

The Hatcher–Waldhausen map is a spectrum map

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Introduction

In this paper we give a manifold model for Waldhausen's A -theory space $A(*)$, geared towards studying the multiplicative spectrum structure induced by smash product of spaces on its one-component. This allows us to infinitely deloop the rational equivalence considered in [Wa2] and [Bö]

$$G/O \rightarrow \Omega Wh^{\text{Diff}}(*)$$

to a spectrum map, when $Wh^{\text{Diff}}(*)$ is made into a spectrum using this multiplicative structure. See Corollary 1.5 for a precise statement. Here G/O is the classifying space for smooth normal invariants, or equivalently the homotopy fiber of the J -homomorphism $BO \rightarrow BG$. $\Omega Wh^{\text{Diff}}(*)$ is a delooping of the stable smooth pseudoisotopy (or concordance) space of a point.

As an application we consider the problem of singly or infinitely delooping Bökstedt's two-complete splitting of the loop space of his model for "étale K -theory of the integers", $\Omega JK(\mathbb{Z})$, defined in [Bö], off from the loop space of algebraic K -theory of the integers, $\Omega K(\mathbb{Z})$.

Let $SG = Q(S^0)_1$ and $K(\mathbb{Z})_1$ denote the multiplicative spectra arising from smash product of spheres and tensor product of $\mathbb{Z}$ -modules respectively. We prove that if the usual map $SG \rightarrow K(\mathbb{Z})_1$ factors over the Adams e -invariant $e : SG \rightarrow J_\infty$, in the stable category, then indeed $JK(\mathbb{Z})_1$ splits off $K(\mathbb{Z})_1$ as two-completed spectra. See Theorem 2.1. In more detail, a modification of Bökstedt's argument provides the fibration sequence of spectra displayed horizontally below,

$$\begin{array}{ccccc}
 C_\infty & \longrightarrow & B \operatorname{fib}(s) & \longrightarrow & JK(\mathbb{Z})_1 \\
 & & \downarrow g & \nearrow \Phi & \\
 & & K(\mathbb{Z})_1 & &
 \end{array}$$

and the projection map factors through $K(\mathbb{Z})_1$. By assuming a null homotopy of the composite $C_\infty \rightarrow K(\mathbb{Z})_1$, we obtain our splitting result.

In the course of lifting Bökstedt's arguments to the spectrum level, we note some extensions of the theorems of Madsen, Snaith and Tornehave [MST] on degree zero spectrum level operations in real or complex K -theory, to operations of more general degree. See Proposition 2.2 and Corollary 2.3.

Using these results, we determine the algebra of spectrum self maps of a component of $JK(\mathbb{Z})$, and prove that twice the section from $\Omega JK(\mathbb{Z})$ to $\Omega K(\mathbb{Z})$ deloops infinitely often. See Theorems 3.3 and 3.5.

1 Multiplicative partitions

In this section we will construct the map $G/O \rightarrow \Omega Wh^{\text{Diff}}(*)$ as a spectrum map, as the map of homotopy fibers of the horizontal maps in a homotopy commutative diagram of spectra:

$$\begin{array}{ccc} BO & \longrightarrow & BG \\ \downarrow & & \downarrow \\ A^S(*)_1 & \longrightarrow & A(*)_1 \end{array}$$

Here G is the monoid of stable self homotopy equivalences of spheres, $A(*)_1$ is the one-component of Waldhausen's A -theory of a point, and $A^S(*)_1$ the corresponding space for stable A -theory [Wa1]. $A(*)_1$ comes equipped with a multiplicative infinite loop space structure, induced from smash product of spaces. As stabilization $A(*)_1 \rightarrow A^S(*)_1$ is coherently compatible with smashing, $A^S(*)_1$ inherits a product from $A(*)_1$. By [Wa3] the functor taking finite based sets to based spaces followed by stabilization induces a spectrum level equivalence $SG = Q(S^0)_1 \rightarrow A(*)_1 \rightarrow A^S(*)_1$.

We will define the multiplicative infinite loop space structure on $Wh^{\text{Diff}}(*)$ by setting $Wh_{\otimes}^{\text{Diff}}(*)$ to be the fiber of the trace map $A(*)_1 \rightarrow A^S(*)_1 \cong SG$.

We proceed to construct a model for the diagram above using a modified version of Waldhausen's manifold models for the spaces related to A -theory [Wa2, Sect. 1]. The original manifold models admit a product up to homotopy, corresponding to a modified smash product of spaces, as indicated in [Wa3, p. 399]. It can be seen that the product given by Waldhausen's construction will be associative up to homotopy. However, it is not clear how to systematically account for all higher coherence homotopies in this set up, which would be necessary for describing an infinite loop space structure on $A(*)_1$ and the related spaces mentioned above.

Similarly, it is not clear how to make Waldhausen's construction using rigid tubes [Wa2, p. 159] of a crucial map $BO \rightarrow T^{\text{Diff}}$ into more than an H -map. The direct sum of subspaces, which gives the composition in a Grassmannian model for BO , will only correspond up to homotopy with the product on the manifold models given by smash product of spaces, and again it is difficult to see how to prove the existence of higher coherence homotopies in the original construction. On the other hand, this will be transparent in our model.

For simplicity we will only consider manifold models for $A(X)$ when $X = *$ is a point, although similar constructions could be made for X an abelian topological monoid. We shall stably approximate Waldhausen's partitions in $D^n \times I$ by partitions in $S^n \times I$, and further by codimension zero submanifolds of S^{n+1} which are standardized near infinity. Here $S^n \times I$ approximates S^{n+1} by extending the boundary components to the origin and infinity, respectively. Our constructions will then be stably equivalent to the original ones.

Definition 1.1. Let $\mathbb{R}^n$ have the supremum metric $\|\vec{x}\| = \sup_i |x_i|$, topologize S^n as the one-point compactification of $\mathbb{R}^n$ based at infinity, and give S^∞ the direct limit topology. For a (large) number δ we call the neighborhood of $*$ in S^∞ consisting of $\vec{x}$ with $\|\vec{x}\| \geq \delta$ a δ -neighborhood of infinity. Let $M' \subset S^n$ be a compact codimension zero submanifold. For a sequence of positive numbers $\varepsilon = (\varepsilon_i)$ we call the neighborhood M of M' in S^∞ consisting of $\vec{x}$ for which there exists a $\vec{y} \in M'$ with $|x_i - y_i| \leq \varepsilon_i$ for all i an ε -thickening of M' . The thickenings of compact manifolds in S^∞ which contain a standard neighborhood of infinity will be our partitions.

We now follow [Wa2, Sect. 1]. Let $\delta \gg 0$ be a fixed number. A *partition* is a subspace $M \subset S^\infty$ which is the union of (1) an ε -thickening of some compact codimension zero submanifold $M' \subset S^n$, for some ε , M' and S^n , and (2) the δ -neighborhood of the base point at infinity. This must be interpreted in a suitable category of manifolds, as explained in [Wa2, Sects. 1 and 6]. We will work with smooth manifolds with corners, i.e. the DIFF category.

Define $\mathcal{P} = \mathcal{P}(*)_\bullet$ to be the simplicial set with k -simplices the locally trivial families of partitions parametrized over a simplex Δ^k . The ε and M' may vary through such a family, but there should exist a global (maximal) n for each simplex. $\mathcal{P}$ can be made into a simplicial category with a morphism from M to N if $M \subseteq N$, and we let $h\mathcal{P}$ denote the simplicial subcategory where the morphisms are (roughly) the homotopy equivalences. Precisely, the morphisms in $h\mathcal{P}$ are $M \rightarrow N$ such that M and N are thickenings of some $M' \subseteq N' \subset S^n$, with both inclusions

$$\partial M' \rightarrow N' - \text{int}(M') \leftarrow \partial N'$$

homotopy equivalences. Let $h\mathcal{P}_k^m$ be the full simplicial subcategory of $h\mathcal{P}$ where M has the homotopy type of a wedge of k m -spheres, or more precisely, the component of $h\mathcal{P}$ containing a thickening of an $M' \subset S^n$ obtained by adding k trivial m -handles to a δ -neighborhood of infinity. Let the simplicial set $\mathcal{P}_k^m$ be given by its objects.

We now discuss stabilization, the additive partial monoid structure on $h\mathcal{P}$, and the coherently associative smash product.

Our thickened manifolds $M \subset S^\infty$ are already stabilized with respect to the containing dimension, but we can also stabilize (the homotopy type of) M itself by suspensions. Suppose M is a partition obtained by adding a δ -neighborhood of infinity to an ε -thickening of $M' \subset S^n$. Let $M'_- \subset \mathbb{R}^n$ be M' with the point at infinity removed. Then the cartesian product $M'_- \times \mathbb{R}^1 \subset \mathbb{R}^n \times \mathbb{R}^1 \cong \mathbb{R}^{n+1}$ is a compact codimension zero submanifold away from a neighborhood of infinity, and the union of its ε -thickening and the δ -neighborhood of infinity is the stabilization of M . This process may actually better be viewed as smashing with S^1 , as we shall see momentarily.

