



The Connes–Moscovici Approach to Cyclic Cohomology for Compact Quantum Groups

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(Received: September 2000)

Abstract. Consider the Hopf algebra $(\mathcal{A}, \Phi)$ of regular functions on a compact quantum group. Let $(\mathcal{A}^o, \Delta)$ denote its maximal dual Hopf algebra. We show that the tensor product Hopf algebra $(\mathcal{H}_2, \Delta_2)$ of $(\mathcal{A}^o, \Delta)$ and its opposite Hopf algebra is endowed with a modular pair (δ, σ) in involution; a notion introduced by A. Connes and J. Moscovici, who associate canonically a cocyclic object to such Hopf algebras. Denote the Hopf cyclic cohomology thus obtained by $HC_{(\delta, \sigma)}^*(\mathcal{H}_2)$. Next we define an action of $(\mathcal{H}_2, \Delta_2)$ on $\mathcal{A}$, and show that the Haar state of $(\mathcal{A}, \Phi)$ is a δ -invariant σ -trace on $\mathcal{A}$ with respect to this action. This gives us a canonical map γ from $HC_{(\delta, \sigma)}^*(\mathcal{H}_2)$ to the ordinary cyclic cohomology of $\mathcal{A}$. We develop a method for computing $HC_{(\delta, \sigma)}^*(\mathcal{H}_2)$, including a Künneth exact sequence for the cyclic cohomology of general cocyclic objects. We recognize the Hopf Hochschild cohomology of any Hopf algebra with a modular pair in involution as a derived functor in the category of bicomodules over the Hopf algebra. Furthermore, we prove a duality theorem to the effect that $HC_{(\delta, \sigma)}^*(\mathcal{H}_2)$ can be computed from a finitely generated free resolution of $\mathcal{A}$ as an $\mathcal{A}$ -bimodule, provided a certain assumption on $(\mathcal{A}, \Phi)$ holds. Unfortunately, we are unable to check this assumption in the case of $\mathcal{A} = SU_q(2)$. Assuming that it holds, we demonstrate how an explicit resolution of $\mathcal{A}$ given by T. Masuda, Y. Nakagami and J. Watanabe can be used to compute both $HC_{(\delta, \sigma)}^*(\mathcal{H}_2)$ and the map γ for $SU_q(2)$. To our surprise $\gamma = 0$ in this case, both on the level of Hochschild cohomology, cyclic cohomology and periodic cyclic cohomology. Finally, we dispense with this assumption in the case of $SU_q(2)$ by replacing $(\mathcal{A}^o, \Delta)$ with the more tractable Hopf subalgebra $(U_q(\mathfrak{sl}_2), \Delta)$. Noticing that the γ -map is well-defined also in this case, we combine our computations with those of M. Crainic to prove that the restricted γ -map is again zero, but this time without any extra assumption on $SU_q(2)$.

Mathematics Subject Classifications (2000): Primary 17B37; secondary 19D55, 16W30, 20G42.

Key words: Hopf algebras, cocyclic objects, Künneth exact sequence.

1. Introduction

Cyclic cohomology plays a fundamental role in noncommutative geometry [3], and has by now acquired a central place in modern cohomology theories [22, 25]. Recall that a functor from the cyclic category Λ to any Abelian

category is called a cocyclic object, and that such an object provides a full-bodied cyclic cohomology theory, including the long exact *IBS*-sequence of Connes.

In their study of the transverse index theorem for foliations, A. Connes and H. Moscovici [4] took advantage of a specific Hopf algebra encountered, to solve a long-standing internal problem of noncommutative geometry. This Hopf algebra was identified with one of S. Majid's bicrossproducts [4, 17], and furthermore, was shown to act on an algebra with a faithful trace. The action and the trace was then used to pullback the ordinary cocyclic object of cochains on the algebra. Consequently, one obtained a cocyclic object associated to the Hopf algebra, defined by formulas that used both the product and the coproduct. It was observed in [4], and emphasized by M. Crainic [7], that the same formulas could be used to give an intrinsic definition of a cocyclic object for any Hopf algebra equipped with a certain character δ , and such Hopf algebras do indeed exist [5]. For commutative or cocommutative Hopf algebras the coinverse is always involutive, in which case the character is just the counit. The Hopf $*$ -algebra $(\mathcal{A}, \Phi)$ of regular functions on a compact quantum group provides another example. The member $\delta = f_{-1}$ of the family $\{f_z\}_{z \in \mathbb{C}}$ governing the modular properties of the Haar state of the quantum group [26] then does the trick.

Later Connes and Moscovici [5] introduced the notion of a Hopf algebra with a modular pair (δ, σ) in involution. Such a pair consists of a character δ and a group-like element σ satisfying certain conditions. They showed that this wider class of Hopf algebras also yields cocyclic objects. It appears that they got this by boldly imagining an action of the Hopf algebra on some algebra, this time coming not with a trace, but instead with a certain KMS-weight, and then studied the pullback of the usual cocyclic object of cochains on the algebra, induced by the action and the KMS-weight.

The so-called γ -map that provided this pullback is pivotal in these investigations, because Hopf algebras (or any of their hybrids) play the role of symmetry in noncommutative geometry. The ingredients necessary to get the γ -map are: (1) a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution, (2) an action $L: \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ of $(\mathcal{H}, \Delta)$ on the algebra $\mathcal{A}$, and (3) a δ -invariant σ -trace τ . The δ -invariance property of the weight τ of $\mathcal{A}$ is an 'integration by parts' formula, whereas the σ -trace property is a 'KMS-condition'. At the time when the first half of this manuscript was written (autumn of 1999), ironically, any construction of the data (1)–(3), beyond the case of τ being a trace, was totally absent. In this paper we shall provide such data.

Now the maximal dual Hopf algebra $(\mathcal{A}^o, \Delta)$ of $(\mathcal{A}, \Phi)$ is a Hopf algebra with a modular pair (I, f_1) in involution. Furthermore, the category of Hopf algebras with modular pairs in involution is invariant under formation of opposite Hopf algebras, and under tensoring Hopf algebras. Thus, if $F: \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o \rightarrow \mathcal{A}_{\text{op}}^o \otimes \mathcal{A}^o$ denotes the flip map, then

$$(\mathcal{H}_2, \Delta_2) = (\mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o, (\iota \otimes F \otimes \iota)(\Delta \otimes \Delta))$$

is a Hopf algebra with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. We show that with respect to the canonical action of $(\mathcal{H}_2, \Delta_2)$ on $\mathcal{A}$, the Haar state of $(\mathcal{A}, \Phi)$ is indeed a δ -invariant σ -trace. From this we get a γ -map from the cyclic cohomology groups associated with this tensor product Hopf algebra to the ordinary cyclic cohomology groups of cochains on $\mathcal{A}$. This is in accordance with the general philosophy of Connes and Moscovici, stating that we are dealing with a theory of characteristic classes for actions of nonunimodular Hopf algebras compatible with the modular theory of weights.

In passing it is perhaps worthwhile mentioning that the Hopf algebra $(\mathcal{H}_2, \Delta_2)$ is one of Majid's bicrossproducts known as the Mirror product $M(\mathcal{A}^o)$ [18]. Both $M(\mathcal{A}^o)$ and the Hopf algebra Connes and Moscovici found in their study of the transverse index theorem for foliations belong to the same bicrossproduct family, given by matched pairs of Hopf algebras [17]. As kindly pointed out to us by the referee, the cyclic cohomology of a bicrossproduct can be reduced to the cyclic cohomology of the associated double crossproduct. Moreover, the double crossproduct associated to $M(\mathcal{A}^o)$ is the Drinfeld quantum double $D(\mathcal{A}^o)$ [17]. So cyclic cohomology of $(\mathcal{H}_2, \Delta_2)$ is related to the cohomology of $D(\mathcal{A}^o)$, but we will have no opportunity to study this relationship any further in this paper.

The price to pay for extending to the bicrossproduct Hopf algebra, is that the γ -map is no longer injective. Also, the cyclic cocycles of its image are not *a priori* left-invariant, and finally, the connection to differential calculi of quantum groups is for the time being not clear. After all, the study of differential calculi of quantum groups has taught us that left-covariant calculi are associated with left-invariant cyclic cocycles that are twisted [10].

Having introduced the γ -map in Theorem 2, we take on the challenge posed to us by the referee: to compute it for quantum $SU(2)$, and in doing so, we develop means for computing the Hopf cyclic cohomology of $(\mathcal{H}_2, \Delta_2)$ in the general case.

An important observation for computing the cohomology of $(\mathcal{H}_2, \Delta_2)$ is that we are dealing with the tensor product of two cocyclic objects; one coming from the Hopf algebra $(\mathcal{A}^o, \Delta)$ and the other coming from the Hopf algebra $(\mathcal{A}_{\text{op}}^o, \Delta)$, both with (I, f_1) as a modular pair in involution. A powerful theory dealing with homology for tensor products of cyclic modules has been long established. It features the Eilenberg–Zilber theorem for Hochschild homology, and the less familiar Künneth exact sequence for cyclic homology [15]. The second of these results follows from a Leibniz rule for the Connes B -operator on the level of Hochschild homology, which can be proved by introducing a cyclic shuffle map in addition to the usual shuffle map for Hochschild homology.

We extend these results to general cocyclic objects (rather than just the Hochschild cocomplex of an algebra), in that we succeed to define a cyclic shuffle map also in this case, a result we have not seen elsewhere and which therefore needs a presentation. Once this is done, we can adjust the arguments in [15] and prove a Leibniz rule for B and an Eilenberg–Zilber theorem for cyclic cohomology, which then yields a Künneth exact sequence for cyclic cohomology. For the

sake of convenience we also write down the familiar Alexander–Whitney map and the shuffle map implementing the isomorphism in the usual Künneth theorem for Hochschild cohomology. Applying these rather general results to the cocyclic object associated to $(\mathcal{H}_2, \Delta_2)$ yields Corollary 1, which in effect reduces the computation of the cohomology of $(\mathcal{H}_2, \Delta_2)$ to that of $(\mathcal{A}^o, \Delta)$.

In the next section we invoke the cobar complex and recognize the Hopf Hochschild cohomology of any Hopf algebra with a modular pair in involution as a derived functor in the category of bicomodules over the Hopf algebra. This allows the Hochschild cohomology of $(\mathcal{A}^o, \Delta)$ with modular pair (I, f_1) in involution to be computed from any injective resolution of $\mathcal{A}^o$ as an $(\mathcal{A}^o, \Delta)$ -bicomodule. In the following section we show how such a resolution can be obtained from a finitely generated free resolution of $\mathcal{A}$ as an $\mathcal{A}$ -bimodule, provided a certain assumption on $(\mathcal{A}, \Phi)$ holds, namely that $\mathcal{A}^o \otimes \mathcal{A}^o$ is injective as an $\mathcal{A}$ -bimodule. However, we have not been able to check that this assumption holds in the case of $SU_q(2)$, so the full machine cannot be applied straightforwardly to the maximal dual Hopf algebra $(\mathcal{A}_q^o, \Delta)$ associated to $SU_q(2)$.

However, the results presented so far are valid for any Hopf subalgebra $(\mathcal{K}, \Delta)$ of $(\mathcal{A}^o, \Delta)$ for which $f_1 \in \mathcal{K}$, and thus in particular for the Hopf algebra $(U_q(\mathfrak{sl}_2), \Delta)$ in the case of $SU_q(2)$. Denote the associated tensor product Hopf algebra of the latter with its opposite Hopf algebra by $(\mathcal{K}_{2q}, \Delta_2)$. Recently, Crainic [7] computed the Hochschild cohomology and periodic cyclic cohomology of $(U_q(\mathfrak{sl}_2), \Delta)$ by different means. This allows us to compute the Hochschild cohomology, the cyclic cohomology and the periodic cyclic cohomology associated to $(\mathcal{K}_{2q}, \Delta_2)$. A careful study of the B -operator for the complexes associated to $(\mathcal{A}_q^o, \Delta)$ yields representatives in the original complexes for the associated cohomology classes, which includes representatives expressed by elements living in $U_q(\mathfrak{sl}_2)$ of the generators of Crainic, together with an element $H \in \mathcal{A}_q^o$. It should be noted that H plays the role of the Cartan element for quantum $SU(2)$, and does not belong to $U_q(\mathfrak{sl}_2)$. Combining these results with the computations in [20] of the ordinary Hochschild and cyclic cohomology of $\mathcal{A}_q$, gives the remarkable result that the γ -map associated to $(U_q(\mathfrak{sl}_2), \Delta)$ for $SU_q(2)$ is zero.

In the final Section 9 we consider a situation where the previously mentioned assumption on $SU_q(2)$ is presumed to hold. We use an explicit free resolution constructed by Masuda *et al.* [20] to compute the Hochschild cohomology associated to $(\mathcal{A}_q^o, \Delta)$ by invoking our duality theorem. Using again detailed knowledge of the B -operator and the IBS-sequence we compute the cyclic cohomology groups as well, and we show that the representatives (involving H and those in $U_q(\mathfrak{sl}_2)$) found above are indeed representatives for all generators of the cohomologies associated to $(\mathcal{A}_q^o, \Delta)$. Combining these results with the computations in [20] for the ordinary Hochschild and cyclic cohomology of $\mathcal{A}_q$, we deduce that also the γ -map associated to $(\mathcal{A}_q^o, \Delta)$ is zero.

Theorem 2 was discovered independently by Connes and Moscovici, who informed us that they had carried out their construction in the summer of 1999. Their

version of this result is contained in Theorem 14 of their paper Cyclic Cohomology and Hopf Symmetry [6]. They call the tensor product Hopf algebra of our paper the *modular square* associated to the compact quantum group. On account of the fully nonunimodular situation present for locally compact quantum groups, their modular square has a modular pair in involution where both the character and the group-like element are nontrivial.

2. Hopf Algebras with Modular Pairs in Involution

In this section we define and study the category of Hopf algebras with modular pairs in involution, and make some elementary observations not stated elsewhere in the literature. We recall the example (Example 1) by Connes and Moscovici concerning Hopf algebras associated to compact quantum groups, and end the section with Example 2, which will be important in the remainder of the paper.

Throughout this section let $(\mathcal{H}, \Delta)$ be a Hopf algebra with comultiplication Δ , coinverse (antipode) S , counit ε , unit I and multiplication $m: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$. If U and V are vector spaces, we denote the flip from $U \otimes V$ to $V \otimes U$ by F . Our basic reference for Hopf algebras is [1].

A *modular pair* for $(\mathcal{H}, \Delta)$ consists of a pair (δ, σ) , where δ is a unital, multiplicative, linear functional on $\mathcal{H}$ and $\sigma \in \mathcal{H}$ satisfies $\Delta(\sigma) = \sigma \otimes \sigma$ and $\delta(\sigma) = 1$. Thus (ε, I) is an example of a modular pair for $(\mathcal{H}, \Delta)$.

Now $\hat{\delta}: \mathcal{H} \rightarrow \mathcal{H}$ given by $\hat{\delta} = (\delta \otimes \iota)\Delta$ is a unital, multiplicative, linear map with inverse $\hat{\delta}^{-1} = (\delta S \otimes \iota)\Delta$. We associate to the modular pair (δ, σ) a *twisted coinverse* $\tilde{S}: \mathcal{H} \rightarrow \mathcal{H}$ defined by $\tilde{S} = (\delta \otimes S)\Delta = S \circ \hat{\delta}$, see [4, 7]. Note that if S is invertible, then $\tilde{S}$ is invertible with inverse $\tilde{S}^{-1} = (S^{-1} \otimes \delta)\Delta$. The requirement $\delta(\sigma) = 1$ implies that $\tilde{S}(\sigma^n) = \sigma^{-n}$ and $\tilde{S}^{-1}(\sigma^n) = \sigma^{-n}$ for any integer n .

We say that the modular pair (δ, σ) is *in involution* if $\tilde{S}^2(x) = \sigma x \sigma^{-1}$ for all $x \in \mathcal{H}$. Since $\text{Ad}: \mathcal{H} \rightarrow \mathcal{H}$ given by $\text{Ad}(x) = \sigma x \sigma^{-1}$ for all $x \in \mathcal{H}$ is an automorphism, we see that a modular pair (δ, σ) is in involution if and only if $S \circ \hat{\delta} \circ S = \text{Ad} \circ \hat{\delta}^{-1}$, and this can only happen if S is bijective. Thus, a modular pair (δ, σ) is in involution if and only if $\tilde{S}^{-2}(x) = \sigma^{-1} x \sigma$ for all $x \in \mathcal{H}$. Clearly, the modular pair (ε, I) is in involution if and only if the coinverse S is involutive.

Suppose $(\mathcal{H}_i, \Delta_i)$ are two Hopf algebras with modular pairs (δ_i, σ_i) in involution and let $\phi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a morphism of Hopf algebras. We say that ϕ is a *morphism* in the category of Hopf algebras with modular pairs in involution if $\delta_2 \phi = \delta_1$ and $\sigma_2 = \phi(\sigma_1)$. Consider the tensor product Hopf algebra $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \times \Delta_2)$ of $(\mathcal{H}_1, \Delta_1)$ and $(\mathcal{H}_2, \Delta_2)$ with comultiplication $\Delta_1 \times \Delta_2 = (\iota \otimes F \otimes \iota)(\Delta_1 \otimes \Delta_2)$. Obviously, the pair $(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)$ is a modular pair in involution for $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \times \Delta_2)$.

