

WHAT ARE RECOLLEMENTS IN STABLE HOMOTOPY THEORY?

CHARLES REZK

I keep forgetting how recollements are supposed to work, so I wrote up this note for myself. This is based on Barwick and Glasman, “A note on stable recollements” [BG16], although I have modified the notation somewhat.

You can think of a recollement (in the context of stable ∞ -categories) as a very nice kind of “short exact sequence” of stable ∞ -categories. However, the notion of recollement comes with a lot of structure, which can express itself in lots of ways. It is a very familiar pattern in stable homotopy theory. It is typically nearby whenever one encounters something called an “fracture square”.

1. DEFINITION OF RECOLLEMENT

In the following, I work with stable ∞ -categories. (In practice, they will be presentable stable ∞ -categories.) By *full subcategory*, I always mean a replete one, i.e., one which is stable under isomorphisms. By a *localization*, I mean any functor which is universally characterized by the collection of morphisms it inverts (so a localization is allowed to be a left adjoint or a right adjoint, or both, or neither).

A **recollement** is a sequence of functors between stable ∞ -categories of the form

$$\mathbf{U} \xrightarrow{i_*} \mathbf{X} \xrightarrow{j^*} \mathbf{Z}$$

with the following properties.

- (1) The functor i^* is fully faithful, and is both a left and right adjoint.
- (2) The functor j^* is a localization, and is both a left and right adjoint.
- (3) The sequence is *exact*, in the sense that $j^*X = 0$ iff X is in the essential image of i_* .

In particular, the above diagram can be extended to one of the form

$$\begin{array}{ccc} & \xleftarrow{i^\#} & \xleftarrow{j_\#} \\ \mathbf{U} & \xrightarrow{i_*} \mathbf{X} & \xrightarrow{j^*} \mathbf{Z} \\ & \xleftarrow{i^!} & \xleftarrow{j_!} \end{array}$$

in which $i^\#$ and $j_\#$ are left adjoints and $i^!$ and $j_!$ are right adjoints. Furthermore, we may identify three full subcategories of $\mathbf{X}$ as essential images of these functors:

$$\mathbf{U} = i_*\mathbf{U}, \quad \mathbf{Z}^\vee = j_\#\mathbf{Z}, \quad \mathbf{Z}^\wedge = j_!\mathbf{Z}.$$

Note that typically $\mathbf{Z}^\vee \neq \mathbf{Z}^\wedge$.

1.1. Remark. I’ve chosen the notation so that i -functors always involve $\mathbf{U}$, while j -functors always involve $\mathbf{Z}$. Also, lower index functors (i_* , $j_\#$, $j_!$) are always fully faithful, while upper index functors ($i^\#$, $i^!$, j^*) are always localizations. Finally, functors decorated with $\#$ are left adjoints, functors decorated with $!$ are right adjoints, while functors decorated with $*$ are both.

My choices probably don’t reflect what people actually use, but it doesn’t seem like there is any consensus about the right notation here.

Given this structure, there are a large number of statements which follow, which can be largely summarized as:

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- all three “horizontal sequences” in the above diagram are exact,
- there are “orthogonality relations” $\mathbf{Z}^\vee \perp \mathbf{U} \perp \mathbf{Z}^\wedge$ between the various subcategories,
- there are cofiber sequences relating the various localization functors, which
- fit into “fracture squares”.

1.2. Proposition. *Given a recollement as above, we have the following properties.*

- (1) *The subcategory $\mathbf{U}$ is a right orthogonal complement of $\mathbf{Z}^\vee$. That is, $X \in \mathbf{U}$ iff $\text{Hom}(M, X) = 0$ for all $M \in \mathbf{Z}^\vee$.*
- (1') *The subcategory $\mathbf{U}$ is a left orthogonal complement of $\mathbf{Z}^\wedge$. That is, $X \in \mathbf{U}$ iff $\text{Hom}(X, N) = 0$ for all $N \in \mathbf{Z}^\wedge$.*
- (2) *The subcategory $\mathbf{Z}^\vee$ is a left orthogonal complement of $\mathbf{U}$. That is, $X \in \mathbf{Z}^\vee$ iff $\text{Hom}(X, U) = 0$ for all $U \in \mathbf{U}$.*
- (2') *The subcategory $\mathbf{Z}^\wedge$ is a right orthogonal complement of $\mathbf{U}$. That is, $X \in \mathbf{Z}^\wedge$ iff $\text{Hom}(U, X) = 0$ for all $U \in \mathbf{U}$.*
- (3) *The sequence of functors $\mathbf{Z} \xrightarrow{j_\#} \mathbf{X} \xrightarrow{i^\#} \mathbf{U}$ is exact. That is, $X \in \mathbf{Z}^\vee$ iff $i^\# X = 0$.*
- (3') *The sequence of functors $\mathbf{Z} \xrightarrow{j!} \mathbf{X} \xrightarrow{i^!} \mathbf{U}$ is exact. That is, $X \in \mathbf{Z}^\wedge$ iff $i^! X = 0$.*
- (4) *There is a natural cofiber sequence $j_\# j^* X \xrightarrow{\epsilon} X \xrightarrow{\eta} i_* i^\# X$, where η and ϵ are unit and counit of the relevant adjunctions.*
- (4') *There is a natural cofiber sequence $i_* i^! X \xrightarrow{\epsilon} X \xrightarrow{\eta} j! j^* X$, where η and ϵ are unit and counit of the relevant adjunctions.*
- (5) *There is a natural pullback square*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & j! j^* X \\ \eta \downarrow & & \downarrow \eta j! j^* \\ i_* i^\# X & \xrightarrow{i_* i^\# \eta} & i_* i^\# j! j^* X \end{array}$$