We can add two partitions if they are disjoint away from the δ -neighborhood of infinity, by forming their union. This provides an additive partial monoid structure

on $\mathcal{P}$ (and similarly for $h\mathcal{P}$), which we can use to group complete the nerve of $\mathcal{P}$, as the inclusion of composable k -tuples of partitions into all k -tuples of partitions is a stable equivalence.

To multiply two partitions together, we will use an identification $\mathbb{R}^\infty \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$, say given by shuffling the coordinates: $((x_i), (y_i)) \mapsto (x_1, y_1, x_2, y_2, \dots)$. This pairing satisfies associativity and commutativity up to the usual coherent isomorphisms. Now suppose we are given two partitions M and N , which are ε_M - and ε_N -thickenings of $M' \subset S^m$ and $N' \subset S^n$ respectively. We shall define their product $M \otimes N$, which has the homotopy type of $M \wedge N$, by thickening the cartesian product of M'_- and N'_- and adding a δ -neighborhood of infinity. Explicitly

$$M' \wedge N' \cong (M'_- \times N'_-)_+ \subset S^\infty \wedge S^\infty$$

is a compact subspace of $S^n \wedge S^m \subset S^\infty \wedge S^\infty \cong S^\infty$, which is a codimension zero submanifold of $S^n \wedge S^m$ away from infinity. We shuffle the thickening distances ε_M and ε_N in the obvious way into ε , and let $M \otimes N$ be the union of the ε -thickening of $M' \wedge N'$ with the standard δ -neighborhood of infinity.

n -fold suspension stabilization may now be realized as forming the $\otimes$ -product with a thickening of $S^n \subset S^\infty$.

We let $\mathcal{P}^m = \coprod_{k \geq 0} \mathcal{P}_k^m$, and denote its nerve as a partial additive monoid by $N_\Gamma(\mathcal{P}^m)$. Consider the double square

$$(1.2) \quad \begin{array}{ccccc} \mathrm{colim}_m \mathcal{P}_1^m & \longrightarrow & \mathrm{colim}_m \mathcal{P}^m & \longrightarrow & \Omega N_\Gamma(\mathrm{colim}_m \mathcal{P}^m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{colim}_m h\mathcal{P}_1^m & \longrightarrow & \mathrm{colim}_m h\mathcal{P}^m & \longrightarrow & \Omega N_\Gamma(\mathrm{colim}_m h\mathcal{P}^m) \end{array}$$

where the direct limits are formed by suspensions. Recall the terminology from [Wa2, Sects. 1 and 3], where $h\underline{P}_k^m(X \times J^n)$ is the simplicial category of parametrized partitions of $X \times J^n \times I$ obtained by adding k trivial m -handles to a standard picture near $X \times J^n \times \{0\}$, under homotopy equivalences. The notations obtained by omitting h , m or k are derived analogously to the cases for $\mathcal{P}$ above. The tube space T^{Diff} is equivalent to $\mathrm{colim}_{m,n} \underline{P}_1^m(D^{m+n})$.

Lemma 1.3. *There is a chain of homotopy equivalences linking the outer rectangle above to the homotopy cartesian square*

$$\begin{array}{ccc} T^{\mathrm{Diff}} & \longrightarrow & A^S(*) \\ \downarrow & & \downarrow \\ BG & \longrightarrow & A(*). \end{array}$$

This is the square of [Wa2, p. 159].

Proof. There are maps

$$h\mathcal{P}_k^m \leftarrow h\underline{P}_k^m(S^n) \rightarrow h\underline{P}_k^m(D^{n+1})$$

which become stable equivalences as n tends to infinity. Here we embed Waldhausen's partitions of $S^n \times I$ into a partition of S^∞ by thinking of $S^n \times I$ as an

annulus missing neighborhoods of the origin and infinity, and then thickening. See also [Wa3, p. 400] for a discussion of this kind of rewriting.

These maps are sufficiently natural when k , m and n vary to assemble into the claimed chains of equivalences. The statement that the square is homotopy cartesian is [Wa2, Proposition 3.1]. $\square$

We have thus recognized the homotopy types in the outer rectangle (1.2). The (coherently) associative and commutative $\otimes$ -product is respected by the maps in that diagram, and gives rise to multiplicative infinite loop space structures on the leftmost spaces, and the one-components of the rightmost spaces. Hence these components of the outer rectangle of (1.2) form a square of infinite loop spaces.

We can also map BO into the model $\operatorname{colim}_m \mathcal{P}_1^m$ for T^{Diff} as a spectrum map, using the additive (Whitney sum) infinite loop space structure on BO . To do this, choose a (small) number $\varepsilon > 0$, and also let ε denote the constant infinite sequence $(\varepsilon, \varepsilon, \dots)$. Take as a model for $BO(m)$ the Grassmannian space of m -planes in R^∞ , and map a vector space V to its ε -thickening in $R^\infty \subset S^\infty$, with a δ -neighborhood of infinity added.

This map takes Whitney sum of vector spaces to the $\otimes$ -product of partitions, and is thus compatible with stabilization in m . This is because the ε -thickening in the supremum metric of the Whitney sum $V \times W \subset \mathbb{R}^\infty \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$ is precisely the cartesian product of the ε -thickenings of V and W . The reason why we use the supremum metric on $\mathbb{R}^\infty$ is to make this map commute “on the nose” with these pairings.

This map is clearly homotopic to that of [Wa2, Proposition 3.2]. We have thus proved:

Theorem 1.4. *There exists a homotopy commutative square of spectra*

$$\begin{array}{ccc} BO & \longrightarrow & BG \\ \downarrow & & \downarrow \\ A^S(*)_1 & \longrightarrow & A(*)_1. \end{array}$$

Here $BO = BO_\oplus$ carries the additive infinite loop space structure from Whitney sum, while BG , $A^S(*)_1 \simeq SG$ and $A(*)_1$ carry the multiplicative infinite loop space structures from smash product. The map $BO \rightarrow BG$ is the delooped J -homomorphism, the map $A^S(*)_1 \rightarrow A(*)_1$ is equivalent to the map induced by including the category of finite based sets into the category of finite spaces, and the map $BG \rightarrow A(*)_1$ includes self homotopy equivalences of spheres into homotopy equivalences of more general spaces. $\square$

Corollary 1.5. *There exists a spectrum map $G/O \rightarrow \Omega Wh_\otimes^{\operatorname{Diff}}$ from the fiber of $BO \rightarrow BG$ to the fiber of $A^S(*)_1 \rightarrow A(*)_1$, which on underlying spaces agrees up to homotopy with Waldhausen’s map [Wa2, Corollary 3.4]. $\square$*

As Waldhausen remarks, A. Hatcher has constructed a similar space level map, which may be the same as this one. K. Igusa has also given a geometric description of such a map.

2 Bökstedt's model for étale K -theory

We apply the result of Sect. 1 to the question of whether Bökstedt's two-complete splitting of the space $\Omega JK(\mathbb{Z})$ off from $\Omega K(\mathbb{Z})$ [Bö] can be lifted to the spectrum level. This turns out to reduce to the old problem of how the cokernel of J maps into $K(\mathbb{Z})$; see Theorem 2.1.

The line of argument follows Bökstedt's paper, with some modifications required to give spectrum level constructions. As all of the homotopies needed for Bökstedt's splittings turn out to lift uniquely to the spectrum level, the added information content of a spectrum level splitting compared to a space level splitting may be viewed as lying entirely in the construction from Sect. 1, and the hypothesis about the cokernel of J .

We will work in the stable category for the remainder of the paper, and all spectra are implicitly completed at the prime two.

The spectrum $JK(\mathbb{Z})$ can be defined as the homotopy fiber of the composite map of spectra

$$kO = \mathbb{Z} \times BO \xrightarrow{\psi^3 - 1} B\mathrm{Spin} \xrightarrow{c} BSU,$$

where ψ^3 is the Adams operation, and c complexification. Its one component $JK(\mathbb{Z})_1$ has a multiplicative infinite loop space structure, given as the homotopy fiber of the composite:

$$BO_{\otimes} \xrightarrow{\psi^3/1} B\mathrm{Spin}_{\otimes} \xrightarrow{c} BSU_{\otimes}$$

We use the notation f/g rather than $f - g$ to denote the difference of two H -maps f and g when we are thinking of the H -group structure of the target as a multiplicative one.

