

MODULI STACKS OF G -CURVES IN HOMOTOPY THEORY AT HEIGHT $p - 1$

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ABSTRACT. Let p be odd and $G' = \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$ the maximal finite subgroup of the Morava stabilizer group at height $p-1$. Inverse Galois theory produces from G' alone a curve X , the unique curve of minimal genus with $\text{Aut}(X) \simeq G'$; its ramification, its field of definition and its equation are consequences of the group, not choices. We prove a G' -equivariant equivalence between the deformations of X and Lubin–Tate space, so that the Lubin–Tate action of G' is the action of $\text{Aut}(X)$ on deformations of the curve. The proof is a coordinate-free Kodaira–Spencer argument reducing to a single character count. The action becomes explicit: G' acts through $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$ shifting and scaling $p+1$ points on $\mathbb{P}^1$. From this we compute $H^*(G', \pi_* E_{p-1})$ and its Tate cohomology. One identity, $\pi^{p-1} = -p$, runs through every section.

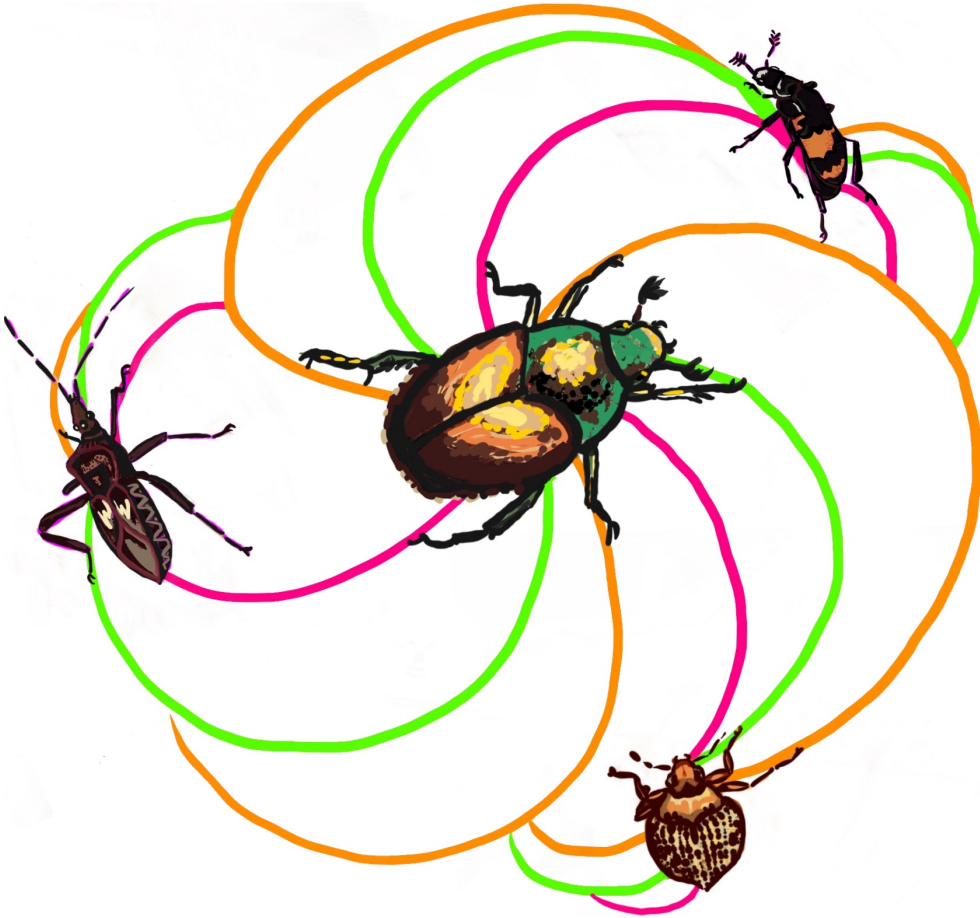


FIGURE 1. Moduli stack of ramification points (the beetles) of a cover. As the beetles walk, the ramification points deform.

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INTRODUCTION

0.1. Motivation: the Lubin–Tate Action, and Dreams of a Perfect World.

Fix an odd prime p and a one-dimensional formal group H_h of height h over a perfect field k of characteristic p . By Lubin–Tate ([LT66a](#)), H_h has a universal deformation over a smooth complete local ring $W(k)[[u_1, \dots, u_{h-1}]]$, and its automorphism group $\mathbb{G}_h := \text{Aut}(H_h)$ — the Morava stabilizer group — acts on the deformation space Def_{H_h} . This is the [Lubin–Tate action](#).

Two spectra carry this arithmetic into homotopy theory. The [Lubin–Tate spectrum](#) E_h is the even periodic ring spectrum with

$$\pi_* E_h = W(k)[[u_1, \dots, u_{h-1}]] [u^{\pm 1}], \quad |u| = -2,$$

whose formal group is the universal deformation of H_h : its homotopy is the Lubin–Tate deformation ring, made periodic. The [Morava \$K\$ -theory](#) $K(h)$ is the residue field of E_h , with $\pi_* K(h) = \mathbb{F}_p[v_h^{\pm 1}]$, $|v_h| = 2(p^h - 1)$; it sees exactly height h and nothing else, and $K(h)$ -localization $L_{K(h)}$ is the operation of restricting the stable homotopy category to its height- h layer. By Goerss–Hopkins–Miller ([GH04](#), [Rez98](#)), E_h admits a unique E_∞ -ring structure, and the Lubin–Tate action lifts, uniquely, to an action of $\mathbb{G}_h$ on E_h by E_∞ -maps. One action, load-bearing in several parts of mathematics at once: the input to the $K(h)$ -local sphere, the absolute Galois action

of the $K(h)$ -local category, the local avatar of Langlands. We take the angles in turn in Subsection 0.2. First, the question that organizes this paper.

The action of the full group $\mathbb{G}_h$ is intractable. The action of its maximal finite subgroup $G' \subseteq \mathbb{G}_h$ is where the interesting homotopy theory lives (Subsection 0.3), and it is where geometry can help. The deformation space is a formal disk, and the G' -action is a nonlinear action of a finite group on a power series ring; such an action becomes understandable the moment the disk is recognized as the formal neighborhood of a point on a moduli space of geometric objects, on which G' acts by honest symmetries. At height 2 the moduli space is the moduli of elliptic curves and the recognition is Serre–Tate. The question is what replaces it.

Question. For which heights h and primes p does the action of the maximal finite subgroup $G' \subseteq \mathbb{G}_h$ on Def_{H_h} admit a geometric model, in the sense of (Ray25): a moduli problem of geometric objects together with a G' -equivariant equivalence onto Def_{H_h} ? And can it be built uniformly, for all interesting h and p at once?

