $\mathrm{CAlg}(\mathrm{Sp}_{T(h)})_{\mathsf{R}/}$, i.e., a finitely presented $T(h)$-local R-$\mathbb{E}_\infty$-algebra, there exists a retract of the map $\mathsf{R} \to \mathsf{A}$.*

Further, similarly to the fact that every nonzero ring admits a map to some algebraically closed field, every nonzero $T(h)$-local $\mathbb{E}_\infty$-ring R admits a map $\mathsf{R} \to E_h(F, \Gamma)$ for some algebraically closed field F. In fact, the redshift inequality $\mathrm{height}_p(\mathsf{K}(\mathsf{R})) \geq \mathrm{height}_p(\mathsf{R}) + 1$[6] from Theorem 9.2 is proved by using such a map $\mathsf{R} \to E_h(F, \Gamma)$ to reduce to the claim that $L_{T(h+1)}\mathsf{K}(E_h(F, \Gamma)) \neq 0$. The latter claim has been proved by Yuan [Yua24]. Similarly, Theorem 9.1 can be proved by verifying the claim that $L_{T(h)} E_{h+1}(F, \Gamma) \neq 0$.

15 The $K(n)$-local category. Truth be told, the discussion of the $K(n)$-local category above is wildly ahistorical. The category $\mathrm{Sp}_{K(h)}$ was originally defined as a Bousfield localization with respect to "Morava K-theory" $K(h) \in \mathrm{Sp}$, a ring spectrum in the same Bousfield class as $E_h(\overline{\mathbb{F}}_p, \Gamma_h) \otimes T(h)$. The role of $E_h(\overline{\mathbb{F}}_p, \Gamma_h)$ as a separable closure $L_{K(h)}\mathbb{S}$ is due to Devinatz and Hopkins [DH04] and Baker and Richter [BR08]. (See also [Mat16b].) The content of Theorem 14.3 is thus in the lift from $\mathrm{Sp}_{K(h)}$ to $\mathrm{Sp}_{T(h)}$.

There has been a great deal of research on $\mathrm{Sp}_{K(n)}$, where the existence of $\mathbb{G}_h$-Galois descent along the map $L_{K(n)}\mathbb{S} \to E_h(\overline{\mathbb{F}}_p, \Gamma_h)$ allows one to consider $\mathrm{Sp}_{K(h)}$ as the category of $\mathbb{G}_h$-equivariant complete $E_h(\overline{\mathbb{F}}_p, \Gamma_h)$-modules. This description provides a bridge from questions about $\mathrm{Sp}_{K(h)}$ to p-adic geometry. Specifically there exists an adic space LT_h for which we have a natural isomorphism $\mathcal{O}^+(LT_h) \simeq LT(\overline{\mathbb{F}}_p, \Gamma_h)$, where $\mathcal{O}^+$ is the sheaf of bounded by 1 analytic functions. Further, LT_h admits a $\mathbb{G}_h$-action which renders this isomorphism $\mathbb{G}_h$-equivariant. This intimate connection between homotopy theory and the p-adic geometry of LT_h was

[6] The inequality $\mathrm{height}_p(\mathsf{K}(\mathsf{R})) \leq \mathrm{height}_p(R) + 1$ is proved in [LMMT20, CMNN20].

used already in 1994 by Hopkins and Gross [HG94], who defined the Gross–Hopkins period map. The recent breakthroughs in p-adic geometry based on pro-étale and perfectoid methods by Scholze and his collaborators [Sch12, Sch14, Sch17, BCK+14, BS13] had allowed one to use this bridge to further the study of $\mathrm{Sp}_{K(h)}$. Using the language of diamonds we can now relate questions about $\mathrm{Sp}_{K(h)}$ to the pro-étale cohomology of certain pro-étale sheaves on the diamond stack

$$\mathfrak{M}_h := [LT_h^\diamond // \mathbb{G}_h]\,.$$

A key result now is the Fargue–Faltings theorem [FGL08] in the version proved by Weinstein and Scholze [SW14] that gives an isomorphism of diamond stacks

$$\mathfrak{M}_h \simeq [\mathcal{H}_h^\diamond // \mathrm{GL}_h(\mathbb{Z}_p)]\,,$$

where $\mathcal{H}_h$ is the Drinfeld upper half space [Dri76].