It will be clear later that the exponential cannibalistic equivalence $\rho^3 : BSO_{\oplus} \rightarrow BSO_{\otimes}$ and the related map $J_{\oplus} \rightarrow J_{\otimes}$ [AP], [MST, Corollary 4.4] induce an equivalence of spectra from the additive zero component $JK(\mathbb{Z})_0$ to $JK(\mathbb{Z})_1$, after passing to one-connected covers. The only difference between the spectra $JK(\mathbb{Z})_0$ and $JK(\mathbb{Z})_1$ is that $BO(1)$ splits off the latter but not the former, just as with BO and $BO_{\otimes}$. Our constructions will relate $JK(\mathbb{Z})_1$ to $K(\mathbb{Z})_1$ and $A(*)_1$ of the previous section, whence it will be natural for us to focus on the multiplicative model for the spectrum structure.

Let $J_{\oplus}$, $J_{\otimes}$, $C_{\oplus}$ and $C_{\otimes}$ be the additive and multiplicative connected image of J and cokernel of J spectra. There is a non-split fibration (see Theorem 3.4)

$$C_{\otimes} \longrightarrow SG \xrightarrow{e} J_{\otimes}$$

and the usual map $SG = Q(S^0)_1 \rightarrow A(*)_1 \rightarrow K(\mathbb{Z})_1$.

Theorem 2.1. *Let all spaces and spectra be completed at the prime two.*

If the composite map $C_{\otimes} \rightarrow SG \rightarrow K(\mathbb{Z})_1$ is null homotopic as a map of spectra, then the natural map $K(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$ is a split surjection in the category of spectra.

If the space of space maps from $C_{\otimes}$ to $K(\mathbb{Z})_1$ is contractible, then the natural map $K(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$ is a split surjection of spaces.

Note that Mitchell [Mit] has proved the space level version of the first hypothesis above, i.e. that the map $SG \rightarrow K(\mathbb{Z})$ of spaces factors through the image of J . The corresponding space level conclusion does not immediately follow, however.

Either of these hypotheses would hold if $K(\mathbb{Z})_1$ were the connected cover of its K -localization, as predicted by the Lichtenbaum–Quillen conjecture [DF]. Imaginably, the second hypothesis could be proved by space level techniques not taking deloopings into account.

Before proving the theorem we give some preparatory extensions of the theorems of Madsen, Snaith and Tornehave [MST], which follow from Adams' presentation in [Ad, Chap. 6].

The original theorems of Madsen, Snaith and Tornehave precisely describe the (degree zero) spectrum maps $BSO \rightarrow BSO$ and $BSU \rightarrow BSU$, which we think of as K -theory operations. In particular such maps are detected on rational homotopy groups. (See [Ad, 6.4.2] and [Ma, V.2.9].) Also, as we are considering the two-complete case, they can all be written as the sum of a scalar times the identity map, plus a map factoring through the Adams operation $\psi^3 - 1$, in both the real and complex cases [MST, 2.3].

Note that by the Hurewicz theorem, any spectrum map $X \rightarrow Y$ has a unique lifting up to homotopy to the k -connected cover of Y , where k is the connectivity of X .

We will use η to denote the essential spectrum map $S^1 \rightarrow S^0$, or the map $BY \rightarrow Y$ induced by smashing with η for any spectrum Y , or even the unique lift of this η mapping into the k -connected cover of Y , where k is the connectivity of BY .

Proposition 2.2. *There are no essential spectrum maps $BSO \rightarrow KO$ in degrees congruent to 3, 5, 6 or 7 modulo 8.*

All spectrum maps $BSO \rightarrow KO$ in degrees congruent to 1 or 2 modulo 8 factor through η or η^2 respectively. In view of the Hurewicz theorem, the same conclusion applies to spectrum maps $BSO \rightarrow \text{Spin}$ or $BBSO \rightarrow \text{Spin}$.

Let $[X, Y]_n$ denote the homotopy classes of spectrum maps of degree n from X to Y , when X and Y are spectra, and let $[X, Y] = [X, Y]_0$.

Proof. The first part is immediate from the universal coefficient theorem [Ad, 6.4.7]

$$[BSO, KO]_* \cong \text{Hom}_{\pi_* KO}(KO_* BSO, \pi_* KO),$$

the fact that $KO_* BSO$ is a free $\pi_* KO$ -module on (countably many) generators in degree zero [Ad, p. 162], and the vanishing of $\pi_* KO$ in the degrees mentioned.

For the second part, note that left composition with η^i for $i = 1$ or 2 defines a natural transformation of degree i of both sides of the formula above. As multiplication by η^i maps $\pi_0 KO$ onto $\pi_i KO$, $\eta_*^i : [BSO, KO]_0 \rightarrow [BSO, KO]_i$ is onto, and the result follows. $\square$

Corollary 2.3. *There are no essential spectrum maps $BSO \rightarrow SU$ or $SU \rightarrow BSO$. A spectrum map $f : BBSO \rightarrow SU$ or $SU \rightarrow BBSO$ is null homotopic if and only if $\pi_*(f; \mathbb{Q}) = 0$; i.e. such maps are detected rationally.*

Proof. For the first part map BSO to or from SU in the fibration sequence [Mil, Sect. 24] or [Ma, Proof of V.5.15]:

$$BSO \xrightarrow{\eta} \text{Spin} \xrightarrow{c} SU \longrightarrow BBSO$$

The result then follows from a little chase using Proposition 2.2.

The second part is handled similarly, using Proposition 2.2 and the fact that a spectrum map $BBSO \rightarrow BBSO$ which is trivial on rational homotopy is null homotopic. $\square$

In the remainder of this section we prove Theorem 2.1. Bökstedt constructs his section $\Omega JK(\mathbb{Z}) \rightarrow \Omega K(\mathbb{Z})$ by establishing a homotopy commutative diagram on the (looped) space level, which is similar to the lower part of our diagram (2.8) displayed after the proof of Lemma 2.7. However, his diagram has vertical morphisms mapping upwards, formed using a non-deloopable section to the unit $e : SG \rightarrow J_{\otimes}$ obtained from a solution to the Adams conjecture. We instead establish (2.8) as a diagram on the spectrum level.

We then turn to the extension problem indicated in the introduction, and show how to reduce it to either of the hypotheses of the theorem, in the spectrum and space level cases respectively. With the notation of (2.8), the problem is to factor $g : B \text{fib}(s) \rightarrow K(\mathbb{Z})_1$ through $B \text{fib}(s) \rightarrow JK(\mathbb{Z})_1$, which can be achieved if we know enough about how the composite $C_{\otimes} \rightarrow B \text{fib}(s) \rightarrow K(\mathbb{Z})_1$ is null homotopic.

Given this construction of a map $h : JK(\mathbb{Z})_1 \rightarrow K(\mathbb{Z})_1$, the necessary adaption of Bökstedt's space level result shows that h is a section to the natural map Φ , by an argument which we explain at the end of this section.

To construct (2.8), first recall the null homotopy of the composite $BSO \rightarrow SG \rightarrow K(\mathbb{Z})_1$:

$$\begin{array}{ccccc}
 BSO & \xrightarrow{Bj} & BSG & & \\
 \downarrow s & & \downarrow i_B & & \\
 SG & \xrightleftharpoons[p_Q]{p_Q} & A(*)_1 & \xrightleftharpoons[i_W]{p_W} & Wh_{\otimes}^{Diff}(*) \\
 & \searrow i_Q & \downarrow l & & \\
 & & K(\mathbb{Z})_1 & &
 \end{array}$$

The commuting square is from Theorem 1.4, and the multiplicative splitting of $A(*)_1$ into SG and $Wh_{\otimes}^{Diff}(*)$ from [Wa1] and [Wa3]. l denotes the linearization map.

Lemma 2.4. *The composite $l \circ i_Q \circ s : BSO \rightarrow SG \rightarrow K(\mathbb{Z})_1$ is null homotopic as a map of spectra.*

Proof. Bökstedt's argument applies unchanged: s is homotopic to the composite $p_Q \circ i_B \circ Bj$, and $l \circ i_B$ is null homotopic. Thus $l \circ i_Q \circ s$ is homotopic to $l \circ i_W \circ p_W \circ i_B \circ Bj$ (up to sign), which factors through the null map $SG \rightarrow A(*)_1 \rightarrow Wh_{\otimes}^{Diff}(*)$. $\square$

Next we consider the map $s : BSO \rightarrow SG$. Bökstedt and Waldhausen proved that the composite $p_Q \circ i_B : BSG \rightarrow A(*)_1 \rightarrow SG$ is homotopic to multiplication by η as a map of spaces [BW]. Presumably this also is true on the spectrum level, and could plausibly be proved by extending the methods of Sect. 1. We will however only need, and prove, the following weaker statements.