We hunger for more than existence. A model found by inspection at each height is a catalogue; a model *forced* by the group is a theorem. In a perfect world the curve is a consequence of the group, the action is a consequence of the curve, and the equivalence is a consequence of a single character count — nothing chosen, everything ordained.

Dream Question. Is there a construction taking as input *only* the abstract group $G' \subseteq \mathbb{G}_h$ and returning its Lubin–Tate action in geometric form:

- a moduli stack $\mathcal{M}$ of decorated curves with G' -action, canonically determined by G' ;
- on which the action is *computable* — explicit generators acting on an explicit representing ring;
- together with a G' -equivariant equivalence of moduli problems

$$\Phi: \mathcal{M} \xrightarrow{\sim} \mathrm{Def}_{H_h},$$

the target carrying the canonical Lubin–Tate action?

This paper treats the first interesting case: $h = p - 1$ for p odd, where $G' = \mathbb{Z}/p \rtimes_{\chi} \mathbb{Z}/(p-1)^2$ with χ surjective onto $\mathbb{F}_p^{\times}$ — the twisted love of multiplication and addition. Why this G' is the one to start with is Subsection 0.3.

Rough Answer. Yes, and the dream version holds. The whole paper is one chain, in which the group is the only input:

$$G' \Rightarrow X \Rightarrow \mathrm{Def}_{(X,G)} \quad \text{and then} \quad \mathrm{Conf}(\mathbb{P}^1, p+1) \xleftarrow{L} \mathrm{Def}_{(X,G)} \xrightarrow{\widehat{\mathcal{F}}^1} \mathrm{Def}_{\widehat{\mathcal{F}}^1(X,G)}^{FG \dim 1}.$$

The double arrows are inverse Galois theory (Part 1): the curve X , its ramification, its field of definition, and finally its equation are consequences of G' , not choices (Theorem A). The functor L (Theorem B.1) makes the model computable: the moduli are configurations of the branch points, and G' acts by shifting and scaling them (Theorem B). The functor $\widehat{\mathcal{F}}^1$ (Theorem C) identifies the model with Lubin–Tate space, by a Kodaira–Spencer argument that reduces to one character count. Composing the two is the title claim.

The paper is an algebraic geometry paper whose theorem is about homotopy theory. It is written to be intelligible to topologists, arithmetic geometers, and algebraic geometers alike.

0.2. Five Angles on One Action. The Lubin–Tate action is one object seen from five directions.

0.2.1. Arithmetic. The action predates its homotopical use. Lubin–Tate formal groups were introduced to make local class field theory explicit (LT65), and the non-abelian version is the Lubin–Tate tower, whose ℓ -adic cohomology realizes the local Langlands and Jacquet–Langlands correspondences for $\mathrm{GL}_h(\mathbb{Q}_p)$ (HT01, Boy99). The stabilizer group $\mathbb{G}_h$ acts compatibly with the Gross–Hopkins period map to $\mathbb{P}^{h-1}$ (HG94) and with the duality between the Lubin–Tate and Drinfeld towers (SW13); the p -adic holography models (GKP⁺17, HMSS18) are built on the Drinfeld side of this duality via Bruhat–Tits buildings and Mumford curves. A statement about the action proved on either side is a statement about the same object.

0.2.2. Physical. A quantum field theory on a genus- g worldsheet has partition function a section of a line bundle on $\mathcal{M}_g$; the curve X is a point of that moduli space, not a worldsheet. At $p = 3$ and $h = 2$, we consider the supersingular elliptic curve $y^2 = x^3 - x$, and $E_2^{hG_{24}} \simeq L_{K(2)}\mathrm{tmf}$: the deformation space of the curve is Lubin–Tate space, and the spectrum that sees it is topological modular forms. Stolz–Teichner (ST11) conjecture that tmf classifies 2-dimensional $\mathcal{N}=(0,1)$ supersymmetric theories up to deformation — the Witten genus (Wit87) is its partition function, and what it misses is torsion. That torsion, 3-locally, is the cohomology of the curve’s automorphism group acting on its own deformation space: the object this paper computes at every p . It is realized by sigma models (GJFW19), detected by a secondary genus (GJF23), and encodes global anomalies of heterotic strings (TY23). For $p \geq 5$ no field theory is known whose partition function sees E_{p-1} ; the Stolz–Teichner ladder has no rung above tmf . This paper does not build that rung. It builds the geometric data any such theory would have to reproduce: the formal neighborhood of a single curve in $\mathcal{M}_g$, with its full symmetry group acting. If the ladder extends, it extends here.

0.2.3. Chromatic. The moduli stack of one-dimensional formal groups controls the stable homotopy category (Qui69, Mor89b, MRW77): height stratifies the stack, the chromatic tower filters the sphere, and chromatic convergence (Rav92) makes the filtration exhaustive. By Devinatz–Hopkins (DH04) the $K(h)$ -local category is the $\mathbb{G}_h$ -equivariant homotopy theory of E_h , and E_h is a pro- $\mathbb{G}_h$ -Galois extension of $L_{K(h)}\mathbb{S}$ (DH04, Rog08). Layer by layer, the Lubin–Tate action is the entire input to the stable homotopy groups of spheres; a finite subgroup $G' \leq \mathbb{G}_h$ cuts out a sub-Galois extension — the one this paper computes at $h = p - 1$.

0.2.4. Hecke. Behrens (Beh06) resolves the $K(2)$ -local sphere at $p = 3$ from TMF and $\mathrm{TMF}_0(\ell)$ via the Hecke operator T_ℓ ; Behrens–Lawson (BL10) run the same program at every height through unitary Shimura varieties, expressing $K(h)$ -locally that topological automorphic forms are products of fixed points $E_h^{h \cdot \mathrm{Aut}(x)}$ over a dense supersingular locus (Nau08). The maximal finite subgroup stabilizes the deepest building vertex, is the automorphism group of the distinguished supersingular point, and is the torsion of the maximal order through which Hecke operators act (Rez06, Zhu19). Running this program at height $p - 1$ needs exactly two local inputs: the group G' , and $\pi_* E_{p-1}$ as a G' -module. The second is supplied by this paper.

0.2.5. *Computational.* E_∞ -rigidity gives fixed points E_h^{hF} for every finite $F \leq \mathbb{G}_h$: higher real K -theories, beginning with $KO_p^\wedge = E_1^{hC_2}$ and $K(2)$ -local TMF. These are the most computable approximations to $L_{K(h)}\mathbb{S}$: a form of $E_4^{hC_8}$ detects the Kervaire classes (HHR16), and $E_{p-1}^{hC_p}$ resolved the Smith–Toda program negatively (Nav10). The return from finite subgroups to $\mathbb{G}_h$ is systematic by Henn (Hen98), with finite resolutions realized at $h = 2$ (GHMR05, BG18); the layers are governed by Tate cohomology $\widehat{H}^*(F, \pi_* E_h)$. Theorem E computes that cohomology at height $p - 1$.