These methods allow us to gain insight into objects such as $\pi_*(L_{K(h)}\mathbb{S})$. For instance, in [BSSW24] we show the following.

Theorem 15.1 (Barthel–Schlank–Stapleton–Weinstein). *There is an isomorphism of graded rings*

$$\pi_*(L_{K(h)}\mathbb{S})) \otimes \mathbb{Q}_p \simeq \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \dots \zeta_h),$$

where $\Lambda_{\mathbb{Q}_p}$ denotes the exterior algebra and $|\zeta_i| = 1 - 2i$.

This result matches suggestion made by Morava [Mor85] and fits a far more general picture envisioned by Hopkins, known as the *Hopkins' chromatic splitting conjecture* [Hov95]. The cases of Theorem 15.1 for $h \le 2$ were previously proven using more direct computational methods [SY95, BGH22, Beh12, GHM14].

16 Lower bounds on homotopy groups. The proof of the failure of cyclotomic descent from Theorem 14.1 is based on finding a ring $\mathsf{R} \in \mathrm{Sp}_{T(h-1)}$ that satisfies cyclotomic descent such that $L_{T(h)}\mathsf{K}(\mathsf{R})$ does *not* satisfy cyclotomic descent, that is, such that the descent map

$$L_{T(h)}\mathsf{K}(\mathsf{R}) \to ((L_{T(h)}\mathsf{K}(\mathsf{R})(\omega_{p^\infty}^{(h)}))^{h\mathbb{Z}_p^\times}$$

is not an isomorphism. Since R is assumed to satisfy cyclotomic descent we can identify the source with $L_{T(h)}\mathsf{K}((\mathsf{R}[\omega_{p^\infty}^{(h-1)}])^{h\mathbb{Z}_p^\times})$, and by Theorem 12.1 we can identify the target with $(L_{T(h)}\mathsf{K}(\mathsf{R}[\omega_{p^\infty}^{(h-1)}]))^{h\mathbb{Z}_p^\times}$. Together, these identify the descent map with the canonical limit comparison map

$$L_{T(h)}\mathsf{K}((\mathsf{R}[\omega_{p^\infty}^{(h-1)}])^{h\mathbb{Z}_p^\times}) \to (L_{T(h)}\mathsf{K}(\mathsf{R}[\omega_{p^\infty}^{(h-1)}]))^{h\mathbb{Z}_p^\times}.$$

Recent advances in the trace-theoretic approach to algebraic K-theory [HW20, NS18, AN21] allow one to show that this map is not an isomorphism for a well-chosen R.

It turns out the classes τ_k are closely related to the failure of cyclotomic descent. For a ring $\mathsf{R} \in \mathrm{Sp}_{T(h)}$, if for instance $\pi_0(\mathsf{R})$ is local Artinian, and R fails cyclotomic descent, then the images of infinitely many $\tau_k \in \pi_1(\mathbb{S}_{T(h)})$ in $\pi_1(\mathsf{R})$ via the unit map $\mathbb{S}_{T(h)} \to \mathsf{R}$ are nonzero. Using these ideas, for any $h > 1$ we choose a specific ring $\mathsf{R} \in \mathrm{Sp}_{T(h-1)}$ and show that the image of $\tau_k \in \pi_1(\mathbb{S}_{T(h)})$ in $\pi_1(T(h) \otimes \mathsf{K}(\mathsf{R}))$ for $k = 1, 2, 3, \dots$ generates an infinite-dimensional $\mathbb{F}_p$-vector space. For example, for $h = 2$ we take $\mathsf{R} \in \mathrm{Sp}_{T(1)}$ to be certain finite Galois extension of $\mathbb{S}_{T(1)}$. These computations employ recent breakthroughs in the analysis of $T(h)$-localized algebraic K-theory [HW22, CMNN20, LMMT20, Lev22].