Let $\zeta_{\otimes} : \text{Spin}_{\otimes} \rightarrow J_{\otimes}$ be the induced map in the Puppe fibration sequence generated by the fibration $J_{\otimes} \rightarrow BO_{\otimes} \rightarrow B\text{Spin}_{\otimes}$ defining $J_{\otimes}$. So $\zeta_{\otimes} \circ \Omega(\psi^3/1) \simeq *$. Let $t_{\otimes} : BSO_{\otimes} \rightarrow J_{\otimes}$ be the composite $t_{\otimes} = \zeta_{\otimes} \circ \eta$, and similarly for $t_{\oplus}$.

Lemma 2.5. *The only essential spectrum map $f : BSO_{\otimes} \rightarrow J_{\otimes}$ is $t_{\otimes}$, which is nontrivial on homotopy. Hence spectrum maps $BSO_{\otimes} \rightarrow J_{\otimes}$ are detected on underlying spaces.*

Corollary 2.6. *The composite $e \circ s : BSO \rightarrow SG \rightarrow J_{\otimes}$ is homotopic as a map of spectra to the composite*

$$BSO \xrightarrow{Bj} BSG \xrightarrow{\eta} SG \xrightarrow{e} J_{\otimes}.$$

Proof. The two maps are homotopic on the space level by Bökstedt and Waldhausen's result quoted above, hence on the spectrum level by the lemma. $\square$

Proof of Lemma 2.5. Consider the diagram:

$$\begin{array}{ccccc} & & \phi & & \\ & & \swarrow & & \searrow \\ BSO_{\otimes} & \xleftarrow{\phi} & BSO_{\otimes} & & \\ \downarrow \eta & & \downarrow f & & \\ O_{\otimes} & \xrightarrow{\Omega(\psi^3/1)} & \text{Spin}_{\otimes} & \xrightarrow{\zeta_{\otimes}} & J_{\otimes} \longrightarrow BO_{\otimes} \end{array}$$

As f is trivial on rational homotopy, and $BO_{\otimes} \cong BSO_{\otimes} \times BO(1)$, the composite of f into $BO_{\otimes}$ is null homotopic. Hence there exists a lift of f into $\text{Spin}_{\otimes}$, which factors through η by Proposition 2.2. Thus there exists a lifting ϕ , which in additive notation can be written as a series

$$\phi \cong \varepsilon(\phi) + \sum_{r,i} a_{r,i}(\psi^{2^r+i} - \psi^i)$$

by [MST, Lemmas 2.2 and 2.11]. Here $a_{r,i}$ is only nonzero for certain odd i , and $\varepsilon(\phi)$ is a scalar multiple of the identity map. Each of the terms on the right factor through $\psi^3/1$, and as $\psi^3/1$ commutes with η , they contribute nothing when composed with $\zeta_{\otimes}$ [Ro]. Explicitly, $\zeta_{\otimes} \circ \eta \circ (\psi^3/1) \simeq \zeta_{\otimes} \circ \Omega(\psi^3/1) \circ \eta \simeq *$. Thus we may make use of the choice in selecting a lifting ϕ of f , and can indeed take ϕ to be a scalar multiple $\varepsilon(\phi)$ of the identity. The lemma follows, as η and thus $t_{\otimes}$ has order two. $\square$

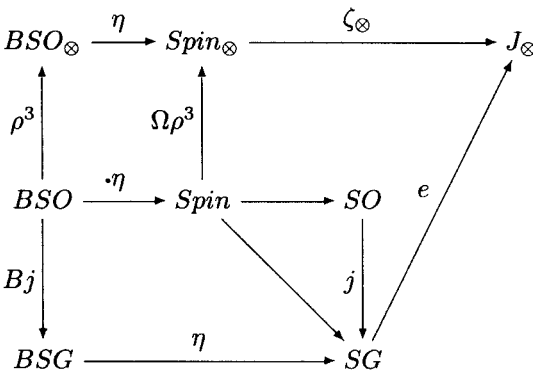
Recall from [Bö] the diagram of fibration sequences in its multiplicative version:

$$\begin{array}{ccccccc} BSO_{\otimes} & \xlongequal{\quad} & BSO_{\otimes} & \longrightarrow & * & \longrightarrow & BBSO_{\otimes} \\ \downarrow \eta & & \downarrow t_{\otimes} & & \downarrow & & \downarrow \eta \\ \text{Spin}_{\otimes} & \xrightarrow{\zeta_{\otimes}} & J_{\otimes} & \longrightarrow & BO_{\otimes} & \xrightarrow{\psi^3/1} & B\text{Spin}_{\otimes} \\ \downarrow c & & \downarrow i_{\otimes} & & \downarrow \parallel & & \downarrow c \\ SU_{\otimes} & \longrightarrow & JK(\mathbb{Z})_1 & \longrightarrow & BO_{\otimes} & \longrightarrow & BSU_{\otimes} \end{array}$$

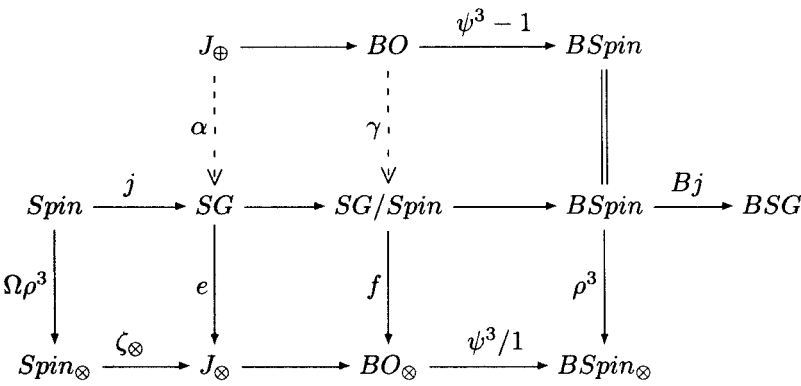
The map $i_\otimes$ is defined by this diagram. To complete the construction of the homotopy commutative diagram (2.8) we need the following:

Lemma 2.7. *The composites $t_\otimes \circ \rho^3$ and $e \circ s$ are homotopic as maps of spectra.*

Proof. We use



which commutes by naturality of multiplication by η and the J -theory diagram of [Ma, p. 107]:



where α and γ cannot be chosen as H -maps. The lower left square can be recognized in the diagram above. $\square$

We proceed to try to produce a map from $JK(\mathbb{Z})_1$ to $K(\mathbb{Z})_1$, not by using the non-deloopable splitting α , but by lifting the map from the (spectrum level) cofiber $g : B\text{fib}(s) \rightarrow K(\mathbb{Z})_1$ over $B\text{fib}(s) \rightarrow JK(\mathbb{Z})_1$. We have constructed the following diagram in the stable category

(2.8)

$$\begin{array}{ccccccc} * & \longrightarrow & C_{\otimes} & \xlongequal{\quad} & C_{\otimes} & & \\ \downarrow & & \downarrow & & \downarrow & \searrow & \\ BSO & \xrightarrow{s} & SG & \longrightarrow & B \operatorname{fib}(s) & \xrightarrow{g} & K(\mathbb{Z})_1 \\ \downarrow \rho^3 \simeq & & \downarrow e & & \downarrow & \nearrow h & \\ BSO_{\otimes} & \xrightarrow{t_{\otimes}} & J_{\otimes} & \xrightarrow{i_{\otimes}} & JK(\mathbb{Z})_1 & & \end{array}$$

where the spectrum map $h : JK(\mathbb{Z})_1 \rightarrow K(\mathbb{Z})_1$ exists as a lifting if and only if the composite $C_{\otimes} \rightarrow B \operatorname{fib}(s) \rightarrow K(\mathbb{Z})_1$ is null homotopic. This is the same map as in the hypothesis of Theorem 2.1. The left part of the diagram is a square of fibrations, and the map $g : B \operatorname{fib}(s) \rightarrow K(\mathbb{Z})_1$ is given by the null homotopy of Lemma 2.4.

We note from Mitchell’s result that $C_{\otimes} \rightarrow K(\mathbb{Z})_1$ is null homotopic on the space level only suffices to define a map from the mapping cone (space level cofiber) of $C_{\otimes} \rightarrow B \operatorname{fib}(s)$ to $K(\mathbb{Z})_1$, not from the spectrum level cofiber $JK(\mathbb{Z})_1$.