Suppose $(\mathcal{H}, \Delta)$ is a Hopf algebra with an invertible coinverse S . Denote by $\mathcal{H}_{\text{op}}$ the algebra $\mathcal{H}$ with the opposite multiplication mF , and denote by Δ_{op} the

opposite comultiplication $F\Delta$. Let $\sigma \in \mathcal{H}$ and let δ be a linear functional on $\mathcal{H}$. Then the following four statements are equivalent:

- (1) (δ, σ) is a modular pair in involution for $(\mathcal{H}, \Delta)$;
- (2) $(\delta S, \sigma)$ is a modular pair in involution for $(\mathcal{H}_{\text{op}}, \Delta)$;
- (3) (δ, σ^{-1}) is a modular pair in involution for $(\mathcal{H}, \Delta_{\text{op}})$;
- (4) $(\delta S, \sigma^{-1})$ is a modular pair in involution for $(\mathcal{H}_{\text{op}}, \Delta_{\text{op}})$.

The next result shows that the category of Hopf algebras with modular pairs in involution is also closed under formation of duals.

PROPOSITION 1. *Suppose $\langle \cdot, \cdot \rangle: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{C}$ is a nondegenerate dual pairing [12] between two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with coinverses S_i . Consider two elements $\delta \in \mathcal{H}_2$ and $\sigma \in \mathcal{H}_1$. Then $(\langle \cdot, \delta \rangle, \sigma)$ is a modular pair in involution for $(\mathcal{H}_1, \Delta_1)$ if and only if $(\langle S_1(\sigma), \cdot \rangle, S_2(\delta))$ is a modular pair in involution for $(\mathcal{H}_2, \Delta_2)$.*

We denote by $\text{Rep } \mathcal{B}$ the category of finite-dimensional $*$ -representations of the unital $*$ -algebra $\mathcal{B}$. We say that an element of $\mathcal{B}$ is *positive* if it can be written as a finite sum of elements of the form b^*b , where $b \in \mathcal{B}$. We conclude this general discussion with a uniqueness result. As usual Tr will denote the operator trace.

PROPOSITION 2 (Roberts and Tuset [21]). *Suppose $(\mathcal{H}, \Delta)$ is a Hopf $*$ -algebra with coinverse S and that $\text{Rep } \mathcal{H}$ separates the elements of $\mathcal{H}$. Then there is at most one positive invertible element $f \in \mathcal{H}$ such that $S^2(x) = fxf^{-1}$ for all $x \in \mathcal{H}$ and $\text{Tr } \pi(f) = \text{Tr } \pi(f^{-1})$ for all representations $\pi \in \text{Rep } \mathcal{H}$. Moreover, the element f is necessarily group-like.*

EXAMPLE 1. Let $(\mathcal{A}, \Phi)$ be the Hopf $*$ -algebra associated to a compact quantum group in the sense of Woronowicz [26, 27]. Let $\{f_z\}_{z \in \mathbb{C}}$ denote the family of functionals on $\mathcal{A}$ describing the modular properties of the Haar state, and let I denote the unit of the algebra $\mathcal{A}$. Then (f_{-1}, I) is a modular pair in involution for $(\mathcal{A}, \Phi)$. The twisted coinverse is given by $\tilde{\kappa} = (f_{-1} \otimes \kappa)\Phi$, where κ is the coinverse of $(\mathcal{A}, \Phi)$.

Denote by $(\mathcal{A}^o, \Delta)$ the maximal Hopf $*$ -algebra dual to $(\mathcal{A}, \Phi)$ [1]. It has product $xy = (x \otimes y)\Phi$ and coproduct $\Delta(x)(a \otimes b) = x(ab)$ for $x, y \in \mathcal{A}^o$ and $a, b \in \mathcal{A}$. Thus $S^2(x) = f_1 x f_{-1}$, where S is the coinverse of $(\mathcal{A}^o, \Delta)$ given by $S(x) = x\kappa$. Note that $f_1^* = f_1$, where $x^*(a) = \overline{x(\kappa(a)^*)}$ is the $*$ -operation for $(\mathcal{A}^o, \Delta)$. Suppose that $(\mathcal{K}, \Delta)$ is a Hopf $*$ -subalgebra of $(\mathcal{A}^o, \Delta)$ containing f_1 . Regard the unit of $\mathcal{A}$ as the linear functional on $\mathcal{K}$ given by $I(x) = x(I)$, so I is just the counit of $(\mathcal{A}^o, \Delta)$. Then (I, f_1) is a modular pair in involution for $(\mathcal{K}, \Delta)$. Here the coinverse S of $(\mathcal{K}, \Delta)$ serves as the twisted coinverse.

Let $\langle \cdot, \cdot \rangle: \mathcal{K} \times \mathcal{A} \rightarrow \mathbb{C}$ denote the dual pairing $\langle x, a \rangle = x(a)$ between $(\mathcal{K}, \Delta)$ and $(\mathcal{A}, \Phi)$. Then $(f_{-1}, I) = (\langle S(f_1), \cdot \rangle, \kappa(I))$ is a modular pair in involution for $(\mathcal{A}, \Phi)$ dual (in the sense of Proposition 1) to the modular pair $(I, f_1) = (\langle \cdot, I \rangle, f_1)$ in involution for $(\mathcal{K}, \Delta)$.

EXAMPLE 2. Here we combine the results in Example 1 with the general discussion preceding it, to construct Hopf $*$ -algebras with modular pairs in involution which will play an important role in the Connes–Moscovici approach to cyclic cohomology for Hopf algebras. Let $\mathcal{H}_2 = \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o$ and $\Delta_2 = \Delta \times \Delta$. Then $(\mathcal{H}_2, \Delta_2)$ is a Hopf algebra with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Note that its coinverse is $S \otimes S^{-1}$, its counit is $I \otimes I$ and its unit is $\varepsilon \otimes \varepsilon$, where ε is the counit of $(\mathcal{A}, \Phi)$.

Of course, any Hopf $*$ -subalgebra $(\mathcal{K}_2, \Delta_2)$ of $(\mathcal{H}_2, \Delta_2)$ that contains σ is also a Hopf algebra with the modular pair $(\delta_{\mathcal{K}_2}, \sigma)$ in involution, where $\delta_{\mathcal{K}_2}$ is the restriction of δ to $\mathcal{K}_2$. Particular examples are $\mathcal{K}_2 = \mathcal{A}^o \otimes \mathbb{C}[f_1]$ and $\mathcal{K}_2 = \mathbb{C}[f_1] \otimes \mathcal{A}_{\text{op}}^o$. In fact, for $\mathcal{K}_2 = \mathcal{A}^o \otimes \mathbb{C}[f_1]$ one may also use $(I \otimes I, f_1 \otimes f_n)$, where n is any integer, as a modular pair in involution. When the quantum group is a group, then $f_1 = \varepsilon$ and $\mathbb{C}[f_1] = \mathbb{C}$. Hence, $\mathcal{K}_2 = \mathcal{A}^o \otimes \mathbb{C}[f_1] = \mathcal{A}^o$ and $\Delta_2 = \Delta$. If it is an affine Lie group, one may restrict further to $U(\mathfrak{g}) \subset \mathcal{A}^o$, where $U(\mathfrak{g})$ is the universal enveloping algebra of the associated Lie algebra $\mathfrak{g}$.

3. Cyclic Cohomology of Hopf Algebras

In this section we include some background material on cocyclic objects and introducing two important examples of such objects; one leading to the ordinary cyclic cohomology $HC^*(\mathcal{A})$ of an algebra $\mathcal{A}$, and the other leading to the Hopf cyclic cohomology $HC_{(\delta, \sigma)}^*(\mathcal{H})$ of a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. Also we include a brief reminder of the Chern character of Connes.

Let $\mathbb{N}_0$ denote the set of nonnegative integers. The *cyclic category* Λ is defined [3, 15, 19] to be the category with objects $[n]$ for all $n \in \mathbb{N}_0$, and morphisms universally generated by the elements

$$\delta_i^n : [n-1] \rightarrow [n], \quad \sigma_i^n : [n+1] \rightarrow [n], \quad \tau_n : [n] \rightarrow [n],$$

where $i \in \{0, 1, \dots, n\}$, satisfying the following relations:

$$\delta_j^n \delta_i^{n-1} = \delta_i^n \delta_{j-1}^{n-1} \quad \text{for } i < j, \quad \sigma_j^n \sigma_i^{n+1} = \sigma_i^n \sigma_{j+1}^{n+1} \quad \text{for } i \leq j,$$

$$\sigma_j^n \delta_i^{n+1} = \delta_i^n \sigma_{j-1}^{n-1} \quad \text{for } i < j, \quad \sigma_j^n \delta_i^{n+1} = 1_n \quad \text{for } i = j \text{ or } i = j+1,$$

$$\sigma_j^n \delta_i^{n+1} = \delta_{i-1}^n \sigma_j^{n-1} \quad \text{for } i > j+1,$$

and

$$\tau_n \delta_i^n = \delta_{i-1}^n \tau_{n-1} \quad \text{for } i \geq 1 \text{ and } \tau_n \delta_0^n = \delta_n^n,$$

$$\tau_n \sigma_i^n = \sigma_{i-1}^n \tau_{n+1} \quad \text{for } i \geq 1 \text{ and } \tau_n \sigma_0^n = \sigma_n^n \tau_{n+1}^2,$$

and $\tau_n^{n+1} = 1_n$.

Here 1_n denotes the identity morphism from $[n]$ to $[n]$. Of course, in the relations above, n has to be taken greater than zero for some of the expressions to make sense, since δ_i^0 and σ_i^{-1} and τ_{-1} are obviously not defined. The morphisms δ_i^n and σ_i^n are referred to as the *coface maps* and *codegeneracy maps*, respectively, and generate a subcategory Δ of Λ called the *cosimplicial category*, or the *ordinal number category*. The morphism τ_n is referred to as the *cyclicity map* and corresponds to a generator of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$.

A *cocyclic object* of any category $\mathcal{C}$ is a covariant functor from Λ to $\mathcal{C}$. By the universal property of the cyclic category Λ , a cocyclic object of a category $\mathcal{C}$ is equivalently described by a quadruple (C^n, d_i^n, s_i^n, t_n) , where the C^n are objects of $\mathcal{C}$, and the d_i^n , s_i^n and t_n are morphisms of $\mathcal{C}$ satisfying the same relations as above when substituted for δ_i^n , σ_i^n and τ_n . The restriction of a cocyclic object to the subcategory $\Delta \subset \Lambda$ is a covariant functor from Δ to $\mathcal{C}$, i.e., a cosimplicial object of $\mathcal{C}$.

Consider a cocyclic object (C^n, d_i^n, s_i^n, t_n) of an Abelian category $\mathcal{C}$. Define morphisms $b_n, b'_n : C^{n-1} \rightarrow C^n$ and $\lambda_n, N^n : C^n \rightarrow C^n$ by

$$b_n = \sum_{i=0}^n (-1)^i d_i^n \quad \text{and} \quad b'_n = \sum_{i=0}^{n-1} (-1)^i d_i^n,$$

$$\lambda_n = (-1)^n t_n \quad \text{and} \quad N^n = \sum_{i=0}^n \lambda_n^i.$$

It is easy to check (by using the defining relations for the morphisms δ_i^n in the category Λ) that (C, b) and (C, b') are complexes, i.e., $b_{n+1}b_n = b'_{n+1}b'_n = 0$ for any $n \in \mathbb{N}_0$. The cohomology of (C, b) is by definition the *Hochschild cohomology* $HH^*(C)$ of the cosimplicial object (C^n, d_i^n, s_i^n) . The complex (C, b') is, however, acyclic with contracting homotopy $s_{n+1} = \sigma_n^n \tau_{n+1}$. Again, only using the relations between the τ 's and δ 's in Λ , one may show that $(1_n - \lambda_n)b_n = b'_n(1_{n-1} - \lambda_{n-1})$ for any $n \in \mathbb{N}$. Therefore we get a subcomplex (C_λ, b) of (C, b) with objects $C_\lambda^n = \ker(1_n - \lambda_n)$. The cohomology of this subcomplex is denoted by $H_\lambda^*(C)$.

Further, it can be shown (by using the relations for the δ 's and τ 's in Λ) that $bN = Nb'$ and $(1 - \lambda)N = N(1 - \lambda) = 0$. From this one obtains a *bicomplex*:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^2 & \xrightarrow{1-\lambda} & C^2 & \xrightarrow{N} & C^2 & \xrightarrow{1-\lambda} & C^2 \xrightarrow{N} \dots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^1 & \xrightarrow{1-\lambda} & C^1 & \xrightarrow{N} & C^1 & \xrightarrow{1-\lambda} & C^1 \xrightarrow{N} \dots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^0 & \xrightarrow{1-\lambda} & C^0 & \xrightarrow{N} & C^0 & \xrightarrow{1-\lambda} & C^0 \xrightarrow{N} \dots \end{array}$$

We denote this bicomplex by (C^{mn}, b, b') , where $C^{mn} = C^n$ for all $m, n \in \mathbb{N}_0$. The cohomology of the total complex of (C^{mn}, b, b') is, by definition, the *cyclic cohomology* $HC^*(C)$ of the cocyclic object (C^n, d_i^n, s_i^n, t_n) . From the bicomplex construction the long exact *IBS*-sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & HC^n & \xrightarrow{I} & HH^n & \xrightarrow{B} & HC^{n-1} \xrightarrow{S} HC^{n+1} \xrightarrow{I} HH^{n+1} \\ & & & & & \xrightarrow{B} & HC^n \longrightarrow \dots \end{array}$$

of Connes follows relatively easily. Here we have used the abbreviations HC^* for $HC^*(C)$ and HH^* for $HH^*(C)$. Recall that the B -operator is given by $B = Ns(1 - \lambda)$ or, more explicitly,

$$B_n = N^n \sigma_n^n \tau_{n+1}(1_{n+1} - \lambda_{n+1}) : C^{n+1} \rightarrow C^n$$

for $n \in \mathbb{N}_0$. We denote the *mixed complex* of Kassel [8] associated to a cocyclic object (C^n, d_i^n, s_i^n, t_n) by (C, b, B) [3, 25].

Suppose that $\mathcal{A}$ is a unital algebra. We recall the basic example of a cocyclic object, namely the one which gives rise to ordinary Hochschild cohomology $HH^*(\mathcal{A})$ and ordinary cyclic cohomology $HC^*(\mathcal{A})$ of $\mathcal{A}$. Let $n \in \mathbb{N}_0$, and denote by C^n the vector space of n -cochains, that is, the set of multilinear maps from the vector space $\mathcal{A}^{n+1}$ to the field of complex numbers. Let $n \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, n\}$. Define linear maps

$$d_i^n : C^{n-1} \rightarrow C^n, \quad s_i^n : C^{n+1} \rightarrow C^n, \quad t_n : C^n \rightarrow C^n$$

as follows:

$$\begin{aligned} (d_i^n \varphi)(a^0, a^1, \dots, a^n) &= \varphi(a^0, \dots, a^i a^{i+1}, \dots, a^n) \quad \text{for } i \leq n-1, \\ (d_n^n \varphi)(a^0, a^1, \dots, a^n) &= \varphi(a^n a^0, a^1, \dots, a^{n-1}), \\ (s_i^n \varphi)(a^0, a^1, \dots, a^n) &= \varphi(a^0, \dots, a^i, I, a^{i+1}, \dots, a^n), \\ (t_n \varphi)(a^0, a^1, \dots, a^n) &= \varphi(a^n, a^0, a^1, \dots, a^{n-1}), \end{aligned}$$

where I is the unit of $\mathcal{A}$, $a^0, \dots, a^n \in \mathcal{A}$ and φ is a cochain. Note that C^{-1} and d_i^0 are not defined, so the range of n must be restricted accordingly for some formulas to make sense. It is straightforward to check that $X = (C^n, d_i^n, s_i^n, t_n)$ is a cocyclic object of the Abelian category of vector spaces.