0.3. **Which Finite Subgroups Matter, and Why.** Three claims of different strengths; the distinction is the honest advertisement for Theorems D and E.

Claim 1 (group theory). $\mathbb{G}_h$ contains p -torsion if and only if $p - 1 \mid h$: an element of order p generates $\mathbb{Q}_p(\zeta_p)$ inside the division algebra D , and $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p - 1$. The maximal finite subgroups are classified.

Lemma 0.1. (Buj12, Hew95) For $p > 2$ and $h = (p - 1)p^{k-1}m$ with $p \nmid m$, $\mathbb{G}_h$ has exactly $k + 1$ conjugacy classes of maximal finite subgroups,

$$\mathbb{Z}/(p^h - 1) \quad \text{and} \quad \mathbb{Z}/p^i \rtimes \mathbb{Z}/(p^{h_i} - 1)(p - 1) \quad (1 \leq i \leq k),$$

where $h_i = h/\varphi(p^i)$. For $i = k$ the subgroup is $\mathbb{Z}/p^k \rtimes \mathbb{Z}/(p - 1)^2$, in a single conjugacy class.

Claim 2 (cohomology). Torsion becomes cohomological structure. Subgroups containing $\mathbb{Z}/p$ have periodic Tate cohomology, which feeds periodicity classes into every descent spectral sequence $H^*(G', \pi_* E_h) \Rightarrow \pi_* E_h^{hG'}$. The shape of Theorem E — a polynomial generator β and an invertible Δ — is this periodicity at height $p - 1$.

Claim 3 (homotopy). For $2p - 2 > h^2 + h$ the $K(h)$ -local category is algebraic (Pst21, BSS20): no exotic phenomena. So subgroups with p -torsion are the only possible carriers of non-algebraic behavior. That they always carry something is not a theorem; it is the verdict of every computed case, most directly Nave’s theorem (Nav10), which is Claim 2 for the wild subgroup C_p alone. Theorems D and E supply the Claim 2 output for the full G' . This is the input any Claim 3 statement at height $p - 1$ must consume.

At $h = p - 1$, the first height with p -torsion, the maximal finite subgroup is $G' = \mathbb{Z}/p \rtimes \mathbb{Z}/(p - 1)^2$, and $\pi_*(E_{p-1}^{hG'})$ is our goal. Heights $p^{k-1}(p - 1)$ for $k \geq 2$ are the next paper.

0.4. **The One Identity.** The construction is organized around one identity of local number theory. In $\mathbb{Z}_p[\zeta_p]$, the element $\pi = \zeta_p - 1$ satisfies

$$\pi^{p-1} = -p \cdot (\text{unit})$$

because $[p](T)/T = ((1 + T)^p - 1)/T$ is Eisenstein of degree $p - 1$. The tame group $C_{(p-1)^2} \subset G'$ acts on a one-dimensional tangent line on both sides of the correspondence — the cotangent line of the curve at its wild point, and the tangent line $\pi_2 E_{p-1}$ of the formal group — and on both sides the conjugation character of the wild C_p is the $(-m)$ -th power of that tangent character. On the curve, $m = p - 1$ is forced by minimality; in $\mathbb{Q}_p(\zeta_p)$, $e = p - 1$ is forced by the Eisenstein polynomial. Same exponent, two computations:

the curve X in characteristic p	the field $\mathbb{Q}_p(\zeta_p)$
tangent character θ of $C_{(p-1)^2}$ at the wild point	tangent character $\tau \mapsto \rho$ of $C_{(p-1)^2} \subset \mathcal{O}_D^\times$ on $\pi_2 E_{p-1}$
conjugation character χ of $C_{(p-1)^2}$ on C_p	cyclotomic character $\tau \mapsto \chi_{\text{cyc}}(\tau)$ on μ_p
ramification break m of $X \rightarrow X/C_p$	ramification index e of $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$
$\chi = \theta^{-m}$ (Lemma 3.4)	$\chi_{\text{cyc}} = \rho^{-(p-1)}$ (Lemma A.1)
$m = r - 1$ for r colliding branch points	$\deg([p](T)/T) = p - 1$

The break of the curve, the exponent of the semidirect product, the degree of the field of definition, and the height are all $p - 1$, for one reason.

Remark 0.2. The constant π with $\pi^{p-1} = -p$, which recurs throughout this paper, is the p -adic counterpart of $2\pi i$ for the additive character: the Artin–Schreier sheaf attached to f is the characteristic- p analogue of the exponential $\mathcal{D}$ -module e^f , and its trace function is the finite-field version of the oscillatory integral $\int e^f$ of a Landau–Ginzburg model with superpotential f .

0.5. Main Results. Notation is fixed in Subsection 0.9, at the end of this introduction. First, the group determines the curve.

Theorem A. (*Tiny and Unique*) *Let p be an odd prime and k a perfect field containing $\mathbb{F}_{p-1}$. Among smooth projective geometrically connected curves over k with a faithful G' -action, the minimal genus is $g = \frac{(p-1)(p-2)}{2}$, attained precisely by the k -forms of*

$$X: y^p - y = z^{p-1}.$$

On the minimal stratum the following are consequences, not hypotheses:

- (1) $X/P \cong \mathbb{P}_k^1$ for P the Sylow p -subgroup;
- (2) $X \rightarrow X/P$ is branched at one point, with break $m = p - 1$;
- (3) X has p -rank 0;
- (4) $\text{Aut}(X) = G'$ for $p \geq 5$, and $\text{Aut}(X, O) = G'$ at $p = 3$.
- (5) If $\wp(k) = k$ and $(k^\times)^{p-1} = k^\times$ — in particular if $k = \bar{k}$ — the curve is unique up to k -isomorphism.

Next, the curve determines the model, and the model is computable: its representing ring is a symmetric algebra on the configuration of branch points, localized away from collisions and completed at the characteristic locus.

Theorem B.1. *There is a G'/G -equivariant isomorphism of moduli problems*

$$L: \text{Def}_{(X, C_{p-1})} \simeq \text{Def}_{(\mathbb{P}^1, p+1)} \simeq \text{Def}_{(\mathbb{A}^1, p)}.$$

Theorem B.2. *The stack $\text{Conf}(\mathbb{A}^1, n)$ is representable, and its representing ring is, as a Σ_n -module, $R \simeq \text{Sym}(\lambda)[N^{-1}] // \mathbb{G}_m$.*

Theorem B. (*Act Up*) *With $\bar{p}$, N , I as in Subsection 0.9 ($n = p$), the formal moduli problem $\text{Def}_{(X, G)}$ is represented by the graded ring*

$$\Lambda \simeq \text{Sym}(\bar{p})[N^{-1}]_I^\wedge,$$

as a G' -representation; equivalently $\text{Def}_{(X, G)} \simeq \text{Spf}(\Lambda) // \mathbb{G}_m$ as formal stacks.