where η and ϵ are unit and counit of the relevant adjunctions.

- (5') *There is a natural pullback square*

$$\begin{array}{ccc} i_* i^! j_\# j^* X & \xrightarrow{\epsilon j_\# j^*} & j_\# j^* X \\ i_* i^* \epsilon \downarrow & & \downarrow \epsilon \\ i_* i^! X & \xrightarrow{\epsilon} & X \end{array}$$

where η and ϵ are unit and counit of the relevant adjunctions.

Proof. I'll prove (1), (2), (3), (4), and (5), as the other statements are analogous. To prove (1), we use $\text{Hom}_{\mathbf{X}}(j_\# M, X) = \text{Hom}_{\mathbf{Z}}(M, j^* X)$, so that X has no maps from objects of $\mathbf{Z}^\vee$ iff $j^* X = 0$, which holds iff $X \in \mathbf{U}$.

Next I claim that for all $X \in \mathbf{X}$ the cofiber Y of the counit map $\epsilon: j_\# j^* X \rightarrow X$ is in $\mathbf{U}$. To see this, note that by properties of the adjunction, the composite of

$$j^* X \xrightarrow{\eta j^*} j^* j_\# j^* X \xrightarrow{j^* \epsilon} j^* X$$

is the identity map, where $\eta: N \rightarrow j^* j_\# N$ is the counit of the adjunction. But since $j_\#$ is fully faithful, we have that η is an isomorphism whenever $N = j^* X$. Thus $j^* \epsilon$ is an equivalence, whence $j^* Y = 0$, so $Y \in \mathbf{U}$ as desired.

To prove (2), note that we have already shown that $\text{Hom}(M, U) = 0$ for all $M \in \mathbf{Z}^\vee$ and $U \in \mathbf{U}$. Now suppose X is such that $\text{Hom}(X, U) = 0$ for all $U \in \mathbf{U}$. Consider the cofiber sequence $j_\# j^* X \rightarrow X \rightarrow Y$, and note that we also have $\text{Hom}_{\mathbf{X}}(j_\# j^* X, U) = \text{Hom}_{\mathbf{Z}}(j^* X, j! U) = 0$ for all

$U \in \mathbf{U}$, whence $\mathrm{Hom}(Y, U) = 0$ for all $U \in \mathbf{U}$. But we know $Y \in \mathbf{U}$, so we conclude that $Y = 0$, so $j_{\#}j^*X \rightarrow X$ is an isomorphism, and thus $X \in \mathbf{Z}^{\vee}$ as desired.

To prove (3), we use $\mathrm{Hom}_{\mathbf{U}}(i^{\#}X, U) = \mathrm{Hom}_{\mathbf{X}}(X, i_*U)$ and (2) to conclude that X is in the essential image of $j_{\#}$ iff $i^{\#}X = 0$.

To prove (4) consider the diagram

$$\begin{array}{ccccc} j_{\#}j^*X & \xrightarrow{\epsilon} & X & \longrightarrow & Y \\ \eta \downarrow & & \eta \downarrow & & \eta \downarrow \approx \\ i_*i^{\#}j_{\#}j^*X & \xrightarrow{i_*i^{\#}\epsilon} & i_*i^{\#}X & \longrightarrow & i_*i^{\#}Y \end{array}$$

where the top row is a cofiber sequence, and the bottom row is obtained by applying $i_*i^{\#}$ to the top row, and so is also a cofiber sequence. We have already observed that $Y \in \mathbf{U}$, and therefore $\eta: Y \rightarrow i_*i^{\#}Y$ is an isomorphism. Finally, $i^{\#}j_{\#} = 0$ by (3) we have that $i_*i^{\#}j_{\#}j^*X = 0$. Thus $i_*i^{\#}X \rightarrow Y$ is an isomorphism, exhibiting the desired cofiber sequence.