Suppose now that $(\mathcal{H}, \Delta)$ is a Hopf algebra with a modular pair (δ, σ) , and retain the notation from Section 2. Let $n \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, n\}$. Define linear maps

$$\delta_i^n : \mathcal{H}^{\otimes(n-1)} \rightarrow \mathcal{H}^{\otimes n}, \quad \sigma_i^n : \mathcal{H}^{\otimes(n+1)} \rightarrow \mathcal{H}^{\otimes n}, \quad \tau_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$$

as follows:

$$\begin{aligned}
\delta_0^n(x^1 \otimes \cdots \otimes x^{n-1}) &= I \otimes x^1 \otimes \cdots \otimes x^{n-1}, \\
\delta_i^n(x^1 \otimes \cdots \otimes x^{n-1}) &= x^1 \otimes \cdots \otimes \Delta x^i \otimes \cdots \otimes x^{n-1} \\
&\quad \text{for } i \in \{1, \dots, n-1\}, \\
\delta_n^n(x^1 \otimes \cdots \otimes x^{n-1}) &= x^1 \otimes \cdots \otimes x^{n-1} \otimes \sigma, \\
\sigma_i^n(x^1 \otimes \cdots \otimes x^{n+1}) &= x^1 \otimes \cdots \otimes \varepsilon(x^{i+1}) \otimes \cdots \otimes x^{n+1}, \\
\tau_n(x^1 \otimes \cdots \otimes x^n) &= (\Delta^{n-1} \tilde{S}(x^1)) \cdot (x^2 \otimes \cdots \otimes x^n \otimes \sigma),
\end{aligned}$$

where $x^1, \dots, x^{n+1} \in \mathcal{H}$. Here $\mathcal{H}^{\otimes(-1)}$ and δ_i^0 are not defined. By $\mathcal{H}^{\otimes 0}$ we mean $\mathbb{C}$. We define $\delta_0^1, \delta_1^1: \mathbb{C} \rightarrow \mathcal{H}$, $\tau_0: \mathbb{C} \rightarrow \mathbb{C}$ and $\sigma_0^0: \mathcal{H} \rightarrow \mathbb{C}$ by setting $\delta_0^1(1) = I$, $\delta_1^1(1) = \sigma$, $\tau_0(1) = 1$ and $\sigma_0^0 = \varepsilon$. The map Δ^n is defined inductively by setting $\Delta^n = (\iota \otimes \Delta) \Delta^{n-1}$ for $n \geq 2$. If we compose the homomorphism $\Delta^{n-1}: \mathcal{H} \rightarrow \mathcal{H}^{\otimes n}$ with the action of $\mathcal{H}^{\otimes n}$ on $\mathcal{H}^{\otimes n}$ given by left multiplication, we get the diagonal action of $\mathcal{H}$ on $\mathcal{H}^{\otimes n}$. Thus, in the definition of τ_n , we have used the diagonal action of $\tilde{S}(x^1)$ on $x^2 \otimes \cdots \otimes x^n \otimes \sigma \in \mathcal{H}^{\otimes n}$. One can check directly [5] that $Y = (\mathcal{H}^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ is a cocyclic object of the Abelian category of vector spaces if and only if the modular pair (δ, σ) is in involution. Hereafter we will always assume that this is the case.

Following Connes and Moscovici we denote the Hochschild cohomology and the cyclic cohomology of the cocyclic object Y by $HH_{(\delta, \sigma)}^*(\mathcal{H})$ and $HC_{(\delta, \sigma)}^*(\mathcal{H})$, respectively, and we call these cohomology groups the (Hopf) Hochschild cohomology and the (Hopf) cyclic cohomology, respectively, of the Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution.

It is known that $H_\lambda^*(\mathcal{A}) \cong HC^*(\mathcal{A})$ and $H_\lambda^*(\mathcal{H}) \cong HC_{(\delta, \sigma)}^*(\mathcal{H})$, whenever the ground field contains the rationals.

We end this preparatory section by recalling the definition of the Chern character [3] in noncommutative geometry, which manifests a profound link between ordinary cyclic cohomology and K -theory. Let φ be an n -dimensional cyclic cocycle, so $\varphi \in C^n$, $\lambda_n(\varphi) = \varphi$ and $b_{n+1}(\varphi) = 0$, and denote by $[\varphi]$ the corresponding cohomology class in $HC^n(\mathcal{A}) = H_\lambda^n(\mathcal{A})$. Let $n, N \in \mathbb{N}_0$ and consider the standard trace Tr on $M_N(\mathbb{C})$. Denote by $\varphi \otimes \text{Tr}$ the $(n+1)$ -multilinear functional on $M_N(\mathcal{A}) = \mathcal{A} \otimes M_N(\mathbb{C})$ given by

$$(\varphi \otimes \text{Tr})(a^0 \otimes x^0, a^1 \otimes x^1, \dots, a^n \otimes x^n) = \varphi(a^0, a^1, \dots, a^n) \text{Tr}(x^0 x^1 \cdots x^n)$$

for all $a^0, \dots, a^n \in \mathcal{A}$ and $x^0, \dots, x^n \in M_N(\mathbb{C})$.

Consider first the case when $n = 2m$ is even. It can be shown that the scalar $(\varphi \otimes \text{Tr})(E, E, \dots, E)$ is invariant under homotopy for idempotents $E^2 = E \in M_N(\mathcal{A})$. Thus, φ provides in this way a numerical invariant for the K -group $K_0(\mathcal{A})$. Moreover, the scalar is independent of what representative φ in $[\varphi]$ is employed. Thus we obtain a bilinear pairing

$$\langle \cdot, \cdot \rangle : HC^{\text{even}}(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{C}$$

given by

$$\langle [\varphi], [E] \rangle = \frac{1}{m!} (\varphi \otimes \text{Tr})(E, E, \dots, E)$$

for all $[\varphi] \in HC^{2m}(\mathcal{A})$ and projections $E \in M_N(\mathcal{A})$. Here $HC^{\text{even}}(\mathcal{A})$ denotes the even cyclic cohomology of $\mathcal{A}$.

Consider next the case when n is odd. For u an invertible element of the unital algebra $M_N(\mathcal{A})$ and φ a cyclic n -cocycle of $\mathcal{A}$, the formula

$$\begin{aligned} \langle [\varphi], [u] \rangle &= \frac{1}{\sqrt{2i}} 2^{-n} \Gamma\left(\frac{n}{2} + 1\right)^{-1} (\varphi \otimes \text{Tr}) \times \\ &\quad \times (u^{-1} - 1, u - 1, u^{-1} - 1, \dots, u - 1) \end{aligned}$$

defines similarly a bilinear pairing

$$\langle \cdot, \cdot \rangle : HC^{\text{odd}}(\mathcal{A}) \times K_1(\mathcal{A}) \rightarrow \mathbb{C}.$$

Here, $HC^{\text{odd}}(\mathcal{A})$ denotes the odd cyclic cohomology of $\mathcal{A}$, Γ is the gamma function and $[u]$ denotes the K -theory class of u in $K_1(\mathcal{A})$.

4. Actions of Hopf Algebras

Here we study actions of Hopf algebras with modular pairs in involution on the algebra of regular functions of a compact quantum group.

Let $(\mathcal{H}, \Delta)$ be a Hopf algebra and let $\mathcal{A}$ be an arbitrary algebra. By an *action* of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$ we mean a linear map $L: \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ such that $L(y \otimes L(x \otimes a)) = L(yx \otimes a)$ and $L(I \otimes a) = a$ and

$$L(x \otimes ab) = \sum_i L(x_i \otimes a) L(y_i \otimes b)$$

for any $a, b \in \mathcal{A}$ and $x, y \in \mathcal{H}$ with $\Delta(x) = \sum_i x_i \otimes y_i$.

Now suppose $L: \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is an action of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$. A linear functional τ on $\mathcal{A}$ is said to be a σ -trace under L if $\tau(ab) = \tau(bL(\sigma \otimes a))$ holds for all $a, b \in \mathcal{A}$. A linear functional τ on $\mathcal{A}$ is said to be δ -invariant under L if $\tau(L(x \otimes a)b) = \tau(aL(\tilde{S}(x) \otimes b))$ holds for all $a, b \in \mathcal{A}$ and $x \in \mathcal{H}$.

In the rest of this section we assume that $(\mathcal{H}, \Delta)$ is a Hopf $*$ -subalgebra of the maximal Hopf $*$ -algebra $(\mathcal{A}^o, \Delta)$ dual to the Hopf $*$ -algebra $(\mathcal{A}, \Phi)$ associated to a compact quantum group with Haar state h . Define $x * a$ and $a * x$ in $\mathcal{A}$ for $x \in \mathcal{A}^o$ and $a \in \mathcal{A}$ by $x * a = (\iota \otimes x)\Phi(a)$ and $a * x = (x \otimes \iota)\Phi(a)$.

Consider the linear map $L: \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ given by $L(x \otimes a) = x * a$ for any $a \in \mathcal{A}$ and $x \in \mathcal{H}$. This is an action of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$. To see this, for $x \in \mathcal{H}$ write

$\Delta(x) = \sum_k x_k \otimes y_k$ and for $a_1, a_2 \in \mathcal{A}$ write $\Phi(a_k) = \sum_i a_i^k \otimes b_i^k$. Then

$$\begin{aligned} x * (a_1 a_2) &= (\iota \otimes x) \Phi(a_1) \Phi(a_2) = \sum_{ij} a_i^1 a_j^2 x(b_i^1 b_j^2) \\ &= \sum_{ij} a_i^1 a_j^2 \Delta(x)(b_i^1 \otimes b_j^2) = \sum_{ijk} a_i^1 a_j^2 x_k(b_i^1) y_k(b_j^2) \\ &= \sum_k \left(\sum_i a_i^1 x_k(b_i^1) \right) \left(\sum_j a_j^2 y_k(b_j^2) \right) = \sum_k (x_k * a_1)(y_k * a_2). \end{aligned}$$

Similarly, one checks that the linear map from $\mathcal{H}_{\text{op}} \otimes \mathcal{A}$ to $\mathcal{A}$ given by $x \otimes a \mapsto a * x$ for any $a \in \mathcal{A}$ and $x \in \mathcal{H}_{\text{op}}$ is an action of $(\mathcal{H}_{\text{op}}, \Delta)$ on $\mathcal{A}$.

Assume that $\phi: \mathcal{H} \rightarrow \mathcal{H}$ is a linear, multiplicative mapping such that $(\phi \otimes \phi) \Delta = \Delta \phi$, whereas $\psi: \mathcal{H} \rightarrow \mathcal{H}$ is a linear, anti-multiplicative map such that $F(\psi \otimes \psi) \Delta = \Delta \psi$. Then it follows that we have actions of

- (1) $(\mathcal{H}, \Delta)$ on $\mathcal{A}$ given by $x \otimes a \mapsto \phi(x) * a$;
- (2) $(\mathcal{H}, \Delta_{\text{op}})$ on $\mathcal{A}$ given by $x \otimes a \mapsto a * \psi(x)$;
- (3) $(\mathcal{H}_{\text{op}}, \Delta)$ on $\mathcal{A}$ given by $x \otimes a \mapsto a * \phi(x)$;
- (4) $(\mathcal{H}_{\text{op}}, \Delta_{\text{op}})$ on $\mathcal{A}$ given by $x \otimes a \mapsto \psi(x) * a$;

where $a \in \mathcal{A}$ and $x \in \mathcal{H}$. Note that we can, for instance, use $\phi = S^{2n} = f_n \cdot f_{-n}$ and $\psi = S^{2n+1} = f_n S(\cdot) f_{-n}$ for any $n \in \mathbb{N}$.

Also, since $x * (a * y) = (x * a) * y$ for any $a \in \mathcal{A}$ and $x, y \in \mathcal{H}$, the actions in items (1) and (4) above can be combined with those in items (2) and (3) to produce actions on $\mathcal{A}$ of the tensor product Hopf algebra of their corresponding Hopf algebras. Thus, for instance, for integers n and m , the formula $T_{n,m}(x \otimes y \otimes a) = S^{2n+1}(x) * a * S^{2m}(y)$, where $x, y \in \mathcal{H}$ and $a \in \mathcal{A}$, defines an action $T_{n,m}$ of the tensor product Hopf algebra $(\mathcal{H}_{\text{op}} \otimes \mathcal{H}_{\text{op}}, \Delta_{\text{op}} \times \Delta)$ on $\mathcal{A}$. For integers n and m , define an automorphism $\theta_{n,m}$ of $\mathcal{A}$ by setting $\theta_{n,m}(a) = f_{-n} * a * f_m$ for any $a \in \mathcal{A}$, and note that $\theta_{n,m} T_{n,m} = T_{0,0}(\iota \otimes \theta_{n,m})$, which means that the actions $T_{n,m}$ are all equivalent to the action $T_{0,0}$. Since $\theta_{n,n} = \kappa^{2n}$, where κ is the coinverse of $(\mathcal{A}, \Phi)$, we see that the intertwiner $\theta_{n,n}$ is a bialgebra morphism. We shall soon need the following not completely standard result.

PROPOSITION 3. *Let notation be as above, with $x, y \in \mathcal{H}$ and $a, b \in \mathcal{A}$. Then*

- (1) $\kappa((h \otimes \iota)((a \otimes I) \Phi(b))) = (h \otimes \iota)(\Phi(a)(b \otimes I))$,
- (2) $\kappa((\iota \otimes h)(\Phi(a)(I \otimes b))) = (\iota \otimes h)((I \otimes a) \Phi(b))$,
- (3) $h((y * a)b) = h(a(S(y) * b))$,
- (4) $h((a * y)b) = h(a(b * S^{-1}(y)))$,
- (5) $h((x * a * y)b) = h(a(S(x) * b * S^{-1}(y)))$.

Proof. Statements (1) and (2) are known as the strong left- and right-invariant properties of the Haar state (for proofs, see [27]). To see that statement (3) holds,

observe that

$$\begin{aligned}
 h((y * a)b) &= h((\iota \otimes y)\Phi(a)b) = y((h \otimes \iota)(\Phi(a)(b \otimes I))) \\
 &= y(\kappa((h \otimes \iota)((a \otimes I)\Phi(b))) \\
 &= S(y)((h \otimes \iota)((a \otimes I)\Phi(b))) \\
 &= h(a(\iota \otimes S(y))(\Phi(b))) = h(a(S(y) * b)).
 \end{aligned}$$

Similarly, to prove statement (4), observe that

$$\begin{aligned}
 h((a * y)b) &= h((y \otimes \iota)\Phi(a)b) = y((\iota \otimes h)(\Phi(a)(I \otimes b))) \\
 &= y(\kappa^{-1}((\iota \otimes h)((I \otimes a)\Phi(b))) \\
 &= S^{-1}(y)((\iota \otimes h)((I \otimes a)\Phi(b))) \\
 &= h(a(S^{-1}(y) \otimes \iota)(\Phi(b))) = h(a(b * S^{-1}(y))).
 \end{aligned}$$

Finally, we get statement (5) from statements (3) and (4) by calculating

$$\begin{aligned}
 h((x * a * y)b) &= h((x * (a * y))b) = h((a * y)(S(x) * b)) \\
 &= h(a((S(x) * b) * S^{-1}(y))) \\
 &= h(a(S(x) * b * S^{-1}(y))).
 \end{aligned}$$

□

Since the coinverse S of $(\mathcal{H}, \Delta)$ is an isomorphism between $(\mathcal{H}, \Delta)$ and $(\mathcal{H}_{\text{op}}, \Delta_{\text{op}})$ and between $(\mathcal{H}, \Delta_{\text{op}})$ and $(\mathcal{H}_{\text{op}}, \Delta)$ as Hopf algebras with modular pairs in involution, and since the corresponding actions of these Hopf algebras on $\mathcal{A}$ are the same under these identifications, we may restrict attention to two cases, namely to the action of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$ given by $x \otimes a \mapsto x * a$ for $a \in \mathcal{A}$ and $x \in \mathcal{H}$, and to the action of $(\mathcal{H}_{\text{op}}, \Delta)$ on $\mathcal{A}$ given by $x \otimes a \mapsto a * x$ for $a \in \mathcal{A}$ and $x \in \mathcal{H}_{\text{op}}$. They yield the tensor product Hopf algebra $(\mathcal{H}_2, \Delta_2)$ from Example 2, and the action $L: \mathcal{H}_2 \otimes \mathcal{A} \rightarrow \mathcal{A}$ on $\mathcal{A}$ given by $L(x \otimes y \otimes a) = x * a * y$ for $a \in \mathcal{A}$ and $x, y \in \mathcal{A}^o$. Recall that $(\mathcal{H}_2, \Delta_2)$, with $\mathcal{H}_2 = \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o$ and $\Delta_2 = \Delta \times \Delta$, has a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution, and coinverse $S \otimes S^{-1}$, where S is the coinverse of $(\mathcal{A}^o, \Delta)$. The twisted coinverse equals $S \otimes S^{-1}$ since $I \otimes I$ is the counit of $(\mathcal{H}_2, \Delta_2)$.

THEOREM 1. *Let notation be as in the preceding paragraph. Then the Haar state h of $(\mathcal{A}, \Phi)$ is a δ -invariant σ -trace for the action L on $\mathcal{A}$ of the tensor product Hopf algebra $(\mathcal{H}_2, \Delta_2)$ with modular pair (δ, σ) in involution.*

Proof. To see that h is a σ -trace, notice that $L(\sigma \otimes a) = f_1 * a * f_1$, and then use the result $h(ab) = h(bf_1 * a * f_1)$ to conclude that $h(ab) = h(bL(\sigma \otimes a))$ for all $a, b \in \mathcal{A}$. To see that h is δ -invariant, let $x \otimes y \in \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o$ and $a, b \in \mathcal{A}$. By Proposition 3, we then get

$$\begin{aligned}
 h(L(x \otimes y \otimes a)b) &= h(x * a * yb) = h(aS(x) * b * S^{-1}(y)) \\
 &= h(aL(S(x) \otimes S^{-1}(y) \otimes b)).
 \end{aligned}$$

□

When the Haar state h is faithful on the enveloping C^* -algebra A of $\mathcal{A}$ (it is always faithful on $\mathcal{A}$), it is a KMS-state [12]. The associated modular group of A in the sense of the Tomita–Takesaki theory [2], when restricted to $\mathcal{A}$, coincides with the one-parameter group $(\sigma_z)_{z \in \mathbb{C}}$ of automorphism of $\mathcal{A}$ defined by setting $\sigma_z(a) = f_{iz} * a * f_{iz}$ for $a \in \mathcal{A}$, $i = \sqrt{-1}$, and $z \in \mathbb{C}$.