By considering certain other fibrations $X \to M$ for M stably framed and with π-finite fibers, we can get the following conclusion.

Theorem 16.1 (Burklund–Carmeli–Hahn–Levi–Schlank–Yanovski). *For $h > 1$ and $i \in \{0, \dots, p-2\}$ the homotopy group $\pi_{2i+1}(T(h))$ is infinite.*

Using similar formulas to those from subsection 8 (such as (8.3) and (8.4)), we get the following.

Theorem 16.2 (BCHLSY). *Let X be a finite CW-complex with $\widetilde{H}_*(X; \mathbb{F}_p) \neq 0$. There exists some constant $C_{X,p} > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{C_{X,p}} \sum_{i=N+1}^{N+C_{X,p}} \dim_{\mathbb{F}_p}((\pi_i^S X)/p) = \infty.$$

The basic maneuver here is to observe that a class $x \in \pi_m(T(h)) = \pi_m(X(h-1)[v_h^{-1}])$ must come from a class $\tilde{x} \in \pi_{m+j|v_h|}(X(h-1))$ for some j. From that point on, $v_h^l \tilde{x} \in \pi_{m+(j+l)|v_h|}(X(h-1))$ is a nonzero class for all $l \geq 0$. When $h = 2$, we can make a more careful analysis of the required j needed for τ_k, to get a more precise estimate.

Theorem 16.3 (BCHLSY). *Let X be a finite CW-complex with $\overline{\mathbb{S}}\langle X\rangle \otimes T(2) \neq 0$ (for example, $X = S^0$). There exists some constant $C_{X,p} > 0$ such that*

$$\frac{1}{C_{X,p}} \sum_{i=N+1}^{N+C_{X,p}} \dim_{\mathbb{F}_p}((\pi_i^S X)/p) \geq O(\log(N)).$$

Note that for $p = 2$ and $X = S^0$, this implies

$$\liminf_{k\to\infty} \left(\frac{1}{k\log(k)} \sum_{n=1}^{k} \log_2(|\pi^S_{(2),n}|) \right) > 0.$$

The previous best lower bounds were due to Oka in [Oka83], who showed

$$\max_{j\leq n}(\dim_{\mathbb{F}_p}(\pi_j^S/p)) \geq \log\log(n).$$

Another bound came from the p-torsion detected at height 1, which implies that the average in Theorem 16.3 is bounded below by a positive constant. In [BS22], Burklund used the Adams spectral sequence to give upper bounds. These give, for example, that there exists some $C > 0$ such that

$$\lim_{k\to\infty} \left(\frac{1}{e^{C(\log k)^3}} \sum_{n=1}^{k} \log_2(|\pi^S_{(2),n}|) \right) = 0.$$

As we can see the lower and upper bounds are still far from each other. There is still great mystery in the behavior of these fundamental objects.

References

[Ada66] J. F. Adams, *On the groups $J(X)$–IV*, Topology, 5 (1966), pp. 21–71, https://doi.org/10.1016/0040-9383(66)90004-8.

[AKN24] B. Antieau, A. Krause, and T. Nikolaus, *On the K-Theory of $\mathbb{Z}/p^n$*, preprint, arXiv:2405.04329, 2024.

[Alg58] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv., 32 (1958), pp. 180–214.

[AN21] B. Antieau and T. Nikolaus, *Cartier modules and cyclotomic spectra*, J. Amer. Math. Soc., 34 (2021), pp. 1–78, https://doi.org/10.1090/jams/951.

[AR02] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory*, Acta Math., 188 (2002), pp. 1–39, https://doi.org/10.1007/BF02392794.

[AR08] C. Ausoni and J. Rognes, *The chromatic red-shift in algebraic K-theory*, Enseign. Math., 54 (2008), pp. 13–15.

[BCK+14] B. Bhatt, C. Caraiani, K. S. Kedlaya, J. Weinstein, and P. Scholze, *Perfectoid spaces*, Math. Surveys Monogr., 242 (2014), https://bookstore.ams.org/surv-242/.