To prove (5), use (4) and (4') to extend the commutative square to one of the form

$$\begin{array}{ccccc} j_{\#}j^*i_*i^!X & \xrightarrow{j_{\#}j^*\epsilon} & j_{\#}j^*X & \xrightarrow{j_{\#}j^*\eta} & j_{\#}j^*j_!j^*X \\ \epsilon i_*i^! \downarrow & & \epsilon \downarrow & & \epsilon j_!j^* \downarrow \\ i_*i^!X & \xrightarrow{\epsilon} & X & \xrightarrow{\eta} & j_!j^*X \\ \eta i_*i^! \downarrow & & \eta \downarrow & & \eta j_!j^* \downarrow \\ i_*i^{\#}i_*i^!X & \xrightarrow{i_*i^{\#}\epsilon} & i_*i^{\#}X & \xrightarrow{i_*i^{\#}\eta} & i_*i^{\#}j_!j^*X \end{array}$$

in which all rows and columns are cofiber sequences. The claim follows since $j^*i_* = 0$, so the upper-left corner is 0. $\square$

We can explicitly reconstruct $\mathbf{X}$ using data based on the subcategories $\mathbf{U}$ and $\mathbf{Z}^{\wedge}$, or $\mathbf{U}$ and $\mathbf{Z}^{\vee}$.

1.3. Proposition. *The functor*

$$\mathbf{Z}^{\wedge} \times_{\mathbf{X}} \mathrm{Fun}(\Delta^1, \mathbf{X}) \times_{\mathbf{X}} \mathbf{U} \rightarrow \mathbf{X}$$

which sends a morphism $Z \rightarrow U$ to its fiber is an equivalence. The functor

$$\mathbf{U} \times_{\mathbf{X}} \mathrm{Fun}(\Delta^1, \mathbf{X}) \times_{\mathbf{X}} \mathbf{Z}^{\vee} \rightarrow \mathbf{X}$$

which sends a morphism $U \rightarrow Z$ to its fiber is an equivalence.

Proof idea. (See Barwick-Glasman, Lemma 9.) By property (4') above, any $X \in \mathbf{X}$ fits in a fiber sequence $X \xrightarrow{\eta} j_!j^*X \rightarrow \Sigma i_*i^!X$ with $j_!j^*X \in \mathbf{Z}^{\wedge}$ and $i_*i^!X \in \mathbf{U}$. Conversely, given any fiber sequence $X \rightarrow Z \rightarrow U$ with $Z \in \mathbf{Z}^{\wedge}$ and $U \in \mathbf{U}$, we have $j_!j^*X \xrightarrow{\sim} j_!j^*Z \approx Z$ since $j^*U = 0$ and $U \approx i_*i^!U \xrightarrow{\sim} \Sigma i_*i^!X$ since $i^!Z = 0$.

Similarly by property (4) above, any $X \in \mathbf{X}$ fits in a fiber sequence $X \xrightarrow{\epsilon} i_*i^{\#}X \rightarrow \Sigma j_{\#}j^*X$ with $i_*i^{\#}X \in \mathbf{U}$ and $j_{\#}j^*X \in \mathbf{Z}^{\vee}$. Conversely, given any fiber sequence $X \rightarrow U \rightarrow Z$ with $U \in \mathbf{U}$ and $Z \in \mathbf{Z}^{\vee}$, we have $i_*i^{\#}X \xrightarrow{\sim} i_*i^{\#}U \approx U$ since $i^{\#}Z = 0$ and $Z \approx j_{\#}j^*Z \xrightarrow{\sim} \Sigma j_{\#}j^*X$ since $j^*U = 0$. $\square$

Thus, for example, we can encode an object of $\mathbf{X}$ by a triple of data

$$(U, Z, \gamma), \quad U \in \mathbf{U}, \quad Z \in \mathbf{Z}^{\wedge}, \quad (\gamma: i^{\#}j_{\#}Z \rightarrow U) \in \mathbf{U},$$

or by a triple of data

$$(U', Z', \gamma'), \quad U' \in \mathbf{U}, \quad Z' \in \mathbf{Z}^{\vee}, \quad (\gamma': U \rightarrow i^!j_!Z') \in \mathbf{U}.$$

2. THE SYMMETRIC MONOIDAL CASE

Let's think about what this looks like when $\mathbf{U}, \mathbf{X}, \mathbf{Z}$ are presentably symmetric monoidal, and the localization functors $i^\# : \mathbf{X} \rightarrow \mathbf{U}$ and $j^* : \mathbf{X} \rightarrow \mathbf{Z}$ are strongly symmetric monoidal. Then we can describe the situation as follows.