However, by “Ganea theory” [Gan] there exists a tower of obstructions to this space level lifting extension problem, lying in the group of homotopy classes of space level maps $C_{\otimes} * \Omega JK(\mathbb{Z})_1 * \dots * \Omega JK(\mathbb{Z})_1$ to $K(\mathbb{Z})_1$. Here $*$ denotes the join of spaces, and there are one or more factors $\Omega JK(\mathbb{Z})_1$. To see this, consider the tower of spaces

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & C(i_2) & \longrightarrow & JK(\mathbb{Z})_1 \\ C_{\otimes} * \Omega JK(\mathbb{Z})_1 * \Omega JK(\mathbb{Z})_1 & \xrightarrow{i_2} & \uparrow & & \parallel \\ & & C(i_1) & \longrightarrow & JK(\mathbb{Z})_1 \\ C_{\otimes} * \Omega JK(\mathbb{Z})_1 & \xrightarrow{i_1} & \uparrow & & \parallel \\ & & C(i_0) & \longrightarrow & JK(\mathbb{Z})_1 \\ C_{\otimes} & \xrightarrow{i_0} & \uparrow & & \parallel \\ & & B \operatorname{fib}(s) & \longrightarrow & JK(\mathbb{Z})_1 \\ & & \downarrow g & & \\ & & K(\mathbb{Z})_1 & & \end{array}$$

where the rows are fibrations and $C(i_s)$ the mapping cone on i_s . The map g extends to $JK(\mathbb{Z})_1$ precisely if all the maps from the fibers to $K(\mathbb{Z})_1$ are null homotopic. Now for any pointed space X , the set of homotopy classes of maps $C_{\otimes} * X \rightarrow K(\mathbb{Z})_1$

is the same as the set of homotopy classes of maps $\Sigma X \rightarrow \text{Map}(C_\otimes, K(\mathbb{Z})_1)$. Hence the second part of Theorem 2.1 follows if the space $\text{Map}(C_\otimes, K(\mathbb{Z})_1)$ is contractible, as in [Bö].

It appears unlikely that Mitchell’s argument will extend to resolve these higher obstruction questions. Let us therefore make the following assumption:

Hypothesis 2.9. The composite spectrum map $C_\otimes \rightarrow SG \rightarrow K(\mathbb{Z})_1$ is null homotopic.

Granted this, a lifting $h : JK(\mathbb{Z})_1 \rightarrow K(\mathbb{Z})_1$ of g exists. Bökstedt defines a ring spectrum map $\Phi : K(\mathbb{Z}) \rightarrow JK(\mathbb{Z})$. In fact there exists a commutative square of rings

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{Z}_3 & \longrightarrow & \mathbb{C} \end{array}$$

where $\mathbb{Z}_3$ denotes the three-adic integers, and a two-complete equivalence $K(\mathbb{Z}_3) \rightarrow K(\mathbb{F}_3)$ induced by the residue field homomorphism [Gab], which composes with a Brauer lift $K(\mathbb{F}_3) \rightarrow K(\mathbb{C})$ to give the map on K -theory induced by the lower horizontal arrow. $JK(\mathbb{Z})$ is a cover of the pullback PB of the induced square

$$\begin{array}{ccccc} K(\mathbb{Z}) & \longrightarrow & PB & \longrightarrow & K(\mathbb{R}) \\ & & \downarrow \lrcorner & & \downarrow \\ & & K(\mathbb{F}_3) & \xleftarrow{\simeq} & K(\mathbb{Z}_3) & \longrightarrow & K(\mathbb{C}) \end{array}$$

and $\Phi : K(\mathbb{Z}) \rightarrow JK(\mathbb{Z})$ is a lift of the map $K(\mathbb{Z}) \rightarrow PB$.

Lemma 2.10. *There is a homotopy commutative diagram of spectra:*

$$\begin{array}{ccccc} SG & \xrightarrow{l \circ i_Q} & K(\mathbb{Z})_1 & & \\ \downarrow e & & \downarrow \Phi & \searrow & \\ J_\otimes & \xrightarrow{i_\otimes} & JK(\mathbb{Z})_1 & \longrightarrow & BO_\otimes \end{array}$$

Presumably there is a similar result for the additive spectra. This diagram fits into the middle lower square of the J -theory diagram, as displayed in the proof of Lemma 2.7.

Proof. The right triangle commutes by the definition of Φ . We next consider the left square.

As in [Bö, Proof of 2.1] we have a covering $\mathbb{Z} \rightarrow JK(\mathbb{Z})_1 \rightarrow PB_1$, and as there are no nontrivial homomorphisms $\pi_1(SG) \cong \mathbb{Z}/2 \rightarrow \mathbb{Z}$ it suffices to compare maps

into PB_1 . This in turn reduces to comparing maps into $BO_\otimes$ and $K(\mathbb{Z}_3)_1 \cong K(\mathbb{F}_3)_1$, as well as the homotopy linking the composites into $BU_\otimes$.

Both composites $SG \rightarrow K(\mathbb{F}_3)_1$ agree with the map given by taking a finite set to the $\mathbb{F}_3$ -vector space it generates, as is seen by the “discrete model” for $J_\otimes$ described e.g. in [Ma, VIII.3.1].

The composites $SG \rightarrow BO_\otimes$ are homotopic on the underlying space level by Bökstedt’s argument, and thus on the spectrum level by [Li] or [Ma, V.7.9].

It remains to compare the homotopies of maps into $BU_\otimes$, or equivalently a lifting $SG \rightarrow U_\otimes$. Now there is only one such, the null map, which follows from the K -theoretic equivalence $SG \rightarrow J_\otimes$ [HS] and the lemma below. $\square$

Lemma 2.11. *There are no essential spectrum maps $f : J_\otimes \rightarrow U_\otimes$.*

Proof. Consider maps of the fibration sequence below to $U_\otimes$:

$$\mathrm{Spin}_\otimes \xrightarrow{\zeta_\otimes} J_\otimes \longrightarrow BO_\otimes$$

$J_\otimes$ is rationally a point, so $f \circ \zeta_\otimes$ is null on rational homotopy. By the second part of Corollary 2.3 (desuspended two degrees) it follows that f extends over $J_\otimes \rightarrow BO_\otimes$. The extension $BO_\otimes \rightarrow U_\otimes$ is trivial, because of the first part of Corollary 2.3 and the absence of maps $BO(1) \rightarrow U_\otimes$ [AH] or $BSO \rightarrow U(1)$. $\square$

We may now prove Theorem 2.1. Let PX denote the contractible path space of a based space X . Again adapting Bökstedt’s proof, we consider the diagram of vertical fibrations

$$\begin{array}{ccccccc} BSO_\otimes & \xleftarrow[\simeq]{\rho^3} & BSO & \longrightarrow & PK(\mathbb{Z})_1 & \xrightarrow{\simeq} & PJK(\mathbb{Z})_1 \\ \downarrow t_\otimes & & \downarrow s & & \downarrow & & \downarrow \\ J_\otimes & \xleftarrow{e} & SG & \longrightarrow & K(\mathbb{Z})_1 & \xrightarrow{\Phi} & JK(\mathbb{Z})_1 \\ \downarrow i_\otimes & & \downarrow & & \parallel & & \parallel \\ JK(\mathbb{Z})_1 & \xleftarrow{} & B \mathrm{fib}(s) & \longrightarrow & K(\mathbb{Z})_1 & \xrightarrow{\Phi} & JK(\mathbb{Z})_1 \end{array}$$

where there exists unique lifting maps from the left to the right by the K -acyclicity of $C_\otimes = \mathrm{fib}(e)$ and the fact that $JK(\mathbb{Z})_1$ is the connected cover of its K -localization. If we assume Hypothesis 2.9 the bottom lifting map may be taken to be $\Phi \circ h$. By Lemma 2.10 the middle lifting map is homotopic to $i_\otimes : J_\otimes \rightarrow JK(\mathbb{Z})_1$, and the three horizontal fibers form a fibration

$$BSO_\otimes \xrightarrow{a} BSO_\otimes \longrightarrow \mathrm{fib}(\Phi \circ h)$$

where $t_\otimes \circ a \simeq t_\otimes$. By [Bö, Lemma 1.10], the map a is a homotopy equivalence of spaces, and thus of spectra. This again uses essentially that we are in a two-complete situation. Hence $\mathrm{fib}(\Phi \circ h)$ is contractible and Φ is a split surjection. This completes the proof of Theorem 2.1.

Remark 2.12. Bökstedt, Hsiang and Madsen [BHM] have constructed a cyclotomic trace map $\text{trc} : K(\mathbb{Z}) \rightarrow TC(\mathbb{Z}, p)$ for each prime p . Suppose, as follows for p odd from Hypothesis 4.15 in [BM], that $TC(\mathbb{Z}, p)$ is the connected cover of its K -localization. Then the analog of Hypothesis 2.9 would be satisfied for maps into $TC(\mathbb{Z}, p)$, and the constructions above may provide a factorization of trc through $\Phi : K(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$. Such a factorization would exist by Thomason's theorem [Th] and the K -locality hypothesis cited.