In words: G' acts on the moduli of the curve by shifting and scaling the p affine branch points. This is the entire wild action of C_p , encoded as combinatorics of ramification — the mechanism is explained in Subsection 5 (for the special fiber) and Theorem 8.20 (for the deformation stack).

We use G' to construct the G' -equivariant functor $\widehat{\mathcal{F}}^1$ by taking the formal group of the Jacobian and projecting onto the idempotent summand cut by the central tame character (Theorem C.1 and Defn 10.3). We show this functor is always surjective, and the injectivity of the functor is the minimal genus theorem seen from the formal group.

Theorem C. (*Equivalent Exchange*) *Let $G := C_{p-1} \subseteq Z(G')$ and X as above. The functor*

$$\widehat{\mathcal{F}}^1: \mathrm{Def}_{(X,G)}^\star \longrightarrow \left(\mathrm{Def}_{\widehat{\mathcal{F}}^1(X,G)}^{FG \dim 1} \right)^\star$$

is a G' -equivariant equivalence of formal moduli problems, where $\widehat{\mathcal{F}}^1(X, G)$ is a one-dimensional formal group of height $p - 1$ and the G' -action on the target is the canonical Lubin–Tate action. Consequently

$$\mathrm{Def}_{(X,G)}^\star // G' \simeq \left(\mathrm{Def}_{\widehat{\mathcal{F}}^1(X,G)}^{FG \dim 1} \right)^\star // G'.$$

Remark 0.3. The graded version is Theorem C(b). Since $\mathrm{Def}_F^\star // \mathbb{G}_h \simeq \widehat{\mathcal{H}}(h)$ is the geometric form of the Devinatz–Hopkins description, Theorem C replaces a sub-Galois extension of $L_{K(h)}\mathbb{S}$ by a moduli problem of curves.

Finally, the payoff: $\pi_* E_{p-1}$ as a G' -representation, and its Tate cohomology — Claim 2 of Subsection 0.3, computed.

Theorem D. (*Up Shot*) *Let $\widetilde{\Lambda} := \mathrm{Sym}(\bar{\rho})$. As G' -representations,*

$$(E_{p-1})_* \simeq \widetilde{\Lambda}[N^{-1}]_I^\wedge =: \Lambda,$$

and

$$H^*(G', E_*) \simeq H^*(G', \widetilde{\Lambda})[\Delta^{-1}]_I^\wedge \simeq H^*(G', \widetilde{\Lambda})[\Delta^{-1}],$$

the disappearance of the completion being Lemma 10.47.

Theorem E. *The Tate cohomology of the G' -module $(E_{p-1})_*$ is that of $\widetilde{\Lambda}[\Delta^{\pm 1}]$:*

$$\widehat{H}^*(G', \widetilde{\Lambda}[\Delta^{\pm 1}]) \simeq \mathbb{Z}_p[\alpha, \beta, \Delta^{\pm 1}]/\alpha^2,$$

with $|\alpha| = (1, 2(p-1))$, $|\beta| = (2, 2p(p-1))$, $|\Delta| = (0, 2p(p-1)^2)$.

0.6. Strategy. Where the theorems come from, as a narrative: begin at $p = 3$, where every step is classical; say exactly what breaks past height 2; show how inverse Galois theory repairs it.

0.6.1. Algebraic Geometry Enters the Fray at Height 2: Elliptic Curves as Configuration Spaces. Elliptic curves appear in homotopy theory because they model the Lubin–Tate action of finite subgroups at heights 1 and 2 (Mor89a, IS05, Bea17).

We stress that what follows is not an analogy but a specialization of Section 0.6.3: the curve $X: y^p - y = x^{p-1}$ of the general construction has genus $\frac{(p-1)(p-2)}{2}$, so at $p = 3$ it is the supersingular elliptic curve $E: y^3 - y = x^2$ with $\mathrm{Aut}_k(E) \simeq C_3 \rtimes C_4$. Everything in this subsection is the $p = 3$ member of the family built in the rest of the paper, and the reader may keep it as a running example.

- Minimality of genus with automorphism group $G' \simeq \mathbb{Z}/3 \rtimes \mathbb{Z}/4$ forces E to be supersingular — formal group of height 2 — and quotienting gives a span of Galois covers

$$\mathbb{P}^1 \xleftarrow{C_2} E \xrightarrow{C_3} \mathbb{P}^1.$$

- By Serre–Tate (CS64), deformations of E coincide with deformations of its formal group.
- The ramification divisor of the C_2 -cover $p_y: E \rightarrow \mathbb{P}^1$ coincides with the level-2 structure on E ; the action of $GL_2(\mathbb{F}_2) \simeq \Sigma_3$ on level structures coincides with the permutation action of Σ_3 on the branch points of p_y , and with the action of $\text{Aut}(E)/C_2$ on the representing ring of $\text{Def}_{(E, C_2)}$.
- Since C_2 is central, $\text{Def}_{(E, C_2)}$ carries an action of the full $G' \simeq \text{Aut}(E) = \text{Aut}(E, C_2)$.

From this: the automorphisms of the elliptic curve acting on its deformation space model the Lubin–Tate action restricted to G' ,

$$\text{Def}_{(E, C_2)}^* // G' \simeq \text{Def}_{\mathcal{F}(E)}^* // G', \quad \text{hence} \quad H^*(G', \mathcal{O}(\text{Def}_{(E, C_2)}^*)) \simeq H^*(G', \pi_*(E_2)).$$

Remark 0.4. The reader might expect the C_4 -cover rather than C_2 ; but the quotient of E by C_4 is not Galois. The choice of the *central* tame subgroup is not a convenience — it is the selection principle that generalizes.

Remark 0.5. A global stack appears naturally here: $\mathcal{M}_{1,1}^N$ completes at (E, L) to this deformation problem, and the global form was used beautifully by Stojanoska (Sto12). We leave the global setting to another place and time.

0.6.2. *What Breaks Past Height 2.* Serre–Tate is gone: no one-dimensional variety has formal group of height $h > 2$ due to Cartier duality, so the model must be a curve whose *Jacobian* contains a one-dimensional height- h formal group compatibly in families — the variety must not split as an algebraic group, yet *must* split on the level of p -divisible groups — and the moduli of the curve must be equivalent to deformations of that one puny summand.

Level structure is gone too: higher-genus curves have no group structure. Curves remain the right vehicle for one reason: they are the varieties whose moduli one can still control after giving up group structure - *ramification data replaces the level structure one gave up*.

It is a lot to ask, and it fully works at $h = p - 1$. Height $p - 1$ deserves its own paper: it is the only level at which the model is literally a *family of curves* — at higher levels the branch points cannot move, and the sequel replaces configurations by a stack of surgeries at fixed branch points.