[BCM20] B. Bhatt, D. Clausen, and A. Mathew, *Remarks on $K(1)$-local K-theory*, Selecta Math., 26 (2020), 39, https://doi.org/10.1007/s00029-020-00566-6.

[BCSY24] T. Barthel, S. Carmeli, T. M. Schlank, and L. Yanovski, *The chromatic Fourier transform*, in Forum of Mathematics, Pi, Vol. 2, Cambridge University Press, 2024, e8.

[Beh12] M. Behrens, *The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited*, Adv. Math., 230 (2012), pp. 458–492, https://doi.org/10.1016/j.aim.2012.02.023.

[BGH22] A. Beaudry, P. G. Goerss, and H.-W. Henn, *Chromatic splitting for the $K(2)$-local sphere at $p = 2$*, Geometry Topol., 26 (2022), pp. 377–476, https://doi.org/10.2140/gt.2022.26.377.

[BHLS23] R. Burklund, J. Hahn, I. Levy, and T. M. Schlank, *k-Theoretic Counterexamples to Ravenel's Telescope Conjecture*, preprint, arXiv:2310.17459, 2023.

[BMCSY25] S. Ben-Moshe, S. Carmeli, T. Schlank, and L. Yanovski, *Descent and cyclotomic redshift for chromatically localized algebraic K-theory*, J. Amer. Math. Soc., 38 (2025), pp. 521–583, https://doi.org/10.1090/jams/1052.

[Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology, 18 (1979), pp. 257–281, https://doi.org/10.1016/0040-9383(79)90018-1.

[BR08] A. Baker and B. Richter, *Galois extensions of Lubin-Tate spectra*, Homology Homotopy Appl., 10 (2008), pp. 27–43.

[Bro62] E. H. Brown, *Cohomology theories*, Ann. of Math. (2), 75 (1962), pp. 467–484, https://doi.org/10.2307/1970209.

[BS13] B. Bhatt and P. Scholze, *The Pro-étale Topology for Schemes*, preprint, arXiv:1309.1198, 2013.

[BS22] R. Burklund and A. Senger, *How Big Are the Stable Homotopy Groups of Spheres?*, preprint, arXiv:2203.00670, 2022.

[BSSW24] T. Barthel, T. M. Schlank, N. Stapleton, and J. Weinstein, *On the Rationalization of the $K(n)$-local Sphere*, preprint, arXiv:2402.00960, 2024.

[BSY22] R. Burklund, T. M. Schlank, and A. Yuan, *The Chromatic Nullstellensatz*, preprint, arXiv:2207.09929, 2022.

[Bur22] R. Burklund, *Multiplicative Structures on Moore Spectra*, preprint, arXiv:2203.14787, 2022.

[CM21] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory*, Invent. Math., 225 (2021), pp. 1–96, https://doi.org/10.1007/s00222-021-01043-3.

[CMM21] D. Clausen, A. Mathew, and M. Morrow, *K-theory and topological cyclic homology of Henselian pairs*, J. Amer. Math. Soc., 34 (2021), pp. 411–473, https://doi.org/10.1090/jams/961.

[CMNN20] D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent and Vanishing in Chromatic Algebraic K-Theory via Group Actions*, preprint, arXiv:2011.08233, 2020.

[CSY21a] S. Carmeli, T. M. Schlank, and L. Yanovski, *Ambidexterity and height*, Adv. Math., 385 (2021), 107763, https://doi.org/10.1016/j.aim.2021.107763.

[CSY21b] S. Carmeli, T. M. Schlank, and L. Yanovski, *Chromatic Cyclotomic Extensions*, preprint, arXiv:2103.02471, 2021.

[CSY22] S. Carmeli, T. M. Schlank, and L. Yanovski, *Ambidexterity in chromatic homotopy theory*, Invent. Math., 228 (2022), pp. 1145–1254, https://doi.org/10.1007/s00222-022-01099-9.