Remark 2.13. Let $G/O_\otimes \simeq BSO_\otimes \times C_\otimes$ be defined as the pullback in a square

$$\begin{array}{ccc} G/O_\otimes & \xrightarrow{\quad} & SG \\ \downarrow & \lrcorner & \uparrow \\ BSO_\otimes & \xrightarrow{\eta} & Spin_\otimes \xrightarrow{\zeta_\otimes} J_\otimes \end{array} \quad \begin{array}{c} \downarrow e \\ J_\otimes \end{array}$$

Miller and Priddy defined $G/O_\oplus$ as the pullback in the analogous additive diagram, where there is no infinite loop section $Spin_\oplus \rightarrow Q(S^0)_0$. They conjectured that $G/O_\oplus \simeq G/O$ as infinite loop spaces [MP, Conjecture A, p. 339]. In view of the splitting above, and Theorem 3.4 below, $G/O_\otimes$ cannot be equivalent to either of these.

Thus there are distinct fibration sequences

$$\begin{aligned} G/O_\oplus &\rightarrow Q(S^0)_0 \rightarrow JK(\mathbb{Z})_0 \\ G/O_\otimes &\rightarrow SG \rightarrow JK(\mathbb{Z})_1 \end{aligned}$$

related to the fundamental E_∞ -ring spectrum map $Q(S^0) \rightarrow K(\mathbb{Z})$.

The composite $\Phi \circ l : A(*)_1 \rightarrow K(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$ induces a map $\Omega Wh_\otimes^{\text{Diff}}(*) \rightarrow G/O_\otimes$ on the fibers of the natural maps from SG into $A(*)_1$ or $JK(\mathbb{Z})_1$ respectively. The composite $G/O \rightarrow \Omega Wh_\otimes^{\text{Diff}}(*) \rightarrow G/O_\otimes$ is a rational equivalence, but fails to be an equivalence on the K -acyclic part, as is seen by mapping $C_\otimes$ into this sequence.

3 Twice a splitting map

In this section we shall consider the spectrum self maps of $JK(\mathbb{Z})$. All spectra are still completed at two, and we continue to work in the stable category.

We will compare the spectrum structures on the additive and multiplicative components $JK(\mathbb{Z})_0$ and $JK(\mathbb{Z})_1$, and note that their one-connected coverings $\widetilde{JK}(\mathbb{Z})_0$ and $\widetilde{JK}(\mathbb{Z})_1$ are equivalent as spectra. We use $\widetilde{JK}(\mathbb{Z})$ to denote the additive version, as a representative for either.

In Proposition 3.2 we prove that spectrum self maps of $\widetilde{JK}(\mathbb{Z})$ are detected on homotopy groups. As an application we determine the algebra of such spectrum self maps, as the quotient algebra of the algebra $\text{End}(BSU)$ of spectrum self maps of BSU determined in [MST], by the closed two-sided ideal generated by an operation $(\psi^3 - 1)(1 + \psi^{-1})$. See Theorem 3.3.

In Theorem 3.4 we give a proof due to Tornehave for the fact that the cofiber map $J_\otimes \rightarrow BC_\otimes$ corresponding to the fibration sequence $C_\otimes \rightarrow SG \rightarrow J_\otimes$ has order

precisely two as a spectrum map. Hence twice the identity map of $J_{\otimes}$ factors through $e : SG \rightarrow J_{\otimes}$, and an explicit factorization can be constructed using Friedlander's spectrum level solution to the complex Adams conjecture.

In a similar manner we can construct a spectrum map $2h : JK(\mathbb{Z})_1 \rightarrow K(\mathbb{Z})_1$. By the results above, this provides a unique infinite delooping of twice Bökstedt's splitting map

$$\Omega h : \Omega\Omega^{\infty} JK(\mathbb{Z}) \rightarrow \Omega\Omega^{\infty} K(\mathbb{Z}) .$$

See Theorem 3.5.

Technically it is easiest to work with the one-connected coverings of the spectra of interest in all these considerations. Finally, in Proposition 3.6 we remark on the modifications needed to consider self maps of a full component of $JK(\mathbb{Z})$.

We use the following notation. A subscript denotes a component, and a tilde denotes the one-connected cover. $B\text{Spin}U$ is the fiber of the essential composite $BU \rightarrow K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}/2, 2)$. $\widetilde{JU}$ is thus the fiber of $\psi^3 - 1 : BU \rightarrow B\text{Spin}U$. $\widetilde{JK}(\mathbb{Z})$ fits into fibration sequences $\widetilde{JK}(\mathbb{Z}) \rightarrow BSO \rightarrow BSU$ and $BSO \rightarrow \widetilde{J} \rightarrow \widetilde{JK}(\mathbb{Z})$. There are similar multiplicative sequences.

Proposition 3.1. *The cannibalistic equivalences $\rho^3 : BSO \rightarrow BSO_{\otimes}$ and $\rho^3 : BSU \rightarrow BSU_{\otimes}$ induce a homotopy equivalence, also denoted ρ^3 , between the one-connected covers of $JK(\mathbb{Z})_0$ and $JK(\mathbb{Z})_1$.*

The two extensions $\widetilde{JK}(\mathbb{Z}) \rightarrow JK(\mathbb{Z})_0 \rightarrow BO(1)$ and $\widetilde{JK}(\mathbb{Z}) \rightarrow JK(\mathbb{Z})_1 \rightarrow BO(1)$ realize the two elements in $[O(1), \widetilde{JK}(\mathbb{Z})] \cong \mathbb{Z}/2$. Only the latter extension is split.

There are no essential spectrum maps $BO(1) \rightarrow \widetilde{JK}(\mathbb{Z})$.

Proof. The cannibalistic equivalences $\rho^3 : BSO \rightarrow BSO_{\otimes}$ and $\rho^3 : BSU \rightarrow BSU_{\otimes}$ commute with the relevant Adams operations and complexification, thus inducing a homotopy equivalence on the homotopy fibers of the horizontal composites in the diagram below. These are the one-connected covers of $JK(\mathbb{Z})_0$ and $JK(\mathbb{Z})_1$.

$$\begin{array}{ccccc} BSO & \xrightarrow{\psi^3 - 1} & B\text{Spin} & \xrightarrow{c} & BSU \\ \downarrow \simeq \rho^3 & & \downarrow \simeq \rho^3 & & \downarrow \simeq \rho^3 \\ BSO_{\otimes} & \xrightarrow{\psi^3/1} & B\text{Spin}_{\otimes} & \xrightarrow{c} & BSU_{\otimes} \end{array}$$

Alternatively one may use the composite $BSO \rightarrow \text{Spin} \rightarrow \widetilde{J}_{\oplus}$ and the corresponding map from the multiplicative case, and see commutation from the J -theory diagram.

From [AH] there are no spectrum maps from the Eilenberg–MacLane spectra $K(\mathbb{Z}, n)$ or $K(\mathbb{Z}/2, n)$ to the periodic spectra K or KO . From the Hurewicz theorem and knowledge of the Steenrod algebra it follows easily that $[O(1), SU] \cong 0$, $[O(1), BSO] \cong \mathbb{Z}/2$, and $[O(1), BSU] \cong \mathbb{Z}/2$, with the two elements in $[O(1), BSO]$ represented by the extensions $BSO \rightarrow BO \rightarrow BO(1)$ and $BSO_{\otimes} \rightarrow BO_{\otimes} \rightarrow BO(1)$, where only the latter is split.

From the extension $SU \rightarrow \widetilde{JK}(\mathbb{Z}) \rightarrow BSO$ we find that $[O(1), \widetilde{JK}(\mathbb{Z})]$ is the kernel of the homomorphism

$$\mathbb{Z}/2 \cong [O(1), BSO] \rightarrow [O(1), BSU] \cong \mathbb{Z}/2 .$$

The map is induced by left composition with $c \circ (\psi^3 - 1) : BSO \rightarrow BSpin \rightarrow BSU$, where $\psi^3 - 1$ extends through BO . So the image of the nonzero element factors through the null homotopic composite $O(1) \rightarrow BSO \rightarrow BO$, hence is zero.

Thus $[O(1), \widetilde{JK}(\mathbb{Z})] \cong \mathbb{Z}/2$. The proof of $[BO(1), \widetilde{JK}(\mathbb{Z})] \cong 0$ is similar but easier.