0.6.3. *Geometric Modelling at Height $p - 1$: Twisted Love of Multiplication and Addition.* At height $p - 1$ the construction from a perfect world exists, birthed by the twisted love of multiplication and addition, $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$. Offered the group $G' \subset \mathbb{G}_h$ as the seed, we sow the model: from the curve X down to the functor inducing the equivalence, following the guide of the elliptic curve. The conceptual insights are two: inverse Galois theory constructs the curve, and ramification data replaces level structure.

- In Part 1 we take X to be the minimal genus curve with automorphism group $G' \simeq C_p \rtimes C_{(p-1)^2}$ (Theorem A); quotienting gives the roof of Galois covers

$$\mathbb{P}^1 \xleftarrow{C_{p-1}} X \xrightarrow{C_p} \mathbb{P}^1.$$

- In Part 9.1 we show the moduli stack $\mathrm{Def}_{(X, C_{p-1})}$ is representable by the ring of the configuration space of p points on $\mathbb{A}^1$, with $\mathrm{Aut}(X)/C_{p-1} \hookrightarrow \Sigma_p$ acting through permutations of the branch points of $q: X \rightarrow X/C_{p-1}$ away from infinity (Theorems B.1, B.2, B).
- In Part 4 we split the p -divisible group of the Jacobian and isolate its one-dimensional block F (Theorem C.1); in Part 5 the functor $\widehat{\mathcal{F}}^1$ selecting that block is shown to be a G' -equivariant equivalence onto Def_F (Theorem C).

Why does the wild subgroup C_p , whose action is the entire difficulty, appear nowhere in the decoration of the curve we deform? Because it cannot: decorating C_p would make its action on the deformation problem *trivial* — inner automorphisms act trivially on deformations of the decorated object — the opposite of the wildly nontrivial Lubin–Tate action. The C_p enters only through its shadow: it normalizes the tame decoration, hence descends to the configuration of branch points, where it acts by the cyclic shift. The entire wild action is combinatorics of ramification. *Decorate the center; let the wild part act through the branch divisor* — this inversion is the mechanism of the paper.

One calibration of what the model is *not*. Sekiguchi–Suwa theory (SS01, MRT14) produces, over $\mathbb{Z}_p[\zeta_p]$, a group scheme interpolating between μ_p and $\mathbb{Z}/p$ — the unified Kummer–Artin–Schreier–Witt isogeny — whose extremal parameter satisfies exactly $\lambda^{p-1} = -p$. But this is the arithmetic of a single point, not a family: its parameters move the local wild structure vertically, while Lubin–Tate space needs $p-2$ horizontal moduli, which come from the configuration of branch points and from nothing else. Sekiguchi–Suwa is the *period* of the story, the identity made flesh; the configuration space is its *moduli*.

0.6.4. *The Proof of Theorem C : Kodaira–Spencer.* The equivalence is a coordinate-free Kodaira–Spencer argument in three parts; Section 10 opens with a roadmap of it. (i) *The functor:* $\widehat{\mathcal{F}}^1$ takes the ψ -eigenpart of the p -divisible group of the Jacobian, for ψ the character of the central G on the extremal differential $\omega_1 = dy/x$; its special fiber is one-dimensional of height $p-1$, and G' embeds in its automorphism group. (ii) *The derivative:* by Grothendieck–Messing and Katz–Oda, the derivative of $\widehat{\mathcal{F}}^1$ at the closed point is cup product with the Hodge line, $H^1(T_X)^G \rightarrow \mathrm{Hom}(k\omega_1, H^1(\mathcal{O}_X)^\psi)$, which is surjective with kernel $H^0(\mathcal{O}_D)^\psi$ for $D = \mathrm{div}(\omega_1)$. The whole proof is therefore one character count on that divisor, and the count comes out to zero because the curve is minimal: *injectivity of the Kodaira–Spencer map is the minimal-genus theorem seen from the formal group*. (iii) Both sides are smooth of dimension $p-2$, so a tangent isomorphism is an equivalence, and the transported action is the Lubin–Tate action because G' sits in $\mathbb{G}_{p-1}$ in a single conjugacy class. An earlier version of this paper proved the theorem the other way round, from an explicit universal curve and a recognition principle; that proof needs an equation, this one needs none, and this one is the one that generalizes (Remark 10.26).

0.7. Prior and Current Work.

What is New. The curve $X: y^p - y = x^{p-1}$ over $\mathbb{F}_{p^{p-1}}$ is the unique curve of minimal genus with automorphism group $G' := C_p \rtimes C_{(p-1)^2}$ (Theorem A): *the group is shown to determine the curve*, and the curve is never chosen. Its deformations, as a curve with its central C_{p-1} action, form a stack that we show is G' -equivariantly

equivalent as deformation stacks to Lubin–Tate space at height $p-1$, and the functor $\widehat{\mathcal{F}}^1$ exhibiting the equivalence is constructed here using idempotents for the first time (Theorem C).

This equivalence is not Serre–Tate. Serre–Tate identifies deformations of a curve with deformations of the p -divisible group of its Jacobian, which here has dimension $(p-1)(p-2)/2$; we identify them with deformations of a single one-dimensional block of height $p-1$ cut out by an idempotent.

Nothing is lifted by hand: the action of $\mathrm{Aut}(X)$ on the universal curve is supplied immediately by deformation theory (Lemma 7.5), not by an equation. This gets rid of the need to find an explicit universal lift of the universal curve and a lift of the action of $\mathrm{Aut}(X)$ to an action on the universal curve, both of which were required by previous work.

The paper also connects $\mathrm{Aut}(X)$ to Σ_p at the level of geometric deformation stacks. *Ramification data replaces level structure:* the branch locus of $X \rightarrow X/C_{p-1}$ is a configuration of p points on the line, deformations of the curve with its group are deformations of that configuration (Theorem 8.20), and $\mathrm{Aut}(X)/C_{p-1} = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ is the affine group inside Σ_p permuting them.

Remark 0.6. The height-one instance is familiar in a different guise: the Σ_3 -action on elliptic curves with full level-2 structure is usually reached through $\mathrm{GL}_2(\mathbb{F}_2) \simeq \Sigma_3$ rather than through its geometric source, the permutation of the three branch points of $E \rightarrow E/C_2$; from the level-2 standpoint, the global moduli of elliptic curves with level-2 structure is what makes the duality of tmf computable (Sto12).