Suppose we have two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with modular pairs (δ_i, σ_i) in involution for $i = 1, 2$. Suppose L_i is an action of $(\mathcal{H}_i, \Delta_i)$ on $\mathcal{A}$. Assume that $\phi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is an isomorphism of the Hopf algebras such that $(\delta_1, \sigma_1) = (\delta_2 \phi, \phi^{-1}(\sigma_2))$ and $L_1 = L_2(\phi \otimes \iota)$. Then τ is a δ_2 -invariant σ_2 -trace under the action L_2 if and only if it is a δ_1 -invariant σ_1 -trace under the action L_1 . Thus, for instance, the Haar state h is a δ -invariant σ -trace for the Hopf algebra $(\mathcal{A}_{\text{op}}^o \otimes \mathcal{A}_{\text{op}}^o, \Delta_{\text{op}} \times \Delta)$ with modular pair $(\delta, \sigma) = (I \otimes I, f_1^{-1} \otimes f_1)$ in involution and its action on $\mathcal{A}$ is given by $x \otimes y \otimes a \mapsto S(x) * a * y$ for $x, y \in \mathcal{A}^o$ and $a \in \mathcal{A}$.

Let τ be a δ -invariant σ -trace for an action L on $\mathcal{A}$ of a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. Suppose that θ is an automorphism of $\mathcal{A}$ such that $\tau\theta = \tau$. Then τ is a δ -invariant σ -trace under the action $\theta^{-1}L(\iota \otimes \theta)$ on $\mathcal{A}$ of $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. Therefore, for the actions $T_{n,m}$ defined above, the Haar state h is a δ -invariant σ -trace because $h\theta = h$ for $\theta = \theta_{n,m}$. Note also, by uniqueness of the Haar state h , that $h\theta = h$ whenever θ is a unital bialgebra automorphism.

5. The γ -Map

Suppose τ is a δ -invariant σ -trace for an action $L: \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ on an algebra $\mathcal{A}$ of a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution.

In the following discussion we use the notation of Section 3. Consider the ordinary cocyclic object $X = (C^n, d_i^n, s_i^n, t_n)$ associated to the algebra $\mathcal{A}$, and the cocyclic object $Y = (\mathcal{H}^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ associated to the Hopf algebra $(\mathcal{H}, \Delta)$ with the modular pair (δ, σ) in involution. For $n \in \mathbb{N}$, define a linear map $\gamma_n: \mathcal{H}^{\otimes n} \rightarrow C^n$ by

$$x^1 \otimes \cdots \otimes x^n \mapsto \gamma_n(x^1 \otimes \cdots \otimes x^n),$$

where

$$\gamma_n(x^1 \otimes \cdots \otimes x^n)(a^0, \dots, a^n) = \tau(a^0 L(x^1 \otimes a^1) \cdots L(x^n \otimes a^n))$$

for $a^0, \dots, a^n \in \mathcal{A}$ and $x^1, \dots, x^n \in \mathcal{H}$. Define also $\gamma_0: \mathcal{H}^{\otimes 0} = \mathbb{C} \rightarrow C^0 = \mathcal{A}'$ by $\gamma_0(1) = \tau$ for $1 \in \mathbb{C}$. (Here $\mathcal{A}'$ is the space of linear functionals on $\mathcal{A}$.) It can then be shown [5] that

$$\gamma_n \delta_i^n = d_i^n \gamma_{n-1}, \quad \gamma_n \sigma_i^n = s_i^n \gamma_{n+1}, \quad \gamma_n \tau_n = t_n \gamma_n$$

for any $n \in \mathbb{N}_0$ and $i \in \{0, \dots, n\}$. Hence, γ_n can be lifted to homomorphisms

$$\gamma_n: HH_{(\delta, \sigma)}^n(\mathcal{H}) \rightarrow HH^n(\mathcal{A}),$$

and

$$\gamma_n : HC_{(\delta, \sigma)}^n(\mathcal{H}) \rightarrow HC^n(\mathcal{A}).$$

It follows directly from the definition of γ that if τ is faithful (in the sense that $\tau(ab) = 0$ for all $b \in \mathcal{A}$ implies $a = 0$), then γ_n is injective on the cochain level if and only if for any $x_i^j \in \mathcal{H}$ the condition

$$\sum_i L(x_i^1 \otimes a^1) \cdots L(x_i^n \otimes a^n) = 0$$

for all $a^1, \dots, a^n \in \mathcal{A}$, implies

$$\sum_i x_i^1 \otimes \cdots \otimes x_i^n = 0.$$

In general the kernels $\ker \gamma_n$ form a cocyclic object. Since L is an action, it follows that $\ker \gamma = \bigoplus_{n \in \mathbb{N}_0} \ker \gamma_n$ is a two-sided ideal of $\bigoplus_{n \in \mathbb{N}_0} \mathcal{H}^{\otimes n}$, where the latter is endowed with the natural algebra structure induced from $\mathcal{H}$.

THEOREM 2. *Consider the cocyclic object $(\mathcal{H}_2^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ associated to the Hopf algebra $(\mathcal{H}_2, \Delta_2)$ with a modular pair (δ, σ) in involution from Theorem 1. Let h denote the Haar state of $(\mathcal{A}, \Phi)$. The formulas $\gamma_0(1) = h$ and*

$$\begin{aligned} & \gamma_n((x^1 \otimes y^1) \otimes \cdots \otimes (x^n \otimes y^n))(a^0, \dots, a^n) \\ &= h(a^0(x^1 * a^1 * y^1) \cdots (x^n * a^n * y^n)) \end{aligned}$$

for

$$a^0, \dots, a^n \in \mathcal{A} \quad \text{and} \quad x^1, y^1, \dots, x^n, y^n \in \mathcal{A}^o$$

and $n \in \mathbb{N}$, define linear maps $\gamma_n: \mathcal{H}_2^{\otimes n} \rightarrow C^n$ such that

$$\gamma_n \delta_i^n = d_i^n \gamma_{n-1}, \quad \gamma_n \sigma_i^n = s_i^n \gamma_{n+1}, \quad \gamma_n \tau_n = t_n \gamma_n$$

for any $n \in \mathbb{N}_0$ and $i \in \{0, \dots, n\}$. Hence the maps γ_n induce natural maps

$$\gamma_n : HH_{(\delta, \sigma)}^n(\mathcal{H}_2) \rightarrow HH^n(\mathcal{A}),$$

and

$$\gamma_n : HC_{(\delta, \sigma)}^n(\mathcal{H}_2) \rightarrow HC^n(\mathcal{A})$$

for all $n \in \mathbb{N}_0$, and similarly for periodic cyclic cohomology.

In this theorem, instead of $\mathcal{H}_2 = \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o$ we could have taken $\mathcal{H}_2 = \mathcal{A}^o \otimes \mathbb{C}[f_1]$, or $\mathcal{H}_2 = \mathbb{C}[f_1] \otimes \mathcal{A}_{\text{op}}^o$, or any Hopf subalgebra $(\mathcal{K}_2, \Delta_2)$ of $(\mathcal{H}_2, \Delta_2)$ such that $\sigma \in \mathcal{K}_2$. Let $\gamma'_n: \mathcal{K}_2^{\otimes n} \rightarrow C^n$ for $n \in \mathbb{N}_0$ denote the corresponding map of

cocyclic objects. The inclusion $j: \mathcal{K}_2 \rightarrow \mathcal{H}_2$ induces an inclusion j of associated cocyclic objects such that $\gamma j = \gamma'$, and thus also a map

$$j: HH_{(\delta, \sigma)}^*(\mathcal{K}_2) \rightarrow HH_{(\delta, \sigma)}^*(\mathcal{H}_2)$$

on the level of Hochschild cohomology such that $\gamma j = \gamma'$. The same is true on the level of cyclic cohomology and periodic cyclic cohomology.

The map γ from Theorem 2 is in general not injective. To see this, consider the case when the compact quantum group $(\mathcal{A}, \Phi)$ equals $(C(G), \Phi)$, where G is a nontrivial finite Abelian group. Then for $s, t \in G$, $s \neq t$, we have for the action L of $(\mathcal{H}_2, \Delta_2)$ on $\mathcal{A} \subset C(G)$ that

$$\begin{aligned} L((\delta_s \otimes \delta_{s^{-1}} - \delta_t \otimes \delta_{t^{-1}}) \otimes f)(u) \\ = (\delta_s * f * \delta_{s^{-1}})(u) - (\delta_t * f * \delta_{t^{-1}})(u) \\ = f(s^{-1}us) - f(t^{-1}ut) = f(u) - f(u) = 0 \end{aligned}$$

for any $f \in \mathcal{A}$ and $u \in G$. Thus, $L((\delta_s \otimes \delta_{s^{-1}} - \delta_t \otimes \delta_{t^{-1}}) \otimes f) = 0$ for all $f \in C(G)$. However, $\delta_s \otimes \delta_{s^{-1}} - \delta_t \otimes \delta_{t^{-1}} \neq 0$, so γ cannot be injective.

In the general case consider the vector space

$$V = \{x \in \mathcal{H} \mid L(x \otimes \mathcal{A}) = 0\} \subset \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o.$$

Clearly, it is a two-sided ideal of $\mathcal{H}_2$ which does not contain the unit. Actually, it is easy to see that

$$V = \left\{ \sum_i x_i \otimes y_i \in \mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o \mid \sum_i y_i \mathcal{A}' x_i = 0 \right\}.$$

6. Tensor Products in Cohomology

Consider two cocyclic objects $(C^n, \delta_i^n, \sigma_i^n, \tau_n)$ and $(C'^n, \delta_i^n, \sigma_i^n, \tau_n)$, where C^n and C'^n are objects in an Abelian category with an appropriately biexact tensor product. (For example, the Abelian category of modules over a fixed field.) The *product* of these two cocyclic objects is the cocyclic object $((C \times C')^n, \delta_i^n, \sigma_i^n, \tau_n)$ with $(C \times C')^n = C^n \otimes C'^n$ and

$$\delta_i^n = \delta_i^n \otimes \delta_i^n, \quad \sigma_i^n = \sigma_i^n \otimes \sigma_i^n, \quad \tau_n = \tau_n \otimes \tau_n.$$

Recall that the tensor product $(C \otimes C', b)$ of the associated complexes (C, b) and (C', b) has coboundary operator

$$b_n = \sum_{i+j=n} (b_i \otimes 1_j + (-1)^i 1_i \otimes b_j).$$

Let $n \in \mathbb{N}_0$ and $i \in \{1, \dots, n\}$. Define $(\bar{\delta})^i: C^{n-i} \rightarrow C^n$ and $(\delta_0)^i: C'^{n-i} \rightarrow C'^n$ by

$$(\bar{\delta})^i = \delta_n^n \delta_{n-1}^{n-1} \dots \delta_{n-i+1}^{n-i+1}, \quad (\delta_0)^i = \delta_0^n \delta_0^{n-1} \dots \delta_0^{n-i+1},$$

and let $(\bar{\delta})^0 = (\delta_0)^0 = 1_n$.

The *Alexander–Whitney map*

$$AW_n : \bigoplus_{i+j=n} C^i \otimes C'^j \rightarrow (C \times C')^n$$

is the natural map of complexes given by

$$AW_n = \bigoplus_{i+j=n} AW_{i,j},$$

where

$$AW_{i,j} : C^i \otimes C'^j \rightarrow (C \times C')^n$$

is defined as

$$AW_{i,j} = (\bar{\delta})^{n-i} \otimes (\delta_0)^{n-j}.$$

The *shuffle map*

$$sh_n : (C \times C')^n \rightarrow \bigoplus_{i+j=n} C^i \otimes C'^j$$

is the natural map of complexes given by

$$sh_n = \bigoplus_{i+j=n} sh_{i,j},$$

where

$$sh_{i,j} : (C \times C')^n \rightarrow C^i \otimes C'^j$$

is defined as

$$sh_{i,j} = \sum_{(\mu, \nu)} \text{sign}(\mu, \nu) \sigma_{\nu_1}^i \cdots \sigma_{\nu_j}^{n-1} \otimes \sigma_{\mu_1}^j \cdots \sigma_{\mu_i}^{n-1}.$$

Here (μ, ν) runs over all (i, j) -shuffles, where by an (i, j) -shuffle we mean a permutation σ on the $i + j$ elements $\{1, \dots, i + j\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+j).$$

We write $\mu_1 = \sigma(1), \dots, \mu_i = \sigma(i)$, and $\nu_1 = \sigma(i+1), \dots, \nu_j = \sigma(i+j)$.

The Eilenberg–Zilber theorem says that the chain maps AW and sh are mutually chain inverse chain equivalences, hence induce inverse isomorphisms on the level of Hochschild cohomology (see theorem 1.6.12 in [15]). Of course this theorem is valid without the presence of the cyclicity maps.

For the convenience of the reader we recall the definition of a cyclic shuffle, from section 4.3.2 of [15]. Let $i, j, n \in \mathbb{N}_0$ with $n = i + j$. Consider the permutation σ on the n elements $\{1, \dots, n\}$ obtained by first performing a cyclic permutation p

times on $\{1, \dots, i\}$ and a cyclic permutation q times on $\{i+1, \dots, i+j\}$ and thereafter applying an (i, j) -shuffle (μ, ν) to the combined result. Such a permutation σ is then said to be an (i, j) -cyclic shuffle if 1 appears before $i+1$ in the resulting sequence. We write

$$\sigma = (\mu, \nu)(1, \dots, i)^p(i+1, \dots, i+j)^q$$

to denote such a cyclic shuffle (where it is assumed that the condition $\sigma(1) \leq \sigma(i+1)$ holds!).

Now define the *cyclic shuffle map*

$$\text{sh}'_n : (C \times C')^n \rightarrow \bigoplus_{i+j=n} C^{i-1} \otimes C'^{j-1}$$

by $\text{sh}'_n = \bigoplus_{i+j=n} \text{sh}'_{i,j}$, where

$$\text{sh}'_{i,j} : (C \times C')^n = C^{i+j} \otimes C'^{i+j} \rightarrow C^{i-1} \otimes C'^{j-1}$$

is defined as

$$\text{sh}'_{i,j} = \sum_{\sigma} \text{sign}(\sigma) \sigma_{i-p-1}^{i-1} \tau_i^{p+1} \sigma_{v_1}^i \cdots \sigma_{v_j}^{n-1} \otimes \sigma_{j-q-1}^{j-1} \tau_j^{q+1} \sigma_{\mu_1}^j \cdots \sigma_{\mu_i}^{n-1}.$$

Here the summation runs over all (i, j) -cyclic shuffles σ of the type

$$\sigma = (\mu, \nu)(1, \dots, i)^p(i+1, \dots, i+j)^q.$$

We are unaware of any previous definition of a cyclic shuffle map in this generality. It is usually defined (see paragraph 4.3.2 in [15]) in the context of the ordinary cyclic complex of an algebra by using a map $\perp$, the contracting homotopy $s_{n+1} = (-1)^{n+1} t_{n+1} s_n^n$ and the shuffle map sh for cyclic modules. The trick we use here is to bypass the map $\perp$, as it is not clear how (or if it is at all possible) to define it in the general case, and we instead define the map sh' directly.

Having the map sh' for cocyclic objects at hand, we will now state several immediate consequences, including a Künneth exact sequence of cyclic cohomology for cocyclic objects. Recall that the B -operator is given by

$$B_n = N^n \sigma_n^n \tau_{n+1} (1_{n+1} - \lambda_{n+1}).$$

When acting on the normalized (sub-)complex associated to a cosimplicial object, any expression with a degeneracy map in the rightmost position vanishes [25]. Thus, the expression for B_n simplifies to $B_n = N^n \sigma_n^n \tau_{n+1}$, so

$$\begin{aligned} B_n &= \sigma_n^n \tau_{n+1} + (-1)^n \sigma_{n-1}^n \tau_{n+1}^2 + \cdots + (-1)^{ni} \sigma_{n-i}^n \tau_{n+1}^{i+1} + \cdots \\ &\quad + (-1)^n \sigma_0^n \tau_{n+1}^{n+1}. \end{aligned}$$

We can now formulate the following important lemma, where we omit indices. The proof requires only a careful inspection of the proof of proposition 4.3.7 in [15], which as written really only applies to the ordinary cyclic complex of an algebra.

LEMMA 1. *When acting in the normalized setting we have:*

- (1) $[b, \text{sh}] = 0$;
- (2) $[B, \text{sh}] + [b, \text{sh}'] = 0$;
- (3) $[B, \text{sh}'] = 0$.