It only remains to note that $JK(\mathbb{Z})_0 \rightarrow BO(1)$ is not split as it factors $JK(\mathbb{Z})_0 \rightarrow BO \rightarrow BO(1)$, while $JK(\mathbb{Z})_1 \rightarrow BO(1)$ is split via the inclusion of 1×1 -matrices $BO(1) \rightarrow K(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$. $\square$

Let v be the self map of $\widetilde{JK}(\mathbb{Z})$ making the diagram below commute

$$\begin{array}{ccccccc} \widetilde{JK}(\mathbb{Z}) & \longrightarrow & BSO & \xrightarrow{\psi^3 - 1} & BSpin & \xrightarrow{c} & BSU \\ \downarrow v & & \downarrow 2 & & \downarrow 2 & \nearrow r & \downarrow 1 + \psi^{-1} \\ \widetilde{JK}(\mathbb{Z}) & \longrightarrow & BSO & \xrightarrow{\psi^3 - 1} & BSpin & \xrightarrow{c} & BSU \end{array}$$

where $\psi^{-1} : BSU \rightarrow BSU$ denotes complex conjugation. $r \circ c = 2$, while $c \circ r = 1 + \psi^{-1}$.

Proposition 3.2. *The rationally trivial spectrum maps $f : \widetilde{JK}(\mathbb{Z}) \rightarrow \widetilde{JK}(\mathbb{Z})$ are precisely the two-adic scalar multiples of v , all of which are distinct. These are all detected on homotopy groups. Hence f is null homotopic if and only if $\pi_*(f) = 0$.*

Proof. Consider the extension $SU \rightarrow \widetilde{JK}(\mathbb{Z}) \rightarrow BSO$. Since there are no essential maps $SU \rightarrow BSO$ (Corollary 2.3), f fits into a ladder

$$\begin{array}{ccccc} SU & \longrightarrow & \widetilde{JK}(\mathbb{Z}) & \longrightarrow & BSO \\ \downarrow & & \downarrow f & & \downarrow \\ SU & \longrightarrow & \widetilde{JK}(\mathbb{Z}) & \longrightarrow & BSO \end{array}$$

which we may extend to the left into a diagram

$$\begin{array}{ccccccc} SO & \xrightarrow{\psi^3 - 1} & Spin & \xrightarrow{c} & SU & \longrightarrow & \widetilde{JK}(\mathbb{Z}) \\ \downarrow x & & \downarrow y & & \downarrow z & & \downarrow f \\ SO & \xrightarrow{\psi^3 - 1} & Spin & \xrightarrow{c} & SU & \longrightarrow & \widetilde{JK}(\mathbb{Z}) \\ & \nearrow \eta & & & \searrow & & \\ & BSO & & & & & BBSO \end{array}$$

which we know commutes if we omit the maps labeled y and u . The bottom part of the diagram is the fibration sequence arising from $U/O \simeq B(\mathbb{Z} \times BO)$.

Let us first assume that f is rationally trivial. Then $z : SU \rightarrow SU$ is rationally trivial in degrees congruent to 1 modulo 4, since $SU \rightarrow \widehat{JK}(\mathbb{Z})$ is a rational equivalence in these degrees. So the composite

$$SU \xrightarrow{z} SU \longrightarrow BBSO$$

is rationally trivial, hence null homotopic by Corollary 2.3. Thus there exists a lifting $u : SU \rightarrow \text{Spin}$ factoring z , with $c \circ u = z$. u is unique as there are no maps $SU \rightarrow BSO$. We define $y = u \circ c : \text{Spin} \rightarrow \text{Spin}$.

We claim that the left square commutes. The composites $(\psi^3 - 1) \circ x$ and $y \circ (\psi^3 - 1)$ become homotopic when continued through c to SU . As there are no choices of liftings $SO \rightarrow BSO$, they are also homotopic already as maps into Spin . This proves the claim, so the whole diagram above commutes.

Next we claim that any map $SU \rightarrow \text{Spin}$ extends over realification $r : SU \rightarrow \text{Spin}$. From the homotopy equivalence from real Bott periodicity, $\Omega O \simeq O/U$, we obtain a fibration sequence

$$B^5(\mathbb{Z} \times BO) \longrightarrow SU \xrightarrow{r} \text{Spin}$$

and by Proposition 2.2 there are no essential spectrum maps $B^5(\mathbb{Z} \times BO) \rightarrow \text{Spin}$, whence such a factorization always exists (uniquely).

In particular u factors as a composite $w \circ r$ for some real operation $w : \text{Spin} \rightarrow \text{Spin}$, which we may write as the sum of a scalar multiple of the identity, and an operation factoring through $\psi^3 - 1$.

With choices of x and z such that u factors through $\psi^3 - 1$, the induced map f is null homotopic. (This follows from a formal chase.) Hence we may alter w by the summand factoring through $\psi^3 - 1$ without altering f . So we may assume u is a scalar multiple nr of the realification map. But then f is precisely nv , where v was defined before the statement of the theorem.

Finally we note that nv is nontrivial on some homotopy groups for n different from zero. In degree $8i - 1$ the group $JK_{8i-1}(\mathbb{Z})$ is cyclic of order $3^{4i} - 1$, which has arbitrarily large two-valuation for suitable i . (To see this, note that $c \circ (\psi^3 - 1)$ induces multiplication by this scalar from $\pi_{8i}(BSO) \cong \mathbb{Z}$ to $\pi_{8i}(BSU) \cong \mathbb{Z}$.) Also $\pi_{8i-1}(nv)$ is multiplication by $2n$, which thus is nonzero for i such that the two-valuation of $2n$ is less than that of $3^{4i} - 1$. Hence the spectrum self maps of $\widehat{JK}(\mathbb{Z})$ are detected on homotopy groups. $\square$

Let $\text{End}(BSU)$ be the algebra of homotopy classes of spectrum maps $BSU \rightarrow BSU$. Its elements are determined in [MST, 2.3 and Theorem 2.9] as the series

$$\varepsilon(\phi) + \sum_{r \geq 0} \sum_{i \geq 0} a_{r,i} (\psi^{2^r+i} - \psi^i)$$

where $\varepsilon(\phi)$ is a scalar multiple of the identity, each $a_{r,i}$ is a two-adic integer, and $a_{r,i} = 0$ if $i \geq 2^{r+1} - 2^r$ or if i is even.

Let $\text{End}(\widehat{JK}(\mathbb{Z}))$ be the algebra of homotopy classes of spectrum maps $\widehat{JK}(\mathbb{Z}) \rightarrow \widehat{JK}(\mathbb{Z})$.

Theorem 3.3. $\text{End}(\widehat{JK}(\mathbb{Z}))$ is the quotient algebra of $\text{End}(BSU)$ by the closed two-sided ideal generated by $(\psi^3 - 1)(1 + \psi^{-1})$.

Proof. With the notation of the preceding proof, every spectrum map $f : \widetilde{JK}(\mathbb{Z}) \rightarrow \widetilde{JK}(\mathbb{Z})$ is determined by two spectrum maps $z : SU \rightarrow SU$ and $x : SO \rightarrow SO$, where in turn z determines x rationally, hence up to homotopy. Homotopy classes of spectrum self maps of SU correspond bijectively to homotopy classes of spectrum self maps of BSU under (de)looping, and similarly in the real case.

The z which determine a null homotopic f are precisely those which factor through $c \circ (\psi^3 - 1)$, i.e. the ideal in $\text{End}(BSU)$ generated by the composite $c \circ (\psi^3 - 1) \circ r$. The result then follows from the relation $c \circ r = 1 + \psi^{-1}$. $\square$

J. Tornehave proved the following result, following a discussion with M. Bökstedt where the author played a minor part.

Theorem 3.4. (*Tornehave*) *The cofiber map $J_\otimes \rightarrow BC_\otimes$ of $e : SG \rightarrow J_\otimes$ has order precisely two as a spectrum map.*

Proof. At odd primes, the map is null homotopic, while at two it is not. Localized at two we have a diagram of spectra

$$\begin{array}{ccccccc} C_\otimes & \longrightarrow & SG & \xrightarrow{e} & J_\otimes & \longrightarrow & BC_\otimes \\ & & \downarrow & & \downarrow & \nearrow & \\ BSO & \longrightarrow & SG/Spin & \xrightarrow{f} & BO_\otimes & & \\ & \searrow c & \uparrow \gamma_{\mathbb{C}} & & \uparrow \rho^3 & & \\ & & BU & \xrightarrow{r} & BSO & & \end{array}$$