- Let $L := i_* i^\# \mathbb{1}$. Then, with respect to the unit map $\mathbb{1} \rightarrow L$ of the adjunction, L is an idempotent ring.
- Let $C := j_\# j^* \mathbb{1}$. Then, with respect to the counit map $C \rightarrow \mathbb{1}$ of the adjunction, C is an idempotent coring.
- These counit and unit maps fit into a cofiber sequence $C \xrightarrow{\epsilon} \mathbb{1} \xrightarrow{\eta} L$.
- We have $\mathbf{U} = \text{Mod}(L)$ and $\mathbf{Z}^\vee = \text{Comod}(C)$, with the inclusions into $\mathbf{X}$ corresponding to the evident forgetful functors.
- The localization monads and comonads are described by

$$\begin{array}{ccc} i_* i^\# = L \otimes - & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & j_\# j^* = C \otimes - \\ i_* i^! = [L, -] & \mathbf{X} & j_! j^* = [C, -] \end{array}$$

so that the cofiber sequences relating them have the form

$$C \otimes X \rightarrow \mathbb{1} \otimes X \rightarrow L \otimes X, \quad [L, X] \rightarrow [\mathbb{1}, X] \rightarrow [C, X].$$

Here I write $[X, Y]$ for an internal function object.

- The arithmetic squares have the form

$$\begin{array}{ccc} X & \longrightarrow & [C, X] \\ \downarrow & & \downarrow \\ L \otimes X & \longrightarrow & L \otimes [C, X] \end{array} \quad \begin{array}{ccc} [L, C \otimes X] & \longrightarrow & C \otimes X \\ \downarrow & & \downarrow \\ [L, X] & \longrightarrow & X \end{array}$$

3. SOME EXAMPLES

Here are some standard examples of recollements which appear in stable homotopy theory. Because full subcategories seem easier to visualize than localizations, I'll often exhibit these diagrammatically in the form:

$$\begin{array}{ccccc} & \xleftarrow{i^\#} & & \xleftarrow{\quad} & \mathbf{Z}^\vee \\ \mathbf{U} & \xrightarrow{\quad} & \mathbf{X} & \xrightarrow{j^*} & \mathbf{Z} \\ & \xleftarrow{i^!} & & \xleftarrow{\quad} & \mathbf{Z}^\wedge \end{array}$$

In this case, the functor $j^* : \mathbf{X} \rightarrow \mathbf{Z}$ may have two different descriptions, according to which subcategory is identified with its codomain.

3.1. *Example* (p -completion). This is:

$$\begin{array}{ccccc} & \xleftarrow{S[\frac{1}{p}] \otimes -} & & \xleftarrow{\quad} & (\text{Mod}_{\mathbb{S}})^{p\text{-tors}} \\ \text{Mod}_{\mathbb{S}[\frac{1}{p}]} & \xrightarrow{\quad} & \text{Mod}_{\mathbb{S}} & \xrightarrow{\quad} & \mathbf{Z}^\vee \\ & \xleftarrow{[\mathbb{S}[\frac{1}{p}], -]} & & \xleftarrow{[\Sigma^{-1}\mathbb{S}/p^\infty, -]} & (\text{Mod}_{\mathbb{S}})_p^\wedge \end{array}$$

where $\mathbf{X} = \text{Mod}_{\mathbb{S}} = \text{Sp}$ is spectra, $\mathbf{U} = \text{Mod}_{\mathbb{S}[\frac{1}{p}]}$ is the full subcategory of spectra on which p is an isomorphism, $\mathbf{Z}^\wedge = (\text{Mod}_{\mathbb{S}})_p^\wedge$ is the full subcategory of p -complete spectra, and $\mathbf{Z}^\vee = (\text{Mod}_{\mathbb{S}})^{p\text{-tors}}$ is

the full subcategory generated under colimits by p -torsion spectra. Then $L = \mathbb{S}[\frac{1}{p}]$ and $C = \Sigma^{-1}\mathbb{S}/p^\infty$. The arithmetic square is

$$\begin{array}{ccc} X & \longrightarrow & X_p \\ \downarrow & & \downarrow \\ \mathbb{S}[\frac{1}{p}] \otimes X & \longrightarrow & \mathbb{S}[\frac{1}{p}] \otimes X_p \end{array}$$

where $X_p = [\Sigma^{-1}\mathbb{S}/p^\infty, X]$ is the p -completion of X .

There is also a less-familiar arithmetic “cofracture square”:

$$\begin{array}{ccc} [\mathbb{S}[\frac{1}{p}], \Sigma^{-1}\mathbb{S}/p^\infty \otimes X] & \longrightarrow & \Sigma^{-1}\mathbb{S}/p^\infty \otimes X \\ \downarrow & & \downarrow \\ [\mathbb{S}[\frac{1}{p}], X] & \longrightarrow & X \end{array}$$