More explicitly, the formula $[B, \text{sh}] + [b, \text{sh}'] = 0$ reads

$$\begin{aligned} \text{sh } B - ((B \otimes 1) + (-1)^{| \cdot |} (1 \otimes B)) \text{sh} \\ = -\text{sh}' b + ((b \otimes 1) + (-1)^{| \cdot |} (1 \otimes b)) \text{sh}'. \end{aligned}$$

An immediate consequence of this formula is that the composite IB -operator $IB_n: HH^{n+1} \rightarrow HH^n$ satisfies a Leibniz rule on the level of Hochschild cohomology.

PROPOSITION 4. *Consider the Hochschild cohomology and the IB -operators associated to the mixed complexes (C, b, B) , (C', b, B) and $(C \times C', b, B)$. Then we have*

$$IB_n = \sum_{i+j=n+1} IB_{i-1} \otimes 1_j + (-1)^i 1_i \otimes IB_{j-1}$$

on the level of Hochschild cohomology.

Following Loday we formulate the Eilenberg–Zilber theorem for cyclic cohomology in this generality. Again we leave out the proof, which given the above definition of the cyclic shuffle map sh' is a direct adaptation of the one for theorem 4.3.8 in [15].

THEOREM 3. *Consider two cocyclic objects $(C^n, \delta_i^n, \sigma_i^n, \tau_n)$ and $(C'^n, \delta_i^n, \sigma_i^n, \tau_n)$, where C^n and C'^n are objects in an Abelian category with a biexact tensor product. Form the mixed complexes $(C \otimes C', b \otimes 1 + 1 \otimes b, B \otimes 1 + 1 \otimes B)$ and $(C \times C', b, B)$. Then there is a canonical isomorphism*

$$\text{Sh} : HC^*(C \times C') \rightarrow HC^*(C \otimes C')$$

induced by the shuffle map sh and the cyclic shuffle map sh' . It commutes with the morphisms B, I and S in Connes long exact sequence.

Proceeding as in [15], we then state the *Künneth exact sequence* of cyclic cohomology. For a proof, consult theorem 4.3.11 in [15] and carry out the necessary trivial modifications of its proof.

Let k be a field, and consider the Abelian category of k -modules, with tensor products formed over k .

THEOREM 4. *Let $(C^n, \delta_i^n, \sigma_i^n, \tau_n)$ and $(C'^n, \delta_i^n, \sigma_i^n, \tau_n)$ be two cocyclic k -modules. Then there is a canonical long exact sequence*

$$\begin{aligned} \dots \longrightarrow HC^{n-1}(C \times C') \xrightarrow{\partial} \bigoplus_{p+q=n-2} HC^p(C) \otimes HC^q(C') \\ \xrightarrow{S \otimes 1 - 1 \otimes S} \bigoplus_{r+s=n} HC^r(C) \otimes HC^s(C') \xrightarrow{i} HC^n(C \times C') \longrightarrow \dots \end{aligned}$$

We now descend to a more concrete setting and formulate the Künneth theorem relevant for the Connes–Moscovici approach to cyclic cohomology for Hopf algebras.

THEOREM 5. *Consider two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with modular pairs (δ_i, σ_i) in involution. Form the tensor product Hopf algebra $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \otimes \Delta_2)$ with modular pair $(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)$ in involution. Then*

$$HH_{(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)}^*(\mathcal{H}_1 \otimes \mathcal{H}_2) \cong HH_{(\delta_1, \sigma_1)}^*(\mathcal{H}_1) \otimes HH_{(\delta_2, \sigma_2)}^*(\mathcal{H}_2),$$

and, moreover, the isomorphism is implemented by the shuffle map sh (going from left to right) with the Alexander–Whitney map AW as its inverse.

Proof. This follows from the Eilenberg–Zilber theorem (see theorem 1.6.12 in [15]) for the two cosimplicial objects $(\mathcal{H}_1^{\otimes n}, \delta_i^n, \sigma_i^n)$ and $(\mathcal{H}_2^{\otimes n}, \delta_i^n, \sigma_i^n)$, together with the obvious identification $(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes n} \cong \mathcal{H}_1^{\otimes n} \otimes \mathcal{H}_2^{\otimes n}$, which we hereafter will suppress. $\square$

The Künneth exact sequence takes on the following form in the context of Hopf cyclic cohomology. To prove it look at the previous proof and apply Theorem 4.

THEOREM 6. *Consider two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with modular pairs (δ_i, σ_i) in involution. Form the tensor product Hopf algebra $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \otimes \Delta_2)$ with modular pair $(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)$ in involution. Then there is a canonical long exact sequence*

$$\begin{aligned} \cdots \longrightarrow HC_{(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)}^{n-1}(\mathcal{H}_1 \otimes \mathcal{H}_2) &\xrightarrow{\partial} \bigoplus_{p+q=n-2} HC_{(\delta_1, \sigma_1)}^p(\mathcal{H}_1) \otimes \\ &\otimes HC_{(\delta_2, \sigma_2)}^q(\mathcal{H}_2) \xrightarrow{S \otimes 1 - 1 \otimes S} \bigoplus_{r+s=n} HC_{(\delta_1, \sigma_1)}^r(\mathcal{H}_1) \otimes HC_{(\delta_2, \sigma_2)}^s(\mathcal{H}_2) \\ &\xrightarrow{i} HC_{(\delta_1 \otimes \delta_2, \sigma_1 \otimes \sigma_2)}^n(\mathcal{H}_1 \otimes \mathcal{H}_2) \longrightarrow \cdots \end{aligned}$$

We state a corollary of these two results which will be applied later in this paper.

COROLLARY 1. *Consider the Hopf algebra $(\mathcal{H}_2, \Delta_2) = (\mathcal{A}^o \otimes \mathcal{A}_{\text{op}}^o, \Delta \times \Delta)$ with a modular pair $(\delta, \sigma) \equiv (I \otimes I, f_1 \otimes f_1)$ in involution. Clearly, $(\mathcal{A}^o, \Delta)$ is a Hopf algebra with (I, f_1) as a modular pair in involution. Then*

$$HH_{(\delta, \sigma)}^n(\mathcal{H}_2) \cong \bigoplus_{i+j=n} HH_{(I, f_1)}^i(\mathcal{A}^o) \otimes HH_{(I, f_1)}^j(\mathcal{A}^o)$$

for $n \in \mathbb{N}_0$. Moreover, the isomorphism is implemented by the shuffle map sh (going from left to right) with the Alexander–Whitney map AW as its inverse. Concerning Hopf cyclic cohomology, there exists a canonical long exact sequence

$$\begin{aligned} \cdots \longrightarrow HC_{(\delta, \sigma)}^{n-1}(\mathcal{H}_2) &\xrightarrow{\partial} \bigoplus_{p+q=n-2} HC_{(I, f_1)}^p(\mathcal{A}^o) \otimes HC_{(I, f_1)}^q(\mathcal{A}^o) \\ &\xrightarrow{S \otimes 1 - 1 \otimes S} \bigoplus_{r+s=n} HC_{(I, f_1)}^r(\mathcal{A}^o) \otimes HC_{(I, f_1)}^s(\mathcal{A}^o) \xrightarrow{i} HC_{(\delta, \sigma)}^n(\mathcal{H}_2) \longrightarrow \cdots \end{aligned}$$

Proof. Observe that the Hopf algebra $(\mathcal{H}_2, \Delta_2)$ with the modular pair (δ, σ) in involution is a tensor product of the Hopf algebra $(\mathcal{A}^o, \Delta)$ with a modular pair (I, f_1) in involution and the Hopf algebra $(\mathcal{A}_{\text{op}}^o, \Delta)$ with a modular pair (I, f_1) in involution. Theorem 5 tells us therefore that

$$HH_{(\delta, \sigma)}^*(\mathcal{H}_2) \cong HH_{(I, f_1)}^*(\mathcal{A}^o) \otimes HH_{(I, f_1)}^*(\mathcal{A}_{\text{op}}^o).$$

But since the cosimplicial objects $((\mathcal{A}^o)^{\otimes n}, \delta_i^n, \sigma_i^n)$ and $((\mathcal{A}_{\text{op}}^o)^{\otimes n}, \delta_i^n, \sigma_i^n)$ are identical (the algebra products only occur in the definition of the cyclicity maps), we clearly have $HH_{(I, f_1)}^*(\mathcal{A}^o) = HH_{(I, f_1)}^*(\mathcal{A}_{\text{op}}^o)$ and the first part of the corollary is proved. The second part of the corollary is similarly a restatement of Theorem 6 in the particular case explained above. Here, however, we note that the $*$ -operation on $\mathcal{A}^o$ implies that $HC_{(I, f_1)}^*(\mathcal{A}^o) \cong HC_{(I, f_1)}^*(\mathcal{A}_{\text{op}}^o)$ as $f_1^* = f_1$. $\square$

The same result holds for $(\mathcal{A}^o, \Delta)$ replaced by any Hopf subalgebra $(\mathcal{K}, \Delta)$ such that $f_1 \in \mathcal{K}$. We end this rather abstract section with some explicit formulas which will be needed later. We skip their straightforward verifications.

PROPOSITION 5. *Let notation be as in the previous corollary. Let $i, j, n \in \mathbb{N}_0$ with $i + j = n$. Consider the Alexander–Whitney map*

$$\text{AW}_{i,j}: (\mathcal{A}^o)^{\otimes i} \otimes (\mathcal{A}^o)^{\otimes j} \longrightarrow \mathcal{H}_2^{\otimes n},$$

and the shuffle map

$$\text{sh}_{i,j}: \mathcal{H}_2^{\otimes n} \longrightarrow (\mathcal{A}^o)^{\otimes i} \otimes (\mathcal{A}^o)^{\otimes j}$$

on the level of complexes. Then

- (1) $\text{AW}_{1,1}(x_0 \otimes y_0) = (x_0 \otimes \varepsilon) \otimes (f_1 \otimes y_0),$
- (2) $\text{AW}_{1,2}(x_0 \otimes y_0 \otimes y_1) = (x_0 \otimes \varepsilon) \otimes (f_1 \otimes y_0) \otimes (f_1 \otimes y_1),$
- (3) $\text{AW}_{2,1}(x_0 \otimes x_1 \otimes y_0) = (x_0 \otimes \varepsilon) \otimes (x_1 \otimes \varepsilon) \otimes (f_1 \otimes y_0),$
- (4) $\text{AW}_{2,2}(x_0 \otimes x_1 \otimes y_0 \otimes y_1) = (x_0 \otimes \varepsilon) \otimes (x_1 \otimes \varepsilon) \otimes (f_1 \otimes y_0) \otimes (f_1 \otimes y_1),$
- (5) $\text{sh}_{1,1}((x_0 \otimes y_0) \otimes (x_1 \otimes y_1)) = I(x_1 y_0) x_0 \otimes y_1 - I(x_0 y_1) x_1 \otimes y_0,$
- (6) $\text{sh}_{1,2}((x_0 \otimes y_0) \otimes (x_1 \otimes y_1) \otimes (x_2 \otimes y_2)) = I(x_1 x_2 y_0) x_0 \otimes y_1 \otimes y_2 - I(x_0 x_2 y_1) x_1 \otimes y_0 \otimes y_2 + I(x_0 x_1 y_2) x_2 \otimes y_0 \otimes y_1,$
- (7) $\text{sh}_{2,1}((x_0 \otimes y_0) \otimes (x_1 \otimes y_1) \otimes (x_2 \otimes y_2)) = I(x_2 y_0 y_1) x_0 \otimes x_1 \otimes y_2 - I(x_1 y_0 y_2) x_0 \otimes x_2 \otimes y_1 + I(x_0 y_1 y_2) x_1 \otimes x_2 \otimes y_0$

for all $x_i, y_i \in \mathcal{A}^o$.

7. The Hochschild Cohomology $HH_{(\delta, \sigma)}^*(\mathcal{H})$

Throughout this section $(\mathcal{H}, \Delta)$ is an arbitrary Hopf algebra with unit I and counit ε . By an $\mathcal{H}$ -bicomodule V with respect to α we mean a vector space V and a linear map

$$\alpha: V \rightarrow \mathcal{H} \otimes V \otimes \mathcal{H}$$

such that $(\iota \otimes \alpha \otimes \iota)\alpha = (\Delta \otimes \iota \otimes \Delta)\alpha$ and $(\varepsilon \otimes \iota \otimes \varepsilon)\alpha = \iota$. If no confusion arises, we simply refer to V as an $\mathcal{H}$ -bicomodule. Suppose V_1 and V_2 are $\mathcal{H}$ -bicomodules with respect to α_1 and α_2 , respectively. An $\mathcal{H}$ -bicomodule *morphism* g is a linear map $g: V_1 \rightarrow V_2$ such that $\alpha_2 g = (\iota \otimes g \otimes \iota)\alpha_1$. We write $\text{Hom}_{\mathcal{H}\text{-bicom}}(V_1, V_2)$ for the vector subspace of $\text{Hom}(V_1, V_2)$ consisting of all such morphisms g . Just as for modules there is a one-to-one correspondence between $\mathcal{H}$ -bicomodules and left comodules over the tensor product Hopf algebra $(\mathcal{H} \otimes \mathcal{H}, \Delta \times \Delta_{\text{op}})$.

The following two examples will be important to us.

EXAMPLE 3. Suppose $\sigma \in \mathcal{H}$ is group-like, so $\Delta\sigma = \sigma \otimes \sigma$ and $\varepsilon(\sigma) = 1$. Define a linear map $\alpha_1: \mathbb{C} \rightarrow \mathcal{H} \otimes \mathbb{C} \otimes \mathcal{H}$ by $\alpha_1(1) = I \otimes 1 \otimes \sigma$. Then, clearly, $\mathbb{C}$ is an $\mathcal{H}$ -bicomodule with respect to α_1 .

EXAMPLE 4. Let $n \in \mathbb{N}_0$ and consider the vector space $\Omega(\mathcal{H})_n = \mathcal{H} \otimes \mathcal{H}^{\otimes n} \otimes \mathcal{H}$. Define a linear map $\alpha_2: \Omega(\mathcal{H})_n \rightarrow \mathcal{H} \otimes \Omega(\mathcal{H})_n \otimes \mathcal{H}$ by

$$\alpha_2(x^0 \otimes x^1 \otimes \cdots \otimes x^n \otimes x^{n+1}) = \Delta x^0 \otimes x^1 \otimes \cdots \otimes x^n \otimes \Delta x^{n+1}$$

for all $x^i \in \mathcal{H}$. By coassociativity of Δ we see that $\Omega(\mathcal{H})_n$ is an $\mathcal{H}$ -bicomodule with respect to α_2 . Note that $\Omega(\mathcal{H})_n$ is an *injective* $\mathcal{H}$ -bicomodule, i.e., the functor $\text{Hom}_{\mathcal{H}\text{-bicom}}(-, \Omega(\mathcal{H})_n)$ is an exact functor.

PROPOSITION 6. Let $n \in \mathbb{N}_0$ and consider the $\mathcal{H}$ -bicomodules $\mathbb{C}$ and $\Omega(\mathcal{H})_n$ introduced in Examples 3 and 4, respectively. Then the linear map

$$F_n: \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n) \longrightarrow \mathcal{H}^{\otimes n}$$

given by

$$F_n(g) = (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon)g(1)$$

for all $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n)$, is bijective. Its inverse F_n^{-1} is given by

$$F_n^{-1}(x^1 \otimes \cdots \otimes x^n) = g$$

for any $x^i \in \mathcal{H}$, where $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n)$ is defined by

$$g(1) = I \otimes x^1 \otimes \cdots \otimes x^n \otimes \sigma.$$

Proof. To see that the linear map F_n^{-1} given by the formula in the proposition is well defined, we need to check that $g: \mathbb{C} \rightarrow \Omega(\mathcal{H})_n$ satisfies $\alpha_2 g = (\iota \otimes g \otimes \iota)\alpha_1$. This goes as follows:

$$\alpha_2 g(1) = \Delta I \otimes x^1 \otimes \cdots \otimes x^n \otimes \Delta \sigma = I \otimes g(1) \otimes \sigma = (\iota \otimes g \otimes \iota)\alpha_1(1).$$

Next

$$\begin{aligned} F_n F_n^{-1}(x^1 \otimes \cdots \otimes x^n) &= (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon) F_n^{-1}(x^1 \otimes \cdots \otimes x^n)(1) \\ &= (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon)(I \otimes x^1 \otimes \cdots \otimes x^n \otimes \sigma) \\ &= x^1 \otimes \cdots \otimes x^n \end{aligned}$$

for all $x^i \in \mathcal{H}$, so $F_n F_n^{-1} = \iota$. To show $F_n^{-1} F_n = \iota$, we claim that for any $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n)$, we have $g(1) = \sum_i I \otimes x_i^1 \otimes \cdots \otimes x_i^n \otimes \sigma$ for some $x_i^k \in \mathcal{H}$. To see this, write $g(1) = \sum_i x_i^0 \otimes \cdots \otimes x_i^{n+1}$ for some $x_i^k \in \mathcal{H}$. Thus,

$$\begin{aligned} \alpha_2 g(1) &= (\iota \otimes g \otimes \iota) \alpha_1(1) \Leftrightarrow \sum_i \Delta x_i^0 \otimes x_i^1 \otimes \cdots \otimes x_i^n \otimes \Delta x_i^{n+1} \\ &= \sum_i I \otimes x_i^0 \otimes \cdots \otimes x_i^{n+1} \otimes \sigma, \end{aligned}$$

and therefore

$$\begin{aligned} g(1) &= (\iota \otimes \varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon \otimes \iota) \left(\sum_i \Delta x_i^0 \otimes x_i^1 \otimes \cdots \otimes x_i^n \otimes \Delta x_i^{n+1} \right) \\ &= \sum_i I \otimes \varepsilon(x_i^0) x_i^1 \otimes x_i^2 \otimes \cdots \otimes x_i^n \varepsilon(x_i^{n+1}) \otimes \sigma, \end{aligned}$$

as claimed.