On Whose Shoulders. The order- p subgroup of $\mathbb{G}_{p-1}$ was first exploited by Ravenel (Rav78). He obtained it from the embedding $\mathbb{Q}_p(\zeta_p) \hookrightarrow D_{p-1}$ into the p -adic division algebra of Hasse invariant $1/(p-1)$. Using nothing about the action except the existence of an element in $\mathbb{G}_{h-1}$ of order p , he detected the odd-primary Arf invariant classes b_i through a homomorphism from the Adams–Novikov E_2 -term to $H^*(\mathbb{Z}/p; \mathbb{F}_{p^{p-1}})$ which, as he notes, he could not construct explicitly. The geometric model here writes that homomorphism down (Theorem D) and identifies its target: b_i maps to $\beta \Delta^{(p^i-1)/(p-1)}$ up to a unit (Remark 10.45), and the exponent is the a_i/p of his inductive differential.

A stack deforming the equation of X , never formally stated, is implicit in Gorbounov–Mahowald (GM00). Their map to Lubin–Tate space is likewise not Serre–Tate; it is built by hand, through Honda’s theory of formal group laws, rather than from idempotents, and they use it to prove an analogue of Theorem D directly rather than to deduce it from an equivalence of moduli problems. Hill’s global stack for EO_{p-1} (Hil06) is built on a Hopf algebroid presented by the same equation $x^p - x = y^{p-1}$; Section 9.3 shows that an ordered, marked version of Hill’s stack has $\mathrm{Def}_{(X, C_{p-1})}$ as a local neighborhood.

Ravenel generalized the Gorbounov–Mahowald construction to higher heights in order to study the $\mathbb{Z}/p$ -action there (Rav07, Rav08). In (Rav07), he showed that the Jacobian of $X_f: y^{p^f-1} = x^p - x$ contains a one-dimensional formal summand of height $(p-1)f$, the analogue of Theorem C.1. His follow up paper (Rav08) constructed by hand an explicit deformation of X_f over a polynomial ring L in $(p-1)f-1$ variables. He conjectured that the Jacobian of this deformation carries the universal one-dimensional formal group of that height; and showed how to isolate a one-dimensional formal group by lifting, through a linear approximation, from a

formal group over $L/(e_1 \cdots e_{(p-1)f-1})^2$ to one over L . The lifting of the $\mathbb{Z}/p$ -action was not addressed.

Finally, the reader may recognize the ring of Theorem E from the Hopkins–Miller computation of the E_2 -term at height $p - 1$, unpublished but recorded by Nave (Nav10) and in Hill’s thesis (Hil06); Nave uses it to prove the non-existence of the Smith–Toda complexes $V((p + 1)/2)$. We recover the ring by a different route, and with the action of the full $C_{(p-1)^2}$ on the grading derived rather than assumed (Lemma 10.34).

Remark 0.7 (Shimura varieties). The stacks of topological automorphic forms (BL10, Beh20, HL10) are built on PEL Shimura varieties for $U(1, h - 1)$. A PEL Shimura variety $\mathcal{S}$ for $U(1, h - 1)$ parametrizes abelian varieties with extra structure whose p -divisible groups are h copies of one one-dimensional height- h formal group, giving for each $s \in \mathcal{S}(k)$ a projection to that formal group. Thus, for that each supersingular point $s \in \mathcal{S}$, we are given $\text{Def}_s \simeq \text{Def}_F$ (Car90, RZ96). There is no clear relation between Jacobian of the universal curve over $\text{Def}_{(X,G)}$ and such a Shimura variety at $h = p - 1$: the bases have the same dimension, but the fibers have dimensions $g = (p - 1)(p - 2)/2$ and $h^2 = (p - 1)^2$. The Shimura variety naturally attached to $\text{Def}_{(X,G)}$ would be a moduli of abelian varieties arising from monodromy datum called $\text{Sh}(\mu_{(p-1)^2}, \mathfrak{f})$ in the sense of (DM86); however, a construction from ramification data along the lines of (LMPT18) does not yet cover this case.

Remark 0.8. The global object Curve_G in which we deform $\text{Def}_{(X,G)}$ is the smooth locus of a Hurwitz stack. This is mentioned briefly in Section 9.3. One could then spectralize the compact stack toward higher height generalization of tmf that is an alternative to taf and the Hopf algebra of (Hil06). We do not discuss this in depth in this paper as it will take us too far afield.

0.8. Outline. Part 1 is the inverse Galois theory: the curve X , its ramification, its field of definition, and finally its equation are consequences of G' , not choices (Theorem A). Part 2 recalls the stacky background and definitions we work with.

Part 3 proves the uniformization L and computes the representing ring with its G' -action (Theorems B.1, B.2, B): the moduli are configurations of the branch points, and G' acts by shifting and scaling them.

Part 4 splits the p -divisible group of the Jacobian and isolates the one-dimensional block of height $p - 1$ (Theorem C.1).

Part 5 constructs $\widehat{\mathcal{F}}^1$ and proves the G' -equivariant equivalence with Lubin–Tate space by Kodaira–Spencer (Theorem C), in ungraded and graded form.

Part 6 combines Theorems B and C to write down the canonical G' -action on Lubin–Tate space (Theorem D) and computes its group and Tate cohomology (Theorem E).

0.9. Conventions and Notation. Fixed for the whole paper, so the reader need not reconstruct it.

- p is an odd prime and $h = p - 1$ unless stated otherwise; k has characteristic p and contains $\mathbb{F}_{p^{p-1}}$; where a statement needs $k = \bar{k}$ we say so.
- X is the minimal genus curve with $\text{Aut}_k(X) \simeq G'$, equation $y^p - y = x^{p-1}$, genus $\frac{(p-1)(p-2)}{2}$. Minimality is Theorem A, not a convention.
- $G' := \text{Aut}_k(X) \simeq C_p \rtimes C_{(p-1)^2}$ is the *large* group, the maximal finite subgroup of $\mathbb{G}_h$; $G := C_{p-1} \subseteq Z(G')$ is the *small*, tame one used to decorate X .

The prime on G' is inherited from the literature and indicates no derived construction.

- $/$ is both a group quotient and the algebraic quotient of a scheme by a finite group (Defn 2.1); $//$ is the stacky quotient.
- Def_Y^* : deformations of Y with morphisms reducing to the identity; Def_Y^G : morphisms reducing to elements of G ; $\text{Def}_Y := \text{Def}_Y^{\text{Aut}(Y)}$. $(-)^{\wedge}$ is completion, at the identity section for a group scheme and at I when written $(-)_I^{\wedge}$.