[DH04] E. S. Devinatz and M. J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology, 43 (2004), pp. 1–47, https://doi.org/10.1016/S0040-9383(03)00029-6.

[DHS88] E. S. Devinatz, M. J. Hopkins, and J. H. Smith, *Nilpotence and stable homotopy theory*, I, Ann. of Math.(2), 128 (1988), pp. 207–241, https://doi.org/10.2307/1971440.

[Dri76] V. G. Drinfel'd, *Coverings of p-adic symmetric regions*, Funct. Anal. Appl., 10 (1976), pp. 107–115, https://doi.org/10.1007/BF01077936.

[ES45] S. Eilenberg and N. E. Steenrod, *Axiomatic approach to homology theory*, Proc. Natl. Acad. Sci. USA, 31 (1945), pp. 117–120, https://doi.org/10.1073/pnas.31.4.117.

[FGL08] L. Fargues, A. Genestier, and V. Lafforgue, *Théorie de messing. L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld: Démonstration du résultat principal*, in L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Progr. Math. 262, Birkhäuser, Basel, 2008, pp. 133–202.

[Fre38] H. Freudenthal, *Über die klassen der sphärenabbildungen* I. *Große dimensionen*, Compositio Math., 5 (1938), pp. 299–314.

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured Ring Spectra, 315 (2004), 22, https://doi.org/10.1017/CBO9780511529955.

[GHM14] P. G. Goerss, H.-W. Henn, and M. Mahowald, *The rational homotopy of the $K(2)$-local sphere and the chromatic splitting conjecture for the prime* 3 *and level* 2, Documenta Math., 19 (2014), pp. 1271–1290, https://doi.org/10.4171/dm/480.

[GS96] J. P. C. Greenlees and H. Sadofsky, *The Tate spectrum of v_n-periodic complex oriented theories*, Math. Z., 222 (1996), pp. 391–406, https://doi.org/10.1007/BF02621873.

[Hah16] J. Hahn, *On the Bousfield Classes of H-ring Spectra*, preprint, arXiv:1612.04386, 2016.

[HG94] M. J. Hopkins and B. H. Gross, *Equivariant vector bundles on the Lubin-Tate moduli space*, Contemporary Math., 158 (1994), pp. 23–23, https://doi.org/10.1090/conm/158/01453.

[Hil70] D. Hilbert, *Über die vollen invariantensysteme*, in Algebra Invariantentheorie Geometrie, Springer, 1970, pp. 287–344.

[HL13] M. Hopkins and J. Lurie, *Ambidexterity in $K(n)$-Local Stable Homotopy Theory*, preprint, 2013.

[Hop31] H. Hopf, *Über die abbildungen der dreidimensionalen sphäre auf die kugelfläche*, Math. Ann., 104 (1931), pp. 637–665, https://doi.org/10.1007/BF01457962.

[Hov95] M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, Contemp. Math., 181 (1995), pp. 225–225, https://doi.org/10.1090/conm/181/02036.

[HS96] M. Hovey and H. Sadofsky, *Tate cohomology lowers chromatic Bousfield classes*, Proc. Amer. Math. Soc., 124 (1996), pp. 3579–3585, https://doi.org/10.1090/S0002-9939-96-03495-8.

[HS98] M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory* II, Ann. of Math. (2), 148 (1998), pp. 1–49, https://doi.org/10.2307/120991.

[HW20] J. Hahn and D. Wilson, *Redshift and Multiplication for Truncated Brown-Peterson Spectra*, preprint, arXiv:2012.00864, 2020.

[HW22] J. Hahn and D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. of Math. (2), 196 (2022), pp. 1277–1351, https://doi.org/10.4007/annals.2022.196.3.6.

[IWX20] D. C. Isaksen, G. Wang, and Z. Xu, *Stable homotopy groups of spheres*, Proc. Natl. Acad. Sci. USA, 117 (2020), pp. 24757–24763, https://doi.org/10.1073/pnas.2012335117.

[Kuh04] N. J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math., 157 (2004), pp. 345–370, https://doi.org/10.1007/s00222-003-0354-z.