3.2. Example (Nilpotent, local, and complete modules). Fix a commutative ring spectrum R , and an ideal $I \subseteq \pi_0 R$ such that $\sqrt{I}$ is the radical of some finitely generated ideal in $\pi_0 R$. Then we have

$$\begin{array}{ccccc} & \xleftarrow{L_I R \otimes -} & & \xleftarrow{\quad} & \text{Mod}_R^{I\text{-nil}} \\ \text{Mod}_R^{I\text{-loc}} & \xrightarrow{\quad} & \text{Mod}_R & \xrightarrow[\llbracket \Gamma_I R, - \rrbracket]{\Gamma_I R \otimes -} & \text{Mod}_R^{I\text{-cpt}} \\ & \xleftarrow{\llbracket L_I R, - \rrbracket} & & \xleftarrow{\quad} & \end{array}$$

where $\mathbf{Z}^\vee = \text{Mod}_R^{I\text{-nil}}$ is the full subcategory of I -nilpotent modules (consisting of M such that $t^{-1}M = \text{colim}_\omega(M, t) = 0$ for all $t \in I$), and $\mathbf{Z}^\wedge = \text{Mod}_R^{I\text{-cpt}}$ is the full subcategory of I -complete modules (consisting of M such that $\lim_\omega^{\text{op}}(M, t) = 0$ for all $t \in I$). To define $\Gamma_I R$ and $L_I R$, choose $\sqrt{I} = \sqrt{(t_1, \dots, t_k)}$, and define $\Gamma_{t_i} R$ by the fiber sequence

$$\Gamma_{t_i} R \rightarrow R \rightarrow t_i^{-1} R.$$

Then $\Gamma_I R := \Gamma_{t_1} R \otimes_R \cdots \otimes_R \Gamma_{t_k} R$, and $L_I R$ is defined by the cofiber sequence

$$\Gamma_I R \rightarrow R \rightarrow L_I R.$$

Objects of $\mathbf{U} = \text{Mod}_R^{I\text{-loc}}$ are called I -local modules, and the functor $L_I M := L_I R \otimes_R M$ is called I -localization, while $M_I^\vee := [\Gamma_I R, M]$ is called I -completion, and $\Gamma_I M := \Gamma_I R \otimes_R M$ is called local cohomology with respect to I .

Note that if $I = (t)$ is a principal ideal, we recover a generalization of the previous example, e.g., $\mathbf{U} = \text{Mod}_R^{(t)\text{-loc}} = \text{Mod}_{R[\frac{1}{t}]}$.

See [Lur12, Ch. 7] for a summary of this, using ideas which go back to Greenlees-May [GM92].

3.3. Example (L_n localization). Fix a prime p , and for $n \geq 0$ let L_n denote Bousfield localization with respect to $K(0) \oplus \cdots \oplus K(n)$. Then we have a recollement of the form

$$\begin{array}{ccccc} & \xleftarrow{L_n \mathbb{S} \otimes -} & & \xleftarrow{\quad} & C_n \mathbb{S} \\ L_n \mathbb{S} & \xrightarrow{\quad} & \text{Sp}_{(p)} & \xrightarrow[\llbracket C_n \mathbb{S}, - \rrbracket]{C_n \mathbb{S} \otimes -} & \text{Sp}_{C_n \mathbb{S}} \\ & \xleftarrow{\llbracket L_n \mathbb{S}, - \rrbracket} & & \xleftarrow{\quad} & \end{array}$$

where $\mathbf{X} = \text{Sp}_{(p)}$ is the category of p -local spectra, $\mathbf{U} = L_n \mathbb{S}$ is the full subcategory of L_n -local spectra, $\mathbf{Z}^\wedge = \text{Sp}_{C_n \mathbb{S}}$ is the full subcategory of $C_n \mathbb{S}$ -local spectra, and $\mathbf{Z}^\vee = C_n \mathbb{S}$ is the full

subcategory spanned by X such that $L_n X = 0$. (Here $C_n X = \text{fiber of } X \rightarrow L_n X$.) Then $L = L_n \mathbb{S}$ and $C = C_n \mathbb{S}$. The arithmetic fracture square is

$$\begin{array}{ccc} X & \longrightarrow & \llbracket C_n \mathbb{S}, X \rrbracket \\ \downarrow & & \downarrow \\ L_n X & \longrightarrow & L_n \llbracket C_n \mathbb{S}, X \rrbracket \end{array}$$

3.4. *Example* (L_n^f localization). Ther is a “finite” analogue of the previous example, which has the form

$$\begin{array}{ccccc} & \xleftarrow{L_n^f \mathbb{S} \otimes -} & & \xleftarrow{\quad} & C_n^f \text{Sp} \\ L_n^f \text{Sp} & \xrightarrow{\quad} & \text{Sp}_{(p)} & \xrightarrow{C_n^f \mathbb{S} \otimes -} & \wr \\ & \xleftarrow{\llbracket L_n^f \mathbb{S}, - \rrbracket} & & \xleftarrow{\llbracket C_n^f \mathbb{S}, - \rrbracket} & \text{Sp}_{C_n^f \mathbb{S}} \end{array}$$

In this case, $\mathbf{Z}^\vee = C_n^f \text{Sp}$ is the full subcategory generated under colimits by F , where F is any choice of a type $n+1$ finite spectrum, in which case $\mathbf{U} = L_n^f \text{Sp}$ is the full subcategory consisting of Y such that $Y \otimes F = 0$.