Representation-theoretic notation for Theorems B, D, E:

- $\lambda := W(k)\{y_0, \dots, y_{n-1}\}/(y_0 + \dots + y_{n-1})$, $|y_i| = -2$: the standard $(n-1)$ -dimensional representation of Σ_n , permuting the basis.
- $\bar{\rho}$: the restriction of λ along $C_n \rtimes \text{Aut}(C_n) \subseteq \Sigma_n$, i.e. the reduced regular representation of C_n with its $\text{Aut}(C_n)$ -action. *Group-action clarification, used silently throughout:* $G' = C_p \rtimes_{\chi} C_{(p-1)^2}$ acts on configurations through $C_p \rtimes \text{Aut}(C_p)$; the central kernel C_{p-1} acts trivially on the configuration and only through the grading, on the invariant differential u . This is the graded content of Theorem C(b).
- $x_j := (1 - \sigma^j)(y_0)$ for σ generating C_n , and $I := (p, x_1, \dots, x_{n-1})$: the G' -equivariant ideal of the characteristic locus — the equivariant replacement for $(p, y_0, \dots, y_{n-1})$, which is not C_p -stable.
- $N := \prod_i \sigma^i(x_0) = x_0 \cdots x_{n-1}$. In centered coordinates the x_i are differences of branch points along the C_p -orbit, so inverting N inverts the adjacent-collision locus, which by transitivity is the whole collision locus: N and the discriminant cut out the same closed set, and N is the equivariant generator of convenient degree. $\Delta := N^{-(p-1)^2}$ carries the inverse exponent deliberately: with $|x_i| = -2$, this makes $|\Delta| = (0, +2p(p-1)^2)$, the periodicity class of Theorem E; at $p = 3$, $|\Delta| = 24$ — the modular Δ , as it must be.

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Part 1. Uniqueness in Inverse Galois Theory, Canonical Designer Curves with Decoration

OVERVIEW

Given a finite group G and a field k , inverse Galois theory asks, and sometimes answers:

Question. Is there a curve X with $\text{Gal}(k(X)/k) \simeq G$?

It rarely asks whether the realization is unique.

Question. Let X be the minimal genus curve such that $\text{Gal}(k(X)/k) \simeq G$. Is X unique?

For the group of this paper G' it is, and the mechanism is pretty: G' carries a character which must be surjective, the ramification break is the exponent through which that character factors, and there is exactly one break small enough to do the job. Minimality of genus does no work beyond pinning down the shape of the cover; after that, the arithmetic of $(p-1)^2$ takes over and writes the equation for us. This is Theorem A: we did not choose $y^p - y = x^{p-1}$. It was ordained by the group.

This Part runs as follows. Section 1 recalls the Kummer/Artin–Schreier dichotomy, and Section 2 the language of algebraic quotients and ramification divisors, for the ease of interested topologists. Section 3 proves uniqueness, with its engine, Lemma 3.4 (two characters, one break), which is the curve-side avatar of $\pi^{p-1} = -p$. Section 4 then computes the ramification of the roof of Galois covers $\mathbb{P}^1 \xleftarrow{C_{p-1}} X \xrightarrow{C_p} \mathbb{P}^1$ that the curve carries (Lemma 4.2), and Section 5 shows how $G'/G =: \text{Aut}^f(X)$ acts on the branch points of $f: X \rightarrow X/G$, giving

$$\text{Aut}^f(X) \hookrightarrow \Sigma_p.$$

That embedding is the decoration we deform in Part 3: the wild group C_p is never decorated — doing so would trivialize its action — and enters only through its permutation of the branch points of the tame quotient.

Convention 1. *Throughout this Part, p is an odd prime, and*

$$G' \simeq C_p \rtimes_{\chi} C_{(p-1)^2}, \quad \chi: C_{(p-1)^2} \twoheadrightarrow \mathbb{F}_p^{\times},$$

is the maximal finite subgroup of $\mathbb{G}_{p-1}$ with nontrivial p -torsion (Theorem 0.1). For a curve X with a faithful G' -action we write P for the Sylow p -subgroup, $H \simeq C_{(p-1)^2}$ for a complement (Schur–Zassenhaus, the orders being coprime), $Y := X/P$ of genus g_Y , $B \subset Y$ the branch locus of $q: X \rightarrow Y$ with $|B| = r$ and breaks m_b , and

$$T := \frac{(p-1)(p-2)}{2}$$

for the target genus.

1. CYCLIC COVERS OF CURVES

Definition 1.1. *A map of curves $f: X \rightarrow Y$ is a **G -Galois cover** if it is a G -Galois extension of function fields, $\text{Gal}_k(k(X)/k(Y)) \simeq G$.*

Remark 1.2. If we were virtuous of heart we would say “generically G -Galois,” as the map is not étale at the ramification points. We drop “generically” in this paper.

Given Y with function field $k(Y)$, the $\mathbb{Z}/n$ -covers of Y are classified according to whether n is prime to $\text{char } k = p$: if $p \nmid n$ the cover is a Kummer extension $x^n = a$ with $a \notin k(Y)^{\times n}$; if $n = p$ it is an Artin–Schreier extension $y^p - y = a$ with $a \notin \wp(k(Y))$, $\wp(b) = b^p - b$. Covers of $\mathbb{A}^1$ correspond to covers of $\mathbb{P}^1$ ramified over infinity, and we switch between them freely. The two basic examples, and the two projections our curve will carry:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^1 \\ \mathbb{Z}/n \downarrow & \lrcorner & \downarrow (-)^n \\ Y & \xrightarrow{a} & \mathbb{A}^1 \end{array} \quad \begin{array}{c} x \\ \downarrow \\ x^n \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \mathbb{A}^1 \\ \mathbb{Z}/p \downarrow & \lrcorner & \downarrow \text{Fr}-\text{Id} \\ Y & \xrightarrow{a} & \mathbb{A}^1 \end{array} \quad \begin{array}{c} x \\ \downarrow \\ x^p - x \end{array}$$

2. RAMIFICATION DIVISORS AND ALGEBRAIC QUOTIENTS OF CURVES

We may consider G -actions on X as equivalent to G -covers with source X , the latter defined through the algebraic quotient; under this equivalence, ramification points are the points with nontrivial stabilizer.

Let X be a scheme of finite type over k with a finite group G acting by k -automorphisms α_g .

Definition 2.1. A *quotient* of X by G is a morphism $f: X \rightarrow Y =: X/G$ which is G -invariant ($f \circ \alpha_g = f$ for all g) and universal for this property: every G -invariant $X \rightarrow Z$ factors uniquely through f . Its fibers are the G -orbits.

Lemma 2.2. Let X be a smooth proper curve over $k = \bar{k}$ and G finite. Every G -action on X induces a G -Galois branched cover of smooth proper curves $f: X \rightarrow X/G$.

Proof. Under the equivalence between smooth proper curves and function fields of transcendence degree one, a G -action on X is a G -action on $\kappa(X)$; the invariants $\kappa(X)^G$ again have transcendence degree one and $\kappa(X)/\kappa(X)^G$ is G -Galois; the curve corresponding to $\kappa(X)^G$ is X/G and the extension is f . $\square$

For a finite cover $f: X \rightarrow Y$ there is an exact sequence $0 \rightarrow f^*\Omega_Y^1 \rightarrow \Omega_X^1 \rightarrow \Omega_{X|Y}^1 \rightarrow 0$ (with $\Omega_X^1 \simeq f^*\Omega_Y^1$ exactly when f is étale).