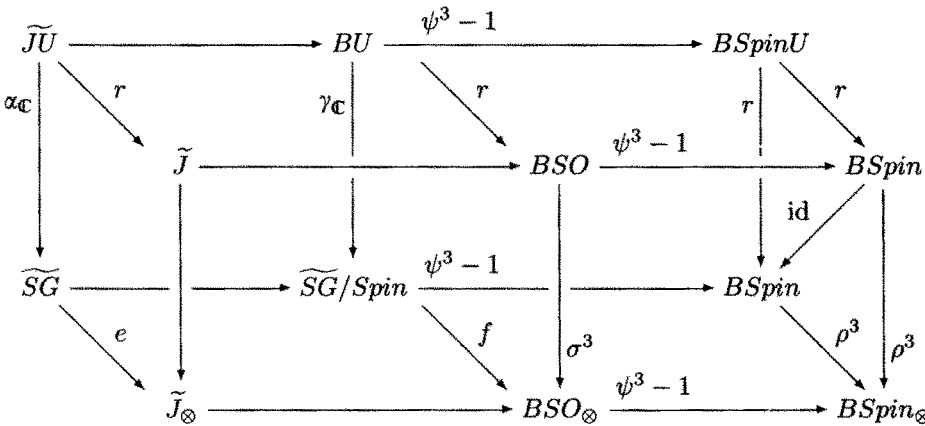
where $\gamma_{\mathbb{C}}$ is Friedlander’s spectrum level lifting of a solution to the Adams conjecture to BU [Fr, Theorem 9.2]. The composite $r \circ c : BSO \rightarrow BSO$ is homotopic to twice the identity, and the composite $SG/Spin \rightarrow BO_\otimes \rightarrow BC_\otimes$ is null homotopic. Thus $BSO \rightarrow BC_\otimes$ on the right side has order two. ρ^3 maps as an equivalence to $BSO_\otimes$, so $BSO_\otimes \rightarrow BC_\otimes$ has order two, and it remains to prove the same for the factor $BO(1) \rightarrow BO_\otimes \rightarrow BC_\otimes$. But this is obvious, as the identity map of $BO(1)$ has order two. Consequently $J_\otimes \rightarrow BO_\otimes \rightarrow BC_\otimes$ also has order two. $\square$

We now turn to infinitely delooping twice Bökstedt’s section. First note that the Adams operations commute with realification when restricted to the additive zero component BU . In particular we have horizontal fibration sequences in the diagram below,

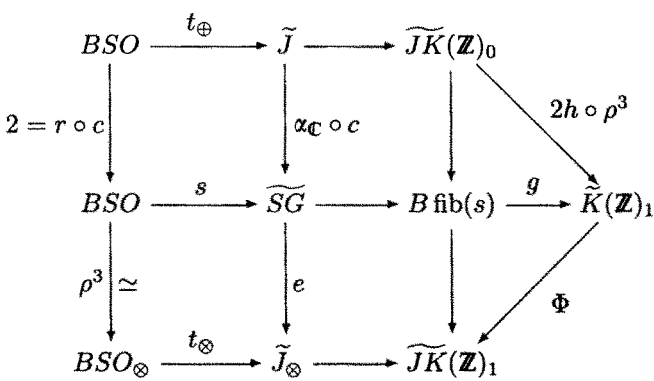
$$\begin{array}{ccccc} \widetilde{J} & \longrightarrow & BSO & \xrightarrow{\psi^3 - 1} & BSpin \\ \downarrow c & & \downarrow c & & \downarrow c \\ \widetilde{JU} & \longrightarrow & BU & \xrightarrow{\psi^3 - 1} & BSpinU \\ \downarrow r & & \downarrow r & & \downarrow r \\ \widetilde{J} & \longrightarrow & BSO & \xrightarrow{\psi^3 - 1} & BSpin \end{array}$$

where the vertical composites are multiplication by two.

Next, Friedlander’s solution to the complex Adams conjecture gives spectrum maps $\alpha_{\mathbb{C}} : \widetilde{JU} \rightarrow \widetilde{SG}$ and $\gamma_{\mathbb{C}} : BU \rightarrow \widetilde{SG}/Spin$, making the following diagram commute:



On the space level, $\alpha_{\mathbb{C}} \circ c \simeq \alpha \circ 2 : \widetilde{J} \rightarrow \widetilde{SG}$, where α is a (non-deloopable) lifting of a solution to the real Adams conjecture. Hence we obtain the diagram of spectra below. The left side consists of horizontal fibration sequences induced by maps assembled from the two preceding diagrams. The lower right hand triangle commutes by Lemma 2.10.



The map denoted $2h \circ \rho^3$ is defined as the composite making the right part of the diagram commute. We can factor it uniquely through the equivalence $\rho^3 : \widetilde{JK}(\mathbb{Z})_0 \rightarrow \widetilde{JK}(\mathbb{Z})_1$ to obtain a spectrum map

$$2h : \widetilde{JK}(\mathbb{Z})_1 \rightarrow \widetilde{K}(\mathbb{Z})_1,$$

with some abuse of notation.

In view of the splittings $JK(\mathbb{Z})_1 \simeq \widetilde{JK}(\mathbb{Z})_1 \times BO(1)$ and $K(\mathbb{Z})_1 \simeq \widetilde{K}(\mathbb{Z})_1 \times BO(1)$, it is clear that $2h$ can be extended to the full component $JK(\mathbb{Z})_1$, mapping arbitrarily on π_1 . Hence we have proved:

Theorem 3.5. *There is a spectrum map $2h : JK(\mathbb{Z})_1 \rightarrow K(\mathbb{Z})_1$ whose underlying map on looped spaces*

$$\Omega\Omega^\infty(2h) : \Omega JK(\mathbb{Z})_1 \rightarrow \Omega K(\mathbb{Z})_1$$

is homotopic to twice Bökstedt's section [Bö, 1.6] to the natural map $\Phi : K(\mathbb{Z}) \rightarrow JK(\mathbb{Z})$. The composite $\Phi \circ 2h$ is homotopic to twice a homotopy equivalence $JK(\mathbb{Z})_1 \rightarrow JK(\mathbb{Z})_1$. $\square$

It remains to discuss which spectrum self maps of $\widetilde{JK}(\mathbb{Z})$ extend to spectrum maps $JK(\mathbb{Z})_i \rightarrow JK(\mathbb{Z})_j$ for $i, j = 0, 1$. We can write such a self map f as the sum of a scalar multiple $\varepsilon(f)$ times the identity, plus an operation factoring through $\psi^3 - 1$.

Proposition 3.6. *A spectrum self map f of $\widetilde{JK}(\mathbb{Z})$ extends to a spectrum map $f' : JK(\mathbb{Z})_i \rightarrow JK(\mathbb{Z})_j$, unless $i = 0, j = 1$ and $\varepsilon(f)$ is a two-adic unit.*

If $j = 0$, such an extension is unique. If $j = 1$ there are two possible extensions, distinguished by the induced map on π_1 . In particular, spectrum maps $JK(\mathbb{Z})_i \rightarrow JK(\mathbb{Z})_j$ are detected on homotopy groups.

Proof. Consider the following diagram, where initially only f is given. An extension f' of f exists precisely if of one of the two possible maps $f'' : O(1) \rightarrow O(1)$ makes the left square commute.

$$\begin{array}{ccccccc} O(1) & \longrightarrow & \widetilde{JK}(\mathbb{Z}) & \longrightarrow & JK(\mathbb{Z})_i & \longrightarrow & BO(1) \\ \downarrow f'' & & \downarrow f & & \downarrow f' & & \downarrow \\ O(1) & \longrightarrow & \widetilde{JK}(\mathbb{Z}) & \longrightarrow & JK(\mathbb{Z})_j & \longrightarrow & BO(1) \end{array}$$

If $i = 1$ we can take f'' to be trivial and have the left square commute, so an extension f' always exists. If $i = 0$, we note that the summand of f factoring through $\psi^3 - 1$ also factors through some map $JK(\mathbb{Z})_i \rightarrow \widetilde{JK}(\mathbb{Z})$, because $\psi^3 - 1$ acts trivially on π_* in the lowest degrees. Hence the extension problem is equivalent to the one where we replace f by the scalar multiple $\varepsilon(f)$ of the identity. Then an f'' making the left square commute exists precisely if $j = 0$, or if $\varepsilon(f)$ is divisible by two. This completes the characterization of when the extension f' exists.

For uniqueness, suppose $f \simeq *$ and $j = 0$. Then f' factors through a map $BO(1) \rightarrow JK(\mathbb{Z})_0$, which must be null homotopic, since otherwise the composite $BO(1) \rightarrow JK(\mathbb{Z})_0 \rightarrow BO(1)$ would provide an impossible splitting. Hence $f' \simeq *$.

On the other hand, if $j = 1$ we are free to alter an extension $f' : JK(\mathbb{Z})_i \rightarrow JK(\mathbb{Z})_1$ by the essential composite $JK(\mathbb{Z})_i \rightarrow BO(1) \rightarrow JK(\mathbb{Z})_1$. This gives two distinct extensions, which are distinguished by their effect on π_1 . $\square$

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