Hence we must show that $F_n^{-1} F_n(g) = g$ for a linear map $g: \mathbb{C} \rightarrow \Omega(\mathcal{H})_n$ such that $g(1) = \sum_i I \otimes x_i^1 \otimes \cdots \otimes x_i^n \otimes \sigma$ for any $x_i^k \in \mathcal{H}$. But

$$F_n^{-1} F_n(g) = F_n^{-1} (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon) g(1) = \sum_i F_n^{-1} (x_i^1 \otimes \cdots \otimes x_i^n) = g,$$

as desired. $\square$

EXAMPLE 4 (continued). Consider the $\mathcal{H}$ -bicomodules $\Omega(\mathcal{H})_n$ for $n \in \mathbb{N}_0$. Let $i \in \{0, 1, \dots, n\}$, and define linear maps

$$d_i^n : \Omega(\mathcal{H})_{n-1} \rightarrow \Omega(\mathcal{H})_n, \quad s_i^n : \Omega(\mathcal{H})_{n+1} \rightarrow \Omega(\mathcal{H})_n$$

by

$$\begin{aligned} d_i^n(x^0 \otimes \cdots \otimes x^n) &= x^0 \otimes \cdots \otimes \Delta x^i \otimes \cdots \otimes x^n, \\ s_i^n(x^0 \otimes \cdots \otimes x^{n+2}) &= x^0 \otimes \cdots \otimes \varepsilon(x^{i+1}) \otimes \cdots \otimes x^{n+2} \end{aligned}$$

for all $x^i \in \mathcal{H}$. It is straightforward to check that $(\Omega(\mathcal{H})_n, d_i^n, s_i^n)$ is a cosimplicial object in the category of $\mathcal{H}$ -bicomodules. Denote by $d^n: \Omega(\mathcal{H})_{n-1} \rightarrow \Omega(\mathcal{H})_n$ the coboundary given by $d^n = \sum_{i=0}^n (-1)^i d_i^n$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}_0$, define linear maps $s^n: \Omega(\mathcal{H})_n \rightarrow \Omega(\mathcal{H})_{n-1}$ by

$$s^n(x^0 \otimes \cdots \otimes x^{n+1}) = \varepsilon(x^0) x^1 \otimes \cdots \otimes x^{n+1}$$

for $x^i \in \mathcal{H}$. It can be checked that $\{s^n\}_{n \in \mathbb{N}_0}$ is a contracting homotopy for the complex $(\Omega(\mathcal{H})_*, d)$. We have thus provided the construction of the *cobar resolution* of $(\mathcal{H}, \Delta)$, which is an injective resolution of $\mathcal{H}$ in the category of $\mathcal{H}$ -bicomodules.

Consider now the vector space $\text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n)$, for $n \in \mathbb{N}_0$, from Proposition 6. We may thus consider the following complex:

$$(\text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_*), D)$$

associated to the cobar resolution $(\Omega(\mathcal{H})_*, d)$. Its associated cosimplicial object has for $n \in \mathbb{N}_0$ and $i \in \{0, \dots, n\}$, linear maps

$$\begin{aligned} D_i^n &: \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n-1}) \longrightarrow \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n), \\ \sum_i^n &: \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n+1}) \longrightarrow \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_n) \end{aligned}$$

given by

$$D_i^n(g)(1) = d_i^n(g(1)), \quad \sum_i^n(g')(1) = s_i^n(g'(1)),$$

where $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n-1})$, $g' \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n+1})$.

LEMMA 2. *Let notation be as above. Suppose the Hopf algebra $(\mathcal{H}, \Delta)$ has a modular pair (δ, σ) in involution and consider the associated cosimplicial object. Then*

$$F_n D_i^n = \delta_i^n F_{n-1}, \quad F_n \sum_i^n = \sigma_i^n F_{n+1}, \quad F_n D^n = b_n F_{n-1}$$

for all $n \in \mathbb{N}_0$ and $i \in \{0, \dots, n\}$. Here $\{F_n\}_{n \in \mathbb{N}_0}$ are the isomorphisms from Proposition 6 and

$$b_n = \sum_{i=0}^n (-1)^i \delta_i^n \quad \text{and} \quad D^n = \sum_{i=0}^n (-1)^i D_i^n,$$

so $D^n(g)(1) = d^n(g(1))$ for all $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n-1})$.

Proof. To show $F_n D_i^n = \delta_i^n F_{n-1}$, consider $g \in \text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_{n-1})$ and write $g(1) = \sum_j I \otimes x_j^1 \otimes \dots \otimes x_j^{n-1} \otimes \sigma$ for $x_j^k \in \mathcal{H}$. Then for $i \in \{1, \dots, n-1\}$,

we get

$$\begin{aligned}
 F_n^{-1} \delta_i^n F_{n-1}(g)(1) &= F_n^{-1} \delta_i^n (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon)(g(1))(1) \\
 &= F_n^{-1} \delta_i^n (\varepsilon \otimes \iota \otimes \cdots \otimes \iota \otimes \varepsilon) \times \\
 &\quad \times \left(\sum_j I \otimes x_j^1 \otimes \cdots \otimes x_j^{n-1} \otimes \sigma \right) (1) \\
 &= F_n^{-1} \delta_i^n \left(\sum_j x_j^1 \otimes \cdots \otimes x_j^{n-1} \right) (1) \\
 &= \sum_j F_n^{-1} (x_j^1 \otimes \cdots \otimes \Delta x_j^i \otimes \cdots \otimes x_j^{n-1}) (1) \\
 &= \sum_j I \otimes x_j^1 \otimes \cdots \otimes \Delta x_j^i \otimes \cdots \otimes x_j^{n-1} \otimes \sigma \\
 &= \sum_j d_i^n (I \otimes x_j^1 \otimes \cdots \otimes x_j^{n-1} \otimes \sigma) \\
 &= d_i^n (g(1)) = D_i^n (g)(1),
 \end{aligned}$$

so $F_n^{-1} \delta_i^n F_{n-1} = D_i^n$. The other relations are proved similarly. $\square$

PROPOSITION 7. *Let $(\mathcal{H}, \Delta)$ be a Hopf algebra with a modular pair (δ, σ) in involution, and let notation be as above. Then*

$$H^*(\mathcal{H}^{\otimes*}, b) \cong H^*(\text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_*), D),$$

or in other words $HH_{(\delta, \sigma)}^*(\mathcal{H}) \cong \text{Ext}_{\mathcal{H}\text{-bicom}}^*(\mathbb{C}, \mathcal{H})$.

Proof. Lemma 2 shows that $F: (\text{Hom}_{\mathcal{H}\text{-bicom}}(\mathbb{C}, \Omega(\mathcal{H})_*), D) \longrightarrow (\mathcal{H}^{\otimes*}, b)$ given by $F = \{F_n\}_{n \in \mathbb{N}_0}$ is an isomorphism of complexes. $\square$

8. The Example $SU_q(2)$

In this section we consider the example quantum $SU_q(2)$, where the parameter $q \in \langle 0, 1 \rangle$. The Hopf $*$ -algebra associated to $SU_q(2)$ is denoted by $(\mathcal{A}_q, \Delta)$. The fundamental unitary corepresentation U of $(\mathcal{A}_q, \Delta)$ is given by

$$U = (U_{ij}) = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

where α and γ are the well-known generators of Woronowicz [26]. The multiplicative functional f_1 is uniquely determined by

$$f_1(\alpha) = q^{-1}, \quad f_1(\alpha^*) = q, \quad f_1(\gamma) = f_1(\gamma^*) = 0, \quad f_1(I) = 1.$$

We need some preliminaries on the quantized universal enveloping Lie algebra $U_q(\mathfrak{sl}_2)$, mainly to fix notation. As usual let $q \in \langle 0, 1 \rangle$ and consider the universal algebra $U_q(\mathfrak{sl}_2)$ with unit ε and (Jimbo) generators e, f, k, k^{-1} satisfying the relations:

$$\begin{aligned} kk^{-1} &= k^{-1}k = \varepsilon, & ke &= qek, & kf &= q^{-1}fk, \\ ef - fe &= \frac{1}{q - q^{-1}}(k^2 - k^{-2}). \end{aligned}$$

It is a Hopf $*$ -algebra with comultiplication $\Delta: U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ uniquely determined by

$$\Delta(k) = k \otimes k, \quad \Delta(e) = e \otimes k^{-1} + k \otimes e, \quad \Delta(f) = f \otimes k^{-1} + k \otimes f.$$

The counit $I: U_q(\mathfrak{sl}_2) \rightarrow \mathbb{C}$ and the coinverse $S: U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2)$ are then given by

$$\begin{aligned} I(k) &= I(k^{-1}) = 1, & I(e) &= I(f) = 0, & S(k) &= k^{-1}, & S(k^{-1}) &= k, \\ S(e) &= -k^{-1}ek = -q^{-1}e, & S(f) &= -k^{-1}fk = -qf. \end{aligned}$$

The $*$ -operation is given by

$$k^* = k, \quad (k^{-1})^* = k^{-1}, \quad e^* = f, \quad f^* = e.$$

It is well known that we have the following Poincaré–Birkhoff–Witt-type linear basis:

$$\{k^r f^m e^n \mid m, n, r \in \mathbb{N}_0\} \cup \{(k^{-1})^s f^m e^n \mid m, n \in \mathbb{N}_0, s \in \mathbb{N}\}$$

for $U(\mathfrak{sl}_2)_q$.

We will regard the Hopf $*$ -algebra $(U_q(\mathfrak{sl}_2), \Delta)$ as a Hopf $*$ -subalgebra of $(\mathcal{A}_q^o, \Delta)$ in such a way that

$$\begin{aligned} k(U_{11}) &= q^{1/2}, & k(U_{22}) &= q^{-1/2}, & k^{-1}(U_{11}) &= q^{-1/2}, \\ k^{-1}(U_{22}) &= q^{1/2}, & k(U_{ij}) &= 0 \quad \text{for } i \neq j, \\ e(U_{ij}) &= \delta_{i1}\delta_{j2}, & f(U_{ij}) &= \delta_{i2}\delta_{j1}. \end{aligned}$$

Under this identification we have $k^{-2} = f_1$.

In our calculations the following will be important. There exist a functional $H \in \mathcal{A}_q^o$ such that

$$\Delta(H) = H \otimes \varepsilon + \varepsilon \otimes H,$$

which is uniquely determined by $H(U_{ij}) = 0$ for $i \neq j$, and by $H(U_{11}) = -1$, $H(U_{22}) = 1$ and $H(I) = 0$. The existence of such a functional can be established by defining it on the Poincaré–Birkhoff–Witt-type linear basis

$$\{\alpha^r \gamma^m (\gamma^*)^n \mid m, n, r \in \mathbb{N}_0\} \cup \{(\alpha^*)^s \gamma^m (\gamma^*)^n \mid m, n \in \mathbb{N}_0, s \in \mathbb{N}\}$$

for $\mathcal{A}_q$ in such a way that $H(ab) = H(a)\varepsilon(b) + \varepsilon(a)H(b)$ for all $a, b \in \mathcal{A}_q$, and such that it has the prescribed values on I and the generators U_{ij} (see also theorem 4.3 in [23]). It can be shown that $e^{hH} = k^2$, where $h = -\ln(q) > 0$, so H does not live in $U_q(\mathfrak{sl}_2)$ (rather, k lives in some h -adic completion of the universal enveloping Lie algebra $U_h(\mathfrak{sl}_2)$ with the (Drinfeld) generators H, e and f , [13]).

Now the identity $(He - eH)(a) = (-2e)(a)$ is easily checked for $a = I$ and $a = U_{ij}$, the latter being the generators for the unital $*$ -algebra $\mathcal{A}_q$. Since

$$\Delta(He - eH) = (He - eH) \otimes k^{-1} + k \otimes (He - eH),$$

we see that both e and $He - eH$ satisfy the same twisted derivation property $X(ab) = X(a)k^{-1}(b) + k(a)X(b)$, and therefore $(He - eH)(a) = (-2e)(a)$ for all $a \in \mathcal{A}_q$. Thus, $He - eH = -2e$. All in all, one gets the following familiar identities:

$$Hk = kH, \quad Hk^{-1} = k^{-1}H, \quad He - eH = -2e, \quad Hf - fH = 2f.$$

Having these elements at hand, we can now say something naively about cohomology in low degrees. Considering the coboundary operator $b_n: (\mathcal{A}_q^o)^{\otimes(n-1)} \rightarrow (\mathcal{A}_q^o)^{\otimes n}$ given by $b_n = \sum_{i=0}^n (-1)^i \delta_i^n$ for $n \in \mathbb{N}$, we get $b_1(1) = \varepsilon - f_1 (\neq 0)$, and therefore $HH_{(I, f_1)}^0(\mathcal{A}_q^o) = \text{Ker}(b_1) = 0$. Similarly, we get $b_2(x) = \varepsilon \otimes x - \Delta(x) + x \otimes f_1$ for all $x \in \mathcal{A}_q^o$, so

$$\text{Ker}(b_2) = \{x \in \mathcal{A}_q^o \mid \Delta(x) = x \otimes f_1 + \varepsilon \otimes x\}.$$

We include here a technique for computing $\text{Ker}(b_2)$, and thus $HC_{(I, f_1)}^1(\mathcal{A}_q^o)$, in the spirit of differential calculi for quantum groups [9, 29].

PROPOSITION 8. *Let notation be as above. Let $\mathcal{R}_q$ denote the two-sided ideal*

$$\mathcal{R}_q = \text{Ker}(\varepsilon) \text{Ker}(f_1) = \{ab \in \mathcal{A}_q \mid \varepsilon(a) = 0, f_1(b) = 0\}$$

of $\mathcal{A}_q$ contained in $\text{Ker}(\varepsilon)$, and let $\mathcal{R}_q^\perp$ denote the vector space of linear functionals on $\mathcal{A}_q$ that annihilate $\mathcal{R}_q$. Then the following holds:

- (1) $\text{Ker}(b_2) = \{x \in \mathcal{R}_q^\perp \mid x(I) = 0\}$.
- (2) $\{f_1, \varepsilon, ek^{-1}, fk^{-1}\}$ is a linear basis for $\mathcal{R}_q^\perp$.
- (3) The set

$$G = \{\alpha^* + q\alpha - (1+q)I, \gamma^2, (\gamma^*)^2, \gamma\gamma^*, (\alpha - I)\gamma, (\alpha - I)\gamma^*\}$$
 is a set of generators for $\mathcal{R}_q$ as a right ideal.
- (4) $\kappa(a)^* \in \mathcal{R}_q$ for all $a \in \mathcal{R}_q$.

Proof. Statement (1) follows from the easy observation that $x \in \mathcal{R}_q^\perp$ if and only if we have $x((a - \varepsilon(a))(b - f_1(b))) = 0$ for all $a, b \in \mathcal{A}_q$.

We see that

$$\Delta(ek^{-1}) = ek^{-1} \otimes f_1 + \varepsilon \otimes ek^{-1}, \quad \Delta(fk^{-1}) = fk^{-1} \otimes f_1 + \varepsilon \otimes fk^{-1},$$

so $\{f_1, \varepsilon, ek^{-1}, fk^{-1}\}$ are linearly independent elements of $\mathcal{R}_q^\perp$. Thus, in particular, we have

$$\mathcal{R}_q \subset \{a \in \mathcal{A}_q \mid f_1(a) = \varepsilon(a) = (ek^{-1})(a) = (fk^{-1})(a) = 0\}.$$

Note that

$$\alpha^* + q\alpha - (1 + q)I = -q(\alpha^* - I)(\alpha - q^{-1}I) + q\gamma^*\gamma \in \mathcal{R}_q,$$

and the same is obviously true for the other five elements of G , so $\mathcal{R} \subset \mathcal{R}_q$, where $\mathcal{R}$ is by definition the right ideal of $\mathcal{A}_q$ generated by G . It suffices to prove that the inclusion $\mathcal{R} \subset \{a \in \mathcal{A}_q \mid f_1(a) = \varepsilon(a) = (ek^{-1})(a) = (fk^{-1})(a) = 0\}$ is in fact an equality, or in other words, that the dimension of the quotient vector space $\mathcal{A}_q/\mathcal{R}$ is not greater than 4.