3.5. *Example* (Monochromatic localization). This is:

$$\begin{array}{ccccc} & \xleftarrow{L_{n-1}^f \mathbb{S} \otimes -} & & \xleftarrow{\quad} & M_n^f \text{Sp} \\ L_{n-1}^f \text{Sp} & \xrightarrow{\quad} & L_n^f \text{Sp} & \xrightarrow{M_n^f} & \wr \\ & \xleftarrow{\llbracket L_{n-1}^f \mathbb{S}, - \rrbracket} & & \xleftarrow{L_{T(n)}} & \text{Sp}_{T(n)} \end{array}$$

where $\mathbf{X} = L_n^f \text{Sp}$ is the full subcategory of L_n^f -local spectra, $\mathbf{U} = L_{n-1}^f \text{Sp}$ is the full subcategory of L_{n-1}^f -local spectra, $\mathbf{Z}^\vee = M_n^f \text{Sp}$ is the full subcategory of n -monochromatic spectra (i.e., L_n^f -local and killed by L_{n-1}^f), and $\mathbf{Z}^\wedge = \text{Sp}_{T(n)}$ is the full subcategory of $T(n)$ -local spectra, where $T(n)$ is a telescope on a type n finite spectrum. Then $L = L_{n-1}^f \mathbb{S}$ and $C = L_n^f C_{n-1}^f \mathbb{S}$. The arithmetic square is

$$\begin{array}{ccc} L_n^f X & \longrightarrow & L_{T(n)} X \\ \downarrow & & \downarrow \\ L_{n-1}^f X & \longrightarrow & L_{n-1}^f L_{T(n)} X \end{array}$$

There is “non-finite” analogue of this example, with L_j^f replaced with L_j , M_j^f replaced by M_j , and $T(j)$ replaced with $K(j)$.

3.6. *Example* (Localization for dualizable rings). Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞ -category such that

- $\mathcal{C}$ is compactly generated, i.e., it is generated under colimits by its compact objects, and
- the compact objects of $\mathcal{C}$ are precisely the dualizable objects.

Let $A \in \text{Alg}(\mathcal{C}^\diamond)$, i.e., an associative ring object¹ in $\mathcal{C}$ whose underlying object is dualizable in $\mathcal{C}$. This gives a recollement of the form

$$\begin{array}{ccccc} & \xleftarrow{U_A \otimes -} & & \xleftarrow{\quad} & \mathcal{C}_{A\text{-tors}} \\ \mathcal{C}[A^{-1}] & \xrightarrow{\quad} & \mathcal{C} & \xrightarrow{V_A \otimes -} & \wr \\ & \xleftarrow{\llbracket U_A, - \rrbracket} & & \xleftarrow{\llbracket V_A, - \rrbracket} & \mathcal{C}_{A\text{-cpt}} \end{array}$$

¹In fact, it should suffice to have a unital object $\eta: \mathbb{1} \rightarrow A$ such that $A \otimes \eta: A \otimes \mathbb{1} \rightarrow A \otimes A$ admits a retraction.

where

- $\mathbf{Z}^\vee = \mathcal{C}_{A\text{-tors}}$ is the full subcategory of A -torsion objects (i.e., the full stable subcategory of $\mathcal{C}$ generated under colimits by objects of the form $A \otimes K$ with $K \in \mathcal{C}^\diamond$),
- $\mathbf{Z}^\wedge = \mathcal{C}_{A\text{-cpt}}$ is the full subcategory of A -complete objects² (consisting of $X \in \mathcal{C}$ such that $[Y, X] = 0$ whenever $A \otimes Y = 0$), and
- $\mathbf{U} = \mathcal{C}[A^{-1}]$ is the full subcategory of A^{-1} -local objects (consisting of $Y \in \mathcal{C}$ such that $[X, Y] = 0$ for all $X \in \mathcal{C}_{A\text{-tors}}$, or equivalently, such that $A \otimes Y = 0$).

The cofiber sequence $V_A \rightarrow \mathbb{1} \rightarrow U_A$ is defined so that

$$V_A = \operatorname{colim}_{\Delta^{\text{op}}}([n] \mapsto (A^{\otimes n+1})^\vee).$$

See Mathew-Naumann-Noel, “Nilpotence and descent in equivariant stable homotopy theory” [MNN15, Sect. 2-3] for more on this.

Example 3.4 is actually a special case of this example, using any finite type $n+1$ spectrum A which admits the structure of an associative ring, e.g., the endomorphism ring of any type $n+1$ finite spectrum, or an appropriately chosen generalized Moore spectrum [Bur22].