[Lev22] I. Levy, *The Algebraic K-Theory of the $K(1)$-local Sphere via TC*, preprint, arXiv:2209.05314, 2022.

[LMMT20] M. Land, A. Mathew, L. Meier, and G. Tamme, *Purity in Chromatically Localized Algebraic K-Theory*, preprint, arXiv:2001.10425, 2020.

[LWX24] W. Lin, G. Wang, and Z. Xu, *On the Last Kervaire Invariant Problem*, preprint, arXiv:2412.10879, 2024.

[Mah81] M. Mahowald, *bo-resolutions*, Pacific J. Math., 92 (1981), pp. 365–383, https://doi.org/10.2140/pjm.1981.92.365.

[Mat16a] A. Mathew, *The Galois group of a stable homotopy theory*, Adv. Math., 291 (2016), pp. 403–541, https://doi.org/10.1016/j.aim.2015.12.017.

[Mil81] H. R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra, 20 (1981), pp. 287–312, https://doi.org/10.1016/0022-4049(81)90064-5.

[Mor85] J. Morava, *Noetherian localisations of categories of cobordism comodules*, Ann. of Math. (2), 121 (1985), pp. 1–39, https://doi.org/10.2307/1971192.

[MR99] M. Mahowald and C. Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Amer. J. Math., 121 (1999), pp. 1153–1177, https://doi.org/10.1353/ajm.1999.0043.

[Nov67] S. P. Novikov, *The methods of algebraic topology from the point of view of cobordism theory*, Izv. Acad. Nauk SSSR Ser. Mat., 31 (1967), pp. 885–951; Math. USSR-Izv, 1 (1967), pp. 827–913.

[NS18] T. Nikolaus and P. Scholze, *On Topological Cyclic Homology*, https://arxiv.org/abs/1707.01799, 2018.

[Oka83] S. Oka, *Small ring spectra and p-rank of the stable homotopy of spheres*, Contemp. Math., 19 (1983), pp. 267–308, https://doi.org/10.1090/conm/019/711058.

[Qui75] D. Quillen, *Higher algebraic K-theory*, in Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974), Volume 1, Canadian Mathermatical Congress, 1975, pp. 171–176.

[Rav84] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math., 106 (1984), pp. 351–414, https://doi.org/10.2307/2374308.

[Rav92] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Princeton University Press, 1992.

[Rav03] D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, American Mathematical Society, 2003.

[Rog08] J. Rognes, *Galois extensions of structured ring spectra/stably dualizable groups: Stably dualizable groups,* Mem. Amer. Math. Soc., 192 (2008).

[Sch12] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci., 116 (2012), pp. 245–313, https://doi.org/10.1007/s10240-012-0042-x.

[Sch14] P. Scholze, *Perfectoid spaces and their applications*, in Proceedings of the International Congress of Mathematicians—Seoul 2014, Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 461–486.

[Sch17] P. Scholze, *Étale Cohomology of Diamonds*, preprint, arXiv:1709.07343, 2017.

[SW14] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Cambridge J. Math., 1 (2014), pp. 145–237.

[SY95] K. Shimomura and A. Yabe, *The homotopy groups* $\pi_*(L_2(S^0))$, Topology, 34 (1995), pp. 261–289, https://doi.org/10.1016/0040-9383(94)00032-G.

[Tho54] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv., 28 (1954), pp. 17–86, https://doi.org/10.1007/BF02566923.

[Tho82] R. W. Thomason, *The Lichtenbaum-Quillen conjecture for* $K/l_*[\beta^{-1}]$, in Current Trends in Algebraic Topology, CMS Conf. Proc. 2, American Mathematical Society, 1982, pp. 117–139.

[Wal84] F. Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy*, in Algebraic Topology (Aarhus, 1982), Lecture Notes in Math. 1051, Springer, Berlin, 1984, pp. 173–195.

[Yua24] A. Yuan, *Examples of chromatic redshift in algebraic* K*-theory*, J. Eur. Math. Soc. (JEMS), (2024), https://doi.org/10.4171/jems.