To this end we introduce an equivalence relation $\sim$ on $\mathcal{A}_q$. Namely, to $a, b \in \mathcal{A}_q$ define $a \sim b \Leftrightarrow a - b \in \mathcal{R}$. We are done if we can show that every element in $\mathcal{A}_q$ is equivalent to a linear combination of the four elements $I, \alpha, \gamma, \gamma^* \in \mathcal{A}_q$. Since $\mathcal{R}$ is a right ideal, we have $a \sim b \Rightarrow ac \sim bc$ for all $a, b, c \in \mathcal{A}_q$. The following is immediate:

$$\begin{aligned} (\gamma^*)^2 &\sim \gamma^2 \sim \gamma^*\gamma = \gamma\gamma^* \sim 0, & q\gamma\alpha &= \alpha\gamma \sim \gamma \\ q\gamma^*\alpha &= \alpha\gamma^* \sim \gamma^*, & \alpha^*\alpha &\sim \alpha\alpha^* \sim I. \end{aligned}$$

Multiplying the equivalence $\alpha^* \sim -q\alpha + (1 + q)I$ from the right by γ, γ^* and α^* gives the equivalences:

$$\begin{aligned} q^{-1}\gamma\alpha^* &= \alpha^*\gamma \sim \gamma, & \gamma^*\alpha^* &= q\alpha^*\gamma^* \sim q\gamma^*, \\ (\alpha^*)^2 &\sim \alpha^* \sim -q\alpha + (1 + q)I. \end{aligned}$$

And finally, multiplying the equivalence $\alpha \sim -q^{-1}\alpha^* + (1 + q^{-1})I$ from the right by α gives $\alpha^2 \sim \alpha$. In effect we have shown that any second-order polynomial in the generators U_{ij} of the unital $*$ -algebra $\mathcal{A}_q$ is equivalent to a linear combination of $I, \alpha, \gamma, \gamma^* \in \mathcal{A}_q$. Using the property $a \sim b \Rightarrow ac \sim bc$ for all $a, b, c \in \mathcal{A}_q$, we conclude by an easy induction argument that the same is true for any polynomial in the generators U_{ij} of the unital $*$ -algebra $\mathcal{A}_q$. $\square$

Thus, $\text{Ker}(b_2)$ has $\{f_1 - \varepsilon, ek^{-1}, fk^{-1}\}$ as a linear basis. Similarly, one may show that H is the only (up to a scalar) linear functional on $\mathcal{A}_q$ such that $\Delta(H) = H \otimes \varepsilon + \varepsilon \otimes H$. It is not difficult to show that

$$\mathcal{R}_q = \text{Ker}(\varepsilon) \text{Ker}(f_1) \cap \text{Ker}(f_1) \text{Ker}(\varepsilon),$$

which is more symmetric looking. In [28] an axiomatic theory for covariant differential $*$ -calculus was set, and has since then been studied intensively [9]. To any compact quantum group $(\mathcal{A}, \Phi)$ a canonical bijection exist between left-covariant differential $*$ -calculus (Γ, d) and right ideals $\mathcal{R} \subset \text{Ker}(\varepsilon)$ of $\mathcal{A}$ such that $\kappa(a)^* \in \mathcal{R}$ for all $a \in \mathcal{R}$. The left-covariant differential $*$ -calculus (Γ_q, d) associated to $\mathcal{R}_q$

with its accompanying Hodge–Laplace theory, Casimir element, Dirac operator etc. has been studied in great detail in [24]. There the aim was to fit it into the scheme of noncommutative geometry of Connes, see also [10]. The calculus (Γ_q, d) is not bicovariant in the sense of [28], and bears only an indirect relation to the bicovariant $4D_+$ -calculus of Woronowicz [28]. In the theory of covariant differential calculi $\text{Ker}(b_2)$ plays the role of the quantum Lie algebra of $(\mathcal{A}_q, \Phi)$ associated to (Γ_q, d) .

Since $\text{Ker}(b_2)$ has $\{f_1 - \varepsilon, ek^{-1}, fk^{-1}\}$ as a linear basis and $\text{Im}(b_1)$ has $f_1 - \varepsilon$ as a linear basis, we see that the two elements $\{[ek^{-1}], [fk^{-1}]\}$ form a linear basis for $HH_{(I, f_1)}^1(\mathcal{A}_q^o)$.

Concerning $HC_{(I, f_1)}^1(\mathcal{A}_q^o)$, note that $\lambda_1: \mathcal{A}_q^o \rightarrow \mathcal{A}_q^o$ is given by

$$\lambda_1(x) = -\tau_1(x) = -\tilde{S}(x)f_1 = -S(x)f_1$$

for all $x \in \mathcal{A}_q^o$. Thus, for any $x \in \text{Ker}(b_2)$, we have $\lambda_1(x) = x$ if and only if $x(I) = 0$. Since $(ek^{-1})(I) = (fk^{-1})(I) = 0$, we therefore see that the two elements $\{[ek^{-1} \oplus 0], [fk^{-1} \oplus 0]\}$ form a linear basis for $HC_{(I, f_1)}^1(\mathcal{A}_q^o)$. (Here $ek^{-1} \oplus 0$ denotes the cocycle in the total complex defining cyclic cohomology corresponding to the sum of the class ek^{-1} in bidegree $(0, 1)$ and the class 0 in bidegree $(0, 1)$ in the corresponding bicomplex. Similarly, for $fk^{-1} \oplus 0$ and more complicated sums.)

To say something about generators in $(\mathcal{A}_q^{o \otimes n}, b)$ for the Hochschild cohomology groups $HH_{(I, f_1)}^2(\mathcal{A}_q^o)$, we need to consider the operator B . Recall that

$$B_1 = N^1 \sigma_1^1 \tau_2 (1_2 - \lambda_2) = N^1 (\sigma_1^1 \tau_2 - \tau_1 \sigma_0^1),$$

so

$$B_1(x \otimes y) = S(x)y - S(y)f_1x - x(I)S(y)f_1 + x(I)y$$

for all $x, y \in \mathcal{A}_q^o$. Recall also that the coboundary operator $b_3: \mathcal{A}_q^o \otimes \mathcal{A}_q^o \rightarrow \mathcal{A}_q^o \otimes \mathcal{A}_q^o$ is given by

$$b_3(x \otimes y) = \varepsilon \otimes x \otimes y - \Delta(x) \otimes y + x \otimes \Delta(y) - x \otimes y \otimes f_1$$

for all $x, y \in \mathcal{A}_q^o$. We need to make a clever guess.

LEMMA 3. *Consider $B_1 = N^1 \sigma_1^1 \tau_2 (1_2 - \lambda_2): \mathcal{A}_q^o \otimes \mathcal{A}_q^o \rightarrow \mathcal{A}_q^o$ and let notation be as above. Then the following holds:*

- (1) $b_3(H \otimes ek^{-1}) = b_3(H \otimes fk^{-1}) = 0$.
- (2) $B_1(H \otimes ek^{-1}) = 2ek^{-1}$.
- (3) $B_1(H \otimes fk^{-1}) = -2fk^{-1}$.

Proof. We have

$$\begin{aligned} b_3(H \otimes ek^{-1}) &= \varepsilon \otimes H \otimes ek^{-1} - \Delta(H) \otimes ek^{-1} + \\ &\quad + H \otimes \Delta(ek^{-1}) - H \otimes ek^{-1} \otimes f_1 \\ &= \varepsilon \otimes H \otimes ek^{-1} - (H \otimes \varepsilon + \varepsilon \otimes H) \otimes ek^{-1} + \\ &\quad + H \otimes (ek^{-1} \otimes f_1 + \varepsilon \otimes ek^{-1}) - H \otimes ek^{-1} \otimes f_1 = 0, \end{aligned}$$

and similarly, we get $b_3(H \otimes fk^{-1}) = 0$. Now

$$\begin{aligned}
 B_1(H \otimes ek^{-1}) &= S(H)ek^{-1} - S(ek^{-1})f_1H - H(I)S(ek^{-1})f_1 + H(I)ek^{-1} \\
 &= -Hek^{-1} - S(k^{-1})S(e)k^{-2}H \\
 &= -Hek^{-1} - k(-k^{-1}ek)k^{-2}H \\
 &= -Hek^{-1} + ek^{-1}H = -Hek^{-1} + eHk^{-1} \\
 &= -(He - eH)k^{-1} = -(-2e)k^{-1} = 2ek^{-1},
 \end{aligned}$$

and similarly, one proves that $B_1(H \otimes fk^{-1}) = -2fk^{-1}$. $\square$

The interesting interpretation of this lemma is that the induced map

$$IB_1 : HH_{(I, f_1)}^2(\mathcal{A}_q^o) \rightarrow HH_{(I, f_1)}^1(\mathcal{A}_q^o)$$

from the long exact IBS -sequence is surjective. Moreover, the equivalence classes $\{[H \otimes ek^{-1}], [H \otimes fk^{-1}]\}$ are linearly independent in $HH_{(I, f_1)}^2(\mathcal{A}_q^o)$ and

$$B_1([H \otimes ek^{-1}]) = 2[ek^{-1} \oplus 0] \quad \text{and} \quad B_1([H \otimes fk^{-1}]) = -2[fk^{-1} \oplus 0].$$

We shall now make use of the following result due to Crainic [7]. Note that $H \in \mathcal{A}_q^o$, but $H \notin U_q(\mathfrak{sl}_2)$.

THEOREM 7. *The Hochschild cohomology groups and the (periodic) cyclic cohomology groups of the Hopf algebra $(U_q(\mathfrak{sl}_2), \Delta)$ with the modular pair (I, f_1) in involution are*

- (1) $HH_{(I, f_1)}^1(U_q(\mathfrak{sl}_2)) \cong \mathbb{C}^2$ with generators $\{[ek^{-1}], [fk^{-1}]\}$.
- (2) $HH_{(I, f_1)}^n(U_q(\mathfrak{sl}_2)) \cong 0$ for $n \neq 1$.
- (3) $HC_{(I, f_1)}^{2n}(U_q(\mathfrak{sl}_2)) \cong 0$ for $n \in \mathbb{N}_0$.
- (4) $HC_{(I, f_1)}^{2n+1}(U_q(\mathfrak{sl}_2)) \cong \mathbb{C}^2$ with generators $\{S^n[ek^{-1} \oplus 0], S^n[fk^{-1} \oplus 0]\}$ for $n \in \mathbb{N}_0$.
- (5) $HP_{(I, f_1)}^0(U_q(\mathfrak{sl}_2)) \cong 0$.
- (6) $HP_{(I, f_1)}^1(U_q(\mathfrak{sl}_2)) \cong \mathbb{C}^2$.

Combining Theorem 7 and Corollary 1 we immediately get the following result:

THEOREM 8. *Let $\mathcal{K}_{2q} = U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)_{\text{op}}$, so $(\mathcal{K}_{2q}, \Delta_2)$ is a Hopf algebra with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Then:*

- (1) $HH_{(\delta, \sigma)}^2(\mathcal{K}_{2q}) \cong \mathbb{C}^4$.
- (2) $HH_{(\delta, \sigma)}^n(\mathcal{K}_{2q}) \cong 0$ for $n \neq 2$.
- (3) $HC_{(\delta, \sigma)}^0(\mathcal{K}_{2q}) \cong 0$.
- (4) $HC_{(\delta, \sigma)}^{2n}(\mathcal{K}_{2q}) \cong \mathbb{C}^4$ for $n \in \mathbb{N}$.
- (5) $HC_{(\delta, \sigma)}^{2n+1}(\mathcal{K}_{2q}) \cong 0$ for $n \in \mathbb{N}_0$.

- (6) $HP_{(\delta,\sigma)}^0(\mathcal{K}_{2q}) \cong \mathbb{C}^4$.
 (7) $HP_{(\delta,\sigma)}^1(\mathcal{K}_{2q}) \cong 0$.

To compute $\gamma' = \gamma j$ we *a priori* need explicit representatives of the generators in the respective original complexes. To find representatives for the generators of $HH_{(\delta,\sigma)}^2(\mathcal{K}_{2q})$ we use the Alexander–Whitney map (cup product)

$$AW_{i,j} : HH_{(I,f_1)}^i(U_q(\mathfrak{sl}_2)) \otimes HH_{(I,f_1)}^j(U_q(\mathfrak{sl}_2)) \longrightarrow HH_{(\delta,\sigma)}^n(\mathcal{K}_{2q}),$$

where $i + j = n \in \mathbb{N}_0$. By Proposition 5 we thus get the following generators of $HH_{(\delta,\sigma)}^2(\mathcal{K}_{2q})$ in $(\mathcal{K}_{2q}^{\otimes 2}, b)$:

$$\begin{aligned} [ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}], \quad [ek^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}], \\ [fk^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}], \quad [fk^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}]. \end{aligned}$$

Now let us consider cyclic cohomology. Note first that the map

$$B_1 : HH_{(\delta,\sigma)}^2(\mathcal{K}_{2q}) \rightarrow HC_{(\delta,\sigma)}^1(\mathcal{K}_{2q})$$

is the zero map because $HC_{(\delta,\sigma)}^1(\mathcal{K}_{2q}) \cong 0$.

PROPOSITION 9. *Let $n \in \mathbb{N}$. The following four elements*

- (1) $S^{n-1}[(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) \oplus (-ek^{-1} \otimes ek^{-1}) \oplus 0]$,
- (2) $S^{n-1}[(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}) \oplus (-ek^{-1} \otimes fk^{-1}) \oplus 0]$,
- (3) $S^{n-1}[(fk^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) \oplus (-fk^{-1} \otimes ek^{-1}) \oplus 0]$,
- (4) $S^{n-1}[(fk^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}) \oplus (-fk^{-1} \otimes fk^{-1}) \oplus 0]$,

are generators in the total complex for $HC_{(\delta,\sigma)}^{2n}(\mathcal{K}_{2q})$.

Proof. For any bicomplex (C^{**}, b, b') associated to a cocyclic object, the complex (C^*, b') is acyclic with contracting homotopy $s_n : C^n \rightarrow C^{n-1}$ given by $s_{n+1} = \sigma_n^n \tau_{n+1}$ for $n \in \mathbb{N}_0$, so $b'_n s_n + s_{n+1} b'_{n+1} = 1_n$ for $n \in \mathbb{N}_0$. Suppose $X \in C^n$ with $b_{n+1}(X) = 0$ and $B_{n-1}(X) = 0$, and set $Y = s_n(1_n - \lambda_n)X \in C^{n-1}$. Then the differential $d_{n+1}^t : \text{Tot}^n \rightarrow \text{Tot}^{n+1}$ of the total complex $\text{Tot}^n = \bigoplus_{i=0}^n C^n$, for $n \in \mathbb{N}_0$, satisfies $d_{n+1}^t(X \oplus Y) = 0$. To see this, first notice that

$$\begin{aligned} -b'_n(Y) &= -b'_n s_n(1_n - \lambda_n)X = (s_{n+1} b'_{n+1} - 1_n)(1_n - \lambda_n)X \\ &= s_{n+1} b'_{n+1}(1_n - \lambda_n)X - (1_n - \lambda_n)X \\ &= s_{n+1}(1_n - \lambda_n)b_{n+1}X - (1_n - \lambda_n)X = -(1_n - \lambda_n)X. \end{aligned}$$

Thus, by definition of d^t , we get

$$\begin{aligned} d_{n+1}^t(X \oplus Y) &= b_{n+1}(X) \oplus ((1_n - \lambda_n)X - b'_n(Y)) \oplus N_{n-1}(Y) \\ &= b_{n+1}(X) \oplus ((1_n - \lambda_n)X - b'_n(Y)) \oplus B_{n-1}(X) \\ &= 0 \oplus 0 \oplus 0 = 0. \end{aligned}$$

Consider the bicomplex $(\mathcal{K}_{2q}^{**}, b, b')$ associated to the Hopf algebra $(\mathcal{K}_{2q}, \Delta_2)$ with the modular pair (δ, σ) in involution. It can be checked that

$$s_2(1_2 - \lambda_2)(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) = -ek^{-1} \otimes ek^{-1}.$$

Thus, the above argument for $X = ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1} \in \mathcal{K}_{2q}^{\otimes 2}$ shows that

$$(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) \oplus (-ek^{-1} \otimes ek^{-1}) \oplus 0$$

is a cocycle in (Tot^{**}, d') . We conclude that it is also a representative for a generator in the total complex for $HC_{(\delta, \sigma)}^2(\mathcal{K}_{2q})$ because $ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}$ lives in the left-most column of the Tzygan bicomplex, so the only part of d'_2 that can hit it is the part coming from b_2 , but we have $[ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}] \neq 0$ in $HH_{(\delta, \sigma)}^2(\mathcal{K}_{2q})$. The same procedure applies to show that the same is true for the other elements in the proposition. $\square$

One can then in principle compute γ' by using an explicit formula for the Haar state h . However, as it turns out, in this specific situation there is no need for that.