3.7. Example (Equivariant spectra and isotropy separation). Let G be a compact Lie group. A **family** $\mathcal{F}$ is a collection of closed subgroups of G such that $H \in \mathcal{F}$ implies $H' \in \mathcal{F}$ whenever $H' \subseteq gHg^{-1}$ for some $g \in G$. Families are in bijective correspondence with subobjects of the terminal object of $\mathcal{S}_G := \operatorname{Fun}(\operatorname{Orb}_G^{\text{op}}, \mathcal{S})$, and given a family $\mathcal{F}$ we write $E\mathcal{F}$ for the corresponding presheaf on Orb_G , characterized by

$$E\mathcal{F}^H \approx \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$

Note that in $\mathcal{S}_G$ the diagonal map of $E\mathcal{F}$ is an equivalence: $E\mathcal{F} \xrightarrow{\sim} E\mathcal{F} \times E\mathcal{F}$. The cofiber of the projection is denoted $\widetilde{E\mathcal{F}}$.

Let $C = \Sigma_+^\infty E\mathcal{F}$ and $L = \Sigma^\infty \widetilde{E\mathcal{F}}$ in Sp_G . Then we get a recollement:

$$\begin{array}{ccc} \operatorname{Sp}_G^{\mathcal{F}\text{-acyc}} & \begin{array}{c} \xleftarrow{\Sigma^\infty \widetilde{E\mathcal{F}} \otimes -} \\ \xrightarrow{\quad \quad \quad} \\ \xleftarrow{\quad \quad \quad} \\ \llbracket \Sigma^\infty \widetilde{E\mathcal{F}}, - \rrbracket \end{array} & \operatorname{Sp}_G \\ & & \begin{array}{c} \xleftarrow{\Sigma_+^\infty E\mathcal{F} \otimes -} \\ \xrightarrow{\quad \quad \quad} \\ \llbracket \Sigma_+^\infty E\mathcal{F}, - \rrbracket \\ \xleftarrow{\quad \quad \quad} \end{array} \\ & & \begin{array}{c} \operatorname{Sp}_G^{\mathcal{F}\text{-cell}} \\ \Downarrow \\ \operatorname{Sp}_G^{\mathcal{F}\text{-Borel}} \end{array} \end{array}$$

We can describe this situation as follows:

- $\mathbf{Z}^\vee = \operatorname{Sp}_G^{\mathcal{F}\text{-cell}}$, the full stable subcategory of G -spectra generated under colimits by $\{\Sigma_+^\infty G/H \mid H \in \mathcal{F}\}$. To see this, note that $E\mathcal{F} \times G/H \xrightarrow{\sim} G/H$ if $H \in \mathcal{F}$ whence $\Sigma_+^\infty G/H \in \mathbf{Z}^\vee$, while $E\mathcal{F}$ itself can be built out of cells based on G/H with $H \in \mathcal{F}$.
- $\mathbf{U} = \operatorname{Sp}_G^{\mathcal{F}\text{-acyc}}$, the full subcategory of G -spectra spanned by X such that $X^H \approx 0$ for all $H \in \mathcal{F}$. (Here $X^H := [\Sigma_+^\infty G/H, X] \in \operatorname{Sp}$.)
- $\mathbf{Z}^\wedge = \operatorname{Sp}_G^{\mathcal{F}\text{-Borel}}$, the full subcategory of G -spectra spanned by X such that $X \xrightarrow{\sim} \llbracket \Sigma_+^\infty E\mathcal{F}, X \rrbracket$.

See Mathew-Naumann-Noel, “Nilpotence and descent in equivariant stable homotopy theory” [MNN17, Sect. 6] for more on this.³

A particular example of this is the family

$$\mathcal{F}_{N \not\leq} := \{H \mid N \not\leq H\}$$

²or A -local objects

³They use different terminology: $\operatorname{Sp}_G^{\mathcal{F}\text{-cell}}$ are what they call the $\mathcal{F}$ -torsion spectra; $\operatorname{Sp}_G^{\mathcal{F}\text{-Borel}}$ are what they call the $\mathcal{F}$ -complete spectra; $\operatorname{Sp}_G^{\mathcal{F}\text{-acyc}}$ are what they call the $\mathcal{F}^{-1}$ -local spectra. That is because this example is also a special case of Example 3.6, using $A = \prod_{H \leq G} (\Sigma_+^\infty G/H)^\vee$.

of subgroups which do not contain N , where $N \trianglelefteq G$ is a closed normal subgroup. Then the functor $\mathrm{Sp}_{G/N} \rightarrow \mathrm{Sp}_G$ induced by restriction along the quotient map $G \rightarrow G/N$ induces an equivalence $\mathrm{Sp}_{G/N} \xrightarrow{\sim} \mathrm{Sp}_G^{\mathcal{F}_{N\mathbb{Z}}}$. (See [LMSM86, Sect. II.9].)