THEOREM 9. *The maps γ' defined in Theorem 2, from Hochschild cohomology, cyclic cohomology and periodic cyclic cohomology for the Hopf algebra $(U_q(\mathfrak{sl}_2), \Delta)$, are all zero for the quantum group $SU_q(2)$.*

Proof. The computations of $HH^*(\mathcal{A}_q)$ in [20] show that $HH^n(\mathcal{A}_q) \neq 0$ only for $n \in \{0, 1\}$. By Theorem 8 it is then evident that $\gamma' = 0$ in the case of Hochschild cohomology. This is also evident in the case of odd cyclic cohomology and for $HC_{(\delta, \sigma)}^0(\mathcal{K}_{2q})$. To see that $\gamma'_2: HC_{(\delta, \sigma)}^2(\mathcal{K}_{2q}) \rightarrow HC^2(\mathcal{A}_q)$ is zero, note that $\gamma' = \gamma j$, so it is sufficient to show that $\gamma_2: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HC^2(\mathcal{A}_q)$ is zero, where $\mathcal{H}_{2q} = \mathcal{A}_q^o \otimes (\mathcal{A}_q^o)_{\text{op}}$. To this end we show that $S: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HC_{(\delta, \sigma)}^4(\mathcal{H}_{2q})$ is zero. We have seen from Lemma 3 that $IB: HH_{(\delta, \sigma)}^3(\mathcal{H}_{2q}) \rightarrow HH_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$ is surjective, so $I: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HH_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$ is surjective. On the other hand, $S: HC_{(\delta, \sigma)}^0(\mathcal{H}_{2q}) \rightarrow HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$ is zero as $HC_{(\delta, \sigma)}^0(\mathcal{H}_{2q}) \cong 0$, so $I: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HH_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$ is, in fact, an isomorphism. Thus, $B: HH_{(\delta, \sigma)}^3(\mathcal{H}_{2q}) \rightarrow HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$ is surjective and $S: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HC_{(\delta, \sigma)}^4(\mathcal{H}_{2q})$ is zero.

Now in [20] it is shown that $S: HC^n(\mathcal{A}_q) \rightarrow HC^{n+2}(\mathcal{A}_q)$ is an isomorphism for all $n \geq 1$. We have just seen that for any $[x] \in HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q})$, we have

$$S\gamma_2([x]) = \gamma_4 S([x]) = \gamma_4(0) = 0.$$

By injectivity of $S: HC^2(\mathcal{A}_q) \rightarrow HC^4(\mathcal{A}_q)$, we therefore get $\gamma_2([x]) = 0$ so $\gamma'_2: HC_{(\delta, \sigma)}^2(\mathcal{K}_{2q}) \rightarrow HC^2(\mathcal{A}_q)$ is zero. To see that $\gamma'_{2n+2}: HC_{(\delta, \sigma)}^{2n+2}(\mathcal{K}_{2q}) \rightarrow HC^{2n+2}(\mathcal{A}_q)$ is zero for $n \in \mathbb{N}$, we again use that $\gamma' = \gamma j$, but this time we show that $j: HC_{(\delta, \sigma)}^{2n+2}(\mathcal{K}_{2q}) \rightarrow HC_{(\delta, \sigma)}^{2n+2}(\mathcal{H}_{2q})$ is zero for $n \in \mathbb{N}$. We observe that j

is the following composition of maps:

$$\begin{aligned} HC_{(\delta, \sigma)}^{2n+2}(\mathcal{K}_{2q}) &\xrightarrow{S^{-n}} HC_{(\delta, \sigma)}^2(\mathcal{K}_{2q}) \xrightarrow{j} HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \\ &\xrightarrow{S} HC_{(\delta, \sigma)}^4(\mathcal{H}_{2q}) \xrightarrow{S^{n-1}} HC_{(\delta, \sigma)}^{2n+2}(\mathcal{H}_{2q}), \end{aligned}$$

and then use the fact that $S: HC_{(\delta, \sigma)}^2(\mathcal{H}_{2q}) \rightarrow HC_{(\delta, \sigma)}^4(\mathcal{H}_{2q})$ is zero, to conclude that $\gamma'_{2n+2}: HC_{(\delta, \sigma)}^{2n+2}(\mathcal{K}_{2q}) \rightarrow HC_{(\delta, \sigma)}^{2n+2}(\mathcal{A}_q)$ is zero for $n \in \mathbb{N}$. It now follows that γ' is zero also on the level of periodic cyclic cohomology. $\square$

And with this ends the story about the rise and fall of the γ' -map.

9. Comments

It is clearly desirable to compute the map γ defined in Theorem 2 for the Hopf algebra $(\mathcal{A}^o, \Delta)$, rather than the restricted map γ' defined in Theorem 2 for the Hopf algebra $(U_q(\mathfrak{sl}_2), \Delta)$, or for any other Hopf subalgebra of $(\mathcal{A}^o, \Delta)$ containing f_1 , because $\gamma' = \gamma j$. In particular, if $\gamma = 0$, then $\gamma' = 0$ for all such Hopf subalgebras.

Concerning the case $SU_q(2)$, it is already clear that γ is zero on the level of Hochschild cohomology, but the situation is still unclear for cyclic cohomology. Below we shall see that $\gamma = 0$ also for cyclic cohomology, and thus periodic cyclic cohomology, provided a certain assumption (to be explained below in Theorem 10) on $\mathcal{A}_q^o$ holds, which we unfortunately have not been able to check.

To this end we shall establish a theorem showing how $HH_{(I, f_1)}^*(\mathcal{A}^o)$ can be computed from a finitely generated free resolution of $\mathcal{A}$ as an $\mathcal{A}$ -bimodule. Throughout this discussion $(\mathcal{A}, \Phi)$ denotes a compact quantum group with maximal dual Hopf algebra $(\mathcal{A}^o, \Delta)$ and modular element $f_1 \in \mathcal{A}^o$, and $\mathcal{B}$ denotes the algebra $\mathcal{A} \otimes \mathcal{A}_{\text{op}}$. We have seen that the computation of $HH_{(\delta, \sigma)}^*(\mathcal{H}_2)$ can be reduced to that of $HH_{(I, f_1)}^*(\mathcal{A}^o)$, and that the latter can be computed from any injective resolution of $\mathcal{A}^o$ as an $\mathcal{A}^o$ -bicomodule. Using this we aim at giving a formula for computing $HH_{(I, f_1)}^*(\mathcal{A}^o)$ from a given finitely generated free resolution

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{m} \mathcal{A} \longrightarrow 0$$

of $\mathcal{A}$ as a left $\mathcal{B}$ -module. (For convenience we may assume that $M_0 = \mathcal{B}$, and that $m: M_0 \rightarrow \mathcal{A}$ is the algebra multiplication.)

Suppose M_{nL} is a finite-dimensional vector space, that $M_n \cong \mathcal{B} \otimes M_{nL}$ as vector spaces, so that $d_n: M_n \rightarrow M_{n-1}$ need only be specified on elements of the type $1_{\mathcal{B}} \otimes z$, where $1_{\mathcal{B}} = I \otimes I$ is the unit of $\mathcal{B}$ and $z \in M_{nL}$. Let $\{e_k^n\}$ be a linear basis of M_{nL} , and let $\{e_k^{n*}\}$ be a basis for the dual vector space M_{nL}^o of M_{nL} such that $e_k^{n*}(e_l^n) = \delta_{kl}$ for all k, l . Note that $M_{0L} \cong \mathbb{C}$ and that $M_0 = \mathcal{B} \otimes M_{0L} \cong \mathcal{B}$. We shall regard $\mathcal{A}^o \otimes \mathcal{A}^o$ as an $\mathcal{A}$ -bimodule with respect to the action $a(\xi \otimes \eta)b = a\xi \otimes \eta b$, where $\xi, \eta \in \mathcal{A}^o$ and $a, b \in \mathcal{A}$, and where by $a\xi, \eta b \in \mathcal{A}^o$ we mean $a\xi(c) = \xi(ca)$ and $\eta b(c) = \eta(bc)$ for $c \in \mathcal{A}$.

THEOREM 10. *Let notation be as in the preceding paragraph. Assume that (M_*, d) is a finitely generated free resolution of $\mathcal{A}$ as a left $\mathcal{B}$ -module. Suppose the $\mathcal{A}$ -bimodule $\mathcal{A}^o \otimes \mathcal{A}^o$ defined above is injective. Then the linear map $\delta_n: M_{(n-1)L}^o \rightarrow M_{nL}^o$ given by*

$$\delta_n(y) = \sum_k (\varepsilon \otimes f_1 \otimes y) d_n(1_{\mathcal{B}} \otimes e_k^n) e_k^{n*}$$

*for all $y \in M_{(n-1)L}^o$ and $n \in \mathbb{N}$, defines a complex (M_{*L}^o, δ) such that its cohomology coincides with $HH_{(I, f_1)}^*(\mathcal{A}^o)$. In fact, we have the following neat formula:*

$$\dim(HH_{(I, f_1)}^n(\mathcal{A}^o)) = \dim(M_{nL}^o) - \dim(\text{Im } \delta_n) - \dim(\text{Im } \delta_{n+1})$$

for all $n \in \mathbb{N}_0$.

Proof. Note that the first formula above is equivalent to the formula

$$\delta_n(y)(z) = (\varepsilon \otimes f_1 \otimes y) d_n(1_{\mathcal{B}} \otimes z)$$

for all $y \in M_{(n-1)L}^o, z \in M_{nL}$ and $n \in \mathbb{N}$.

By Proposition 7, we have $HH_{(I, f_1)}^*(\mathcal{A}^o) \cong H^*(\text{Hom}_{\mathcal{A}^o\text{-bicom}}(\mathbb{C}, M_*^o), D^o)$, where

$$\mathcal{A}^o \xrightarrow{\Delta} M_0^o \xrightarrow{d_1^o} M_1^o \xrightarrow{d_2^o} M_2^o \longrightarrow \dots$$

is any injective resolution of $\mathcal{A}^o$ as an $\mathcal{A}^o$ -bicomodule. This means that the sequence above is exact and that the vector spaces M_n^o , for $n \in \mathbb{N}_0$, are injective $\mathcal{A}^o$ -bicomodules. As usual $\mathbb{C}$ is regarded as an $\mathcal{A}^o$ -bicomodule with respect to $\alpha_1: \mathbb{C} \rightarrow \mathcal{A}^o \otimes \mathbb{C} \otimes \mathcal{A}^o$ given by $\alpha_1(1) = \varepsilon \otimes 1 \otimes f_1$, and $M_0^o = \mathcal{A}^o \otimes \mathcal{A}^o$ is regarded as an $\mathcal{A}^o$ -bicomodule with respect to $\alpha_2: M_0^o \rightarrow \mathcal{A}^o \otimes M_0^o \otimes \mathcal{A}^o$ given by $\alpha_2(\xi \otimes \eta) = \Delta(\xi) \otimes \Delta(\eta)$ for $\xi, \eta \in \mathcal{A}^o$.

To provide such a resolution, define the vector space M_n^o by

$$M_n^o = \text{Hom}_{\mathcal{A}\text{-bimod}}(M_n, \mathcal{A}^o \otimes \mathcal{A}^o)$$

for all $n \in \mathbb{N}_0$. Now M_n^o is an $\mathcal{A}^o$ -bicomodule in an obvious way. Since M_n is a finitely generated free $\mathcal{A}$ -bimodule, M_n^o is a finite sum of copies of $\mathcal{A}^o \otimes \mathcal{A}^o$, hence is injective. By hypothesis $\text{Hom}_{\mathcal{A}\text{-bimod}}(-, \mathcal{A}^o \otimes \mathcal{A}^o)$ preserves exactness, so the complex $\mathcal{A}^o \rightarrow M_*^o$ is also exact. Thus, we obtain an injective resolution of $\mathcal{A}^o$ as an $\mathcal{A}^o$ -bicomodule. Let $n \in \mathbb{N}_0$. As $\dim(M_{nL}) < \infty$, we see that

$$\begin{aligned} M_n^o &= \text{Hom}_{\mathcal{A}\text{-bimod}}(\mathcal{A} \otimes M_{nL} \otimes \mathcal{A}, \mathcal{A}^o \otimes \mathcal{A}^o) \cong \text{Hom}_{\mathbb{C}}(M_{nL}, \mathcal{A}^o \otimes \mathcal{A}^o) \\ &\cong \mathcal{A}^o \otimes \text{Hom}_{\mathbb{C}}(M_{nL}, \mathbb{C}) \otimes \mathcal{A}^o = \mathcal{A}^o \otimes M_{nL}^o \otimes \mathcal{A}^o \end{aligned}$$

as $\mathcal{A}^o$ -bicomodules. Hence, by Proposition 6, we get

$$\text{Hom}_{\mathcal{A}^o\text{-bicom}}(\mathbb{C}, M_*^o) \cong \text{Hom}_{\mathcal{A}^o\text{-bicom}}(\mathbb{C}, \mathcal{A}^o \otimes M_{*L}^o \otimes \mathcal{A}^o) \cong M_{*L}^o$$

as vector spaces, under which the coboundary D^o becomes δ , by Lemma 2. Concerning the neat formula observe that

$$HH_{(I, f_1)}^n(\mathcal{A}^o) \cong \text{Ker } \delta_{n+1} / \text{Im } \delta_n$$

for all $n \in \mathbb{N}_0$. □

Let us now return to the case $SU_q(2)$. In the rest of this section we *assume* that the $\mathcal{A}_q$ -bimodule $\mathcal{A}_q^o \otimes \mathcal{A}_q^o$ defined above is injective. In [20] an explicit and rather complicated free resolution

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{m} \mathcal{A}_q \longrightarrow 0$$

of $\mathcal{A}_q$ as a left $\mathcal{B}$ -module (here m is the multiplication) was constructed by hand. In that paper $\alpha = y$, $\gamma = u$, $\alpha^* = x$, $\gamma^* = -q^{-1}v$ and $\mu = q$. Now by Theorem 10, one can easily write down the complex (M_{*L}^o, δ) and compute $HH_{(I, f_1)}^*(\mathcal{A}_q^o)$. The result of this somewhat lengthy computation is the following.

THEOREM 11. *If our assumption holds for $SU_q(2)$, then:*

- (1) $HH_{(I, f_1)}^1(\mathcal{A}_q^o) \cong HH_{(I, f_1)}^2(\mathcal{A}_q^o) \cong \mathbb{C}^2$.
- (2) $HH_{(I, f_1)}^n(\mathcal{A}_q^o) \cong 0$ for all $n \notin \{1, 2\}$.

Thus the generators for $HH_{(I, f_1)}^1(\mathcal{A}_q^o)$ are $\{[ek^{-1}], [fk^{-1}]\}$, whereas those of $HH_{(I, f_1)}^2(\mathcal{A}_q^o)$ are $\{[H \otimes ek^{-1}], [H \otimes fk^{-1}]\}$.

Remark 1. If one takes the limit $q \rightarrow 1$ in (M_{*L}^o, δ) and then compute its cohomology one gets

- (1) $HH_{(I, f_1)}^0(\mathcal{A}_q^o) \cong HH_{(I, f_1)}^3(\mathcal{A}_q^o) \cong \mathbb{C}$,
- (2) $HH_{(I, f_1)}^1(\mathcal{A}_q^o) \cong HH_{(I, f_1)}^2(\mathcal{A}_q^o) \cong \mathbb{C}^3$,
- (3) $HH_{(I, f_1)}^n(\mathcal{A}_q^o) \cong 0$, for $n \geq 4$,

which agrees with the classical case.

Using Theorem 11, the *IBS*-sequence and Lemma 3, one can furthermore check that $HC_{(I, f_1)}^1(\mathcal{A}_q^o) \cong \mathbb{C}^2$ is the only nonvanishing component in cyclic cohomology, with generators $\{[ek^{-1} \oplus 0], [fk^{-1} \oplus 0]\}$, so $HP_{(I, f_1)}^*(\mathcal{A}_q^o) \cong 0$.

By the Künneth formula for Hopf Hochschild cohomology (Corollary 1), we see that $HH_{(\delta, \sigma)}^n(\mathcal{H}_{2q})$ is $\mathbb{C}^4$ for $n \in \{2, 4\}$ and $\mathbb{C}^8$ for $n = 3$ and otherwise zero. Corollary 1 tells us, moreover, that $HC_{(\delta, \sigma)}^n(\mathcal{H}_{2q})$ is $\mathbb{C}^4$ for $n \in \{2, 3\}$ and otherwise zero. In particular, the map $I: HC_{(\delta, \sigma)}^n \rightarrow HH_{(\delta, \sigma)}^n$ is injective. Hence $HP_{(\delta, \sigma)}^*(\mathcal{H}_{2q}) \cong 0$. Using the Alexander–Whitney map and the shuffle map from Proposition 5, it is a straightforward exercise to find explicit generators for these cohomologies in the, respectively, original complexes. However, in this case these generators are not needed. By arguments similar to those in the proof of Theorem 9, using that S acts injectively on $HH^*(\mathcal{A}_q)$ in positive degrees, one can conclude

that $\gamma = 0$ for the Hopf algebra $(\mathcal{A}_q^o, \Delta)$ associated to $SU_q(2)$ also on the level of cyclic cohomology and periodic cyclic cohomology, provided our assumption on $SU_q(2)$ holds.

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