3.8. *Example* (Equivariant spectra for $G = C_p$). Let $G = C_p$ be a cyclic group of prime order. Then $\mathcal{F}_{C_p\mathbb{Z}} = \{e\}$, and so $E\mathcal{F}_{C_p\mathbb{Z}} = EC_p$, the free contractible C_p -space. With this family the previous example specializes to a recollement of the form

$$\begin{array}{ccc} \mathrm{Sp}_{C_p}^{\mathrm{u-acyc}} & \begin{array}{c} \xleftarrow{\Sigma^\infty \widetilde{EC}_p \otimes -} \\ \xrightarrow{\quad \quad \quad} \\ \xleftarrow{\quad \quad \quad} \\ \xrightarrow{\quad \quad \quad} \\ \xleftarrow{\quad \quad \quad} \end{array} & \mathrm{Sp}_{C_p} \\ & & \begin{array}{c} \xleftarrow{\Sigma_+^\infty EC_p \otimes -} \\ \xrightarrow{\quad \quad \quad} \\ \xleftarrow{\quad \quad \quad} \\ \xrightarrow{\quad \quad \quad} \\ \xleftarrow{\quad \quad \quad} \end{array} \begin{array}{c} \mathrm{Sp}_{C_p}^{\mathrm{free}} \\ \Downarrow \\ \mathrm{Sp}_{C_p}^{\mathrm{Borel}} \end{array} \end{array}$$

where

- $\mathbf{Z}^\vee = \mathrm{Sp}_{C_p}^{\mathrm{free}}$ is the full subcategory of free C_p -spectra (i.e., the full stable subcategory generated under colimits by $\Sigma_+^\infty C_p/e$),
- $\mathbf{Z}^\wedge = \mathrm{Sp}_{C_p}^{\mathrm{Borel}}$ is the full subcategory of Borel C_p -spectra (i.e., the X such that $X \xrightarrow{\sim} \llbracket \Sigma_+^\infty EC_p, X \rrbracket$), and
- $\mathbf{U} = \mathrm{Sp}_{C_p}^{\mathrm{u-acyc}}$ is the full subcategory of C_p -spectra whose underlying ordinary spectrum is contractible.

Furthermore, we have identifications

$$\mathbf{U} \approx \mathrm{Sp}_{C_p}^{\mathrm{u-acyc}} \approx \mathrm{Sp}, \quad \mathbf{Z} \approx \mathrm{Sp}_{C_p}^{\mathrm{free}} \approx \mathrm{Sp}_{C_p}^{\mathrm{Borel}} \approx \mathrm{Sp}^{BC_p} \approx \mathrm{Mod}_{\mathbb{S}[C_p]}.$$

That is, $\mathbf{Z}$ is the category of spectra equipped with a C_p -action, in which case $j^*: \mathrm{Sp}_{C_p} \rightarrow \mathrm{Sp}^{BC_p}$ the underlying spectrum functor (i.e., $j^* = [\Sigma_+^\infty C_p/e, -]$), the functor $j_\#: \mathrm{Sp}^{BC_p} \rightarrow \mathrm{Sp}_{C_p}$ is given by $\Sigma_+^\infty C_p/e \otimes_{\mathbb{S}[C_p]} -$, and $j_!: \mathrm{Sp}^{BC_p} \rightarrow \mathrm{Sp}_{C_p}$ is “Borelification”. Furthermore, the functor $i^!: \mathrm{Sp}_{C_p} \rightarrow \mathrm{Sp}$ coincides with the C_p -fixed point functor (i.e., $i^! = [\mathbb{1}, -]$).

The associated arithmetic square is

$$\begin{array}{ccc} X & \xrightarrow{\quad \quad \quad} & \llbracket \Sigma_+^\infty EC_p, X \rrbracket \\ \downarrow & & \downarrow \\ \Sigma^\infty \widetilde{EC}_p \otimes X & \longrightarrow & \Sigma^\infty \widetilde{EC}_p \otimes \llbracket \Sigma_+^\infty EC_p, X \rrbracket \end{array}$$

Taking C_p -fixed points of this square gives a pullback square in Sp of the form

$$\begin{array}{ccc} X^{C_p} & \longrightarrow & X^{hC_p} \\ \downarrow & & \downarrow \\ \Phi^{C_p} X & \longrightarrow & X^{tC_p} \end{array}$$

where X^{hC_p} is the C_p -fixed points of the underlying spectrum with C_p -action, Φ^{C_p} is geometric fixed points, and X^{tC_p} is the Tate spectrum. (Much of this can be found in Dennis Nardin, “Introduction to equivariant homotopy theory” [Nar20].)

In particular, we can explicitly reconstruct Sp_{C_p} using (1.3), so that an object $X \in \mathrm{Sp}_{C_p}$ can be encoded by triple of data

$$(U, Z, \gamma), \quad U \in \mathrm{Sp}, \quad Z \in \mathrm{Sp}^{BC_p}, \quad (\gamma: U \rightarrow Z^{tC_p}) \in \mathrm{Sp}.$$

Under this encoding, $U = \Phi^{C_p} X$ and $Z = [\Sigma_+^\infty C_p/e, X]$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN, URBANA, IL
 Email address: rezk@illinois.edu