Definition 2.3. The *ramification divisor* of a dominant morphism $f: X \rightarrow Y$ of curves is $\mathfrak{R} = \sum_P \text{length}(\Omega_{X/Y})_P [P]$, the sum over closed points; the *branch divisor* is its image $B := f(\mathfrak{R})$.

When f is a tame Galois quotient with stabilizers G_v , $\text{length}(\Omega_{X/Y})_v = |G_v| - 1 = e_v - 1$. When f is wild the length is larger than $e_v - 1$ and is computed by the higher ramification groups: with $G_i(v)$ the lower-numbered filtration,

$$2g_X - 2 = |G| (2g_Y - 2) + \sum_{v \in X} \sum_{i \geq 0} (|G_i(v)| - 1), \quad (1)$$

the wild Riemann–Hurwitz formula (Sti09) (a consequence of Hurwitz Genus Formula + Hilbert’s difference formula). For a $\mathbb{Z}/p$ -cover with a single break m at v the inner sum is $(m + 1)(p - 1)$, which is the only case of (1) we use in this paper.

Lemma 2.4 (Riemann–Hurwitz, sheaf form). For $f: X \rightarrow Y$ a finite map of curves with ramification divisor $\mathfrak{R}$,

$$\Omega_X^1 \simeq \mathcal{O}_X(\mathfrak{R}) \otimes f^*\Omega_Y^1.$$

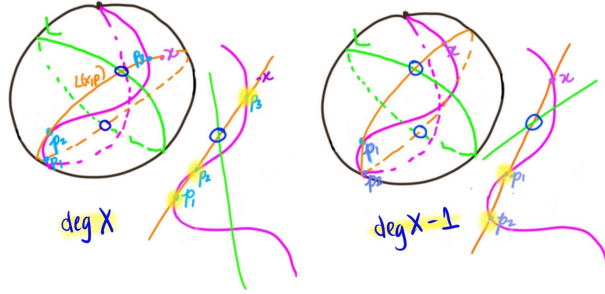
Proof. By definition $\mathcal{O}_{\mathfrak{R}} \simeq \Omega_{X|Y}^1$, so $0 \rightarrow f^*\Omega_Y \rightarrow \Omega_X \rightarrow \mathcal{O}_{\mathfrak{R}} \rightarrow 0$ is exact. Twisting by the line bundle $\Omega_X^\vee$ gives $f^*\Omega_Y \otimes \Omega_X^\vee \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\mathfrak{R}}$, which is the sequence $\mathcal{O}_X(-\mathfrak{R}) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\mathfrak{R}}$ of the divisor; hence $f^*\Omega_Y \otimes \Omega_X^\vee \simeq \mathcal{O}_X(-\mathfrak{R})$, and dualizing gives the claim. $\square$

A practical tool for ramification of plane curves is projection from a point.

Construction 1. For $X \subset \mathbb{P}^2$, a point $x \in \mathbb{P}^2$, and a line $L \subset \mathbb{P}^2$, let $\pi_x: X \rightarrow L$ send p to $L(x, p) \cap L$, where $L(x, p)$ is the line through x and p .

Lemma 2.5. If X has degree d , then $\deg \pi_x = d$ for $x \notin X$ and $\deg \pi_x = d - 1$ for $x \in X$.

Proof. For $x \notin X$ a line through x meets X in d points. For $x \in X$ the map is defined on $X \setminus \{x\}$ and extends over x along the tangent line; each line through x now meets X in d points *one of which is x itself*, so the degree drops by one.



$\square$

Definition 2.6 (Ramification break). Let L/K be a finite Galois extension of local fields with Galois group $G = \text{Gal}(L/K)$, valuation ring $\mathcal{O}_L$, and uniformizer π_L . For $i \geq -1$ the *lower-numbering ramification groups* are

$$G_i := \{ \sigma \in G \mid v_L(\sigma(\pi_L) - \pi_L) \geq i + 1 \},$$

a decreasing filtration $G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq \dots$ with $G_i = 1$ for $i \gg 0$. An integer $b \geq 0$ is a *ramification break* (or *jump*) of L/K in the lower numbering if

$$G_b \neq G_{b+1}.$$

Equivalently, writing $\varphi_{L/K}$ for the Herbrand function and $G^u := G_{\psi_{L/K}(u)}$ for the upper-numbering filtration, a real number $u \geq 0$ is a break in the upper numbering if $G^u \neq G^{u+\varepsilon}$ for all $\varepsilon > 0$; the two sets of breaks correspond under $u = \varphi_{L/K}(b)$.

3. UNIQUENESS: THE CURVE FROM THE GROUP

We prove that $y^p - y = z^{p-1}$ is the unique curve of minimal genus admitting a faithful action of $G' = \mathbb{Z}/p \rtimes_{\chi} \mathbb{Z}/(p-1)^2$ with χ surjective, uniformly for all odd p including $p = 3$. No hypothesis on the ramification or on the quotient X/P is imposed: the break $m = p - 1$, the rationality of X/P , the vanishing of the p -rank, and the minimal field of definition $\mathbb{F}_{p^{p-1}}$ are consequences of the group.

Theorem A. Let p be an odd prime and k a perfect field containing $\mathbb{F}_{p^{p-1}}$. Among smooth projective geometrically connected curves over k admitting a faithful G' -action, the minimal genus is

$$g = T = \frac{(p-1)(p-2)}{2},$$

attained precisely by the k -forms of

$$X: \quad y^p - y = z^{p-1}.$$

On the minimal stratum the following are consequences, not hypotheses:

- (1) $X/P \cong \mathbb{P}_k^1$;
- (2) $X \rightarrow X/P$ is branched at exactly one point, with break $m = p - 1$;
- (3) X has p -rank 0;
- (4) for $p \geq 5$, $\text{Aut}(X) = G'$; for $p = 3$, $\text{Aut}(X, O) = G'$ for the origin O of the elliptic curve X .

If moreover $\wp(k) = k$ and $(k^\times)^{p-1} = k^\times$ — in particular if $k = \bar{k}$ — the curve is unique up to k -isomorphism.

Remark 3.1 ($p = 3$). We use the phrasing “faithful action” to speak of the $p = 3$ and $p \geq 5$ case in the same breath. Since $\text{Aut}(E)$ is infinite for an elliptic curve, we consider instead $\text{Aut}(E, O)$ in this case (When $T = 1$ and $X = E: y^3 - y = x^2$ is the supersingular elliptic curve $j = 0$, with of order 12). The phrasing “faithful action” is what makes the statement uniform, since $\text{Aut}(E)$ itself contains the translations. The p -rank computation independently confirms uniqueness: characteristic 3 has one supersingular j -invariant.

Remark 3.2 (minimality is essential). Without it the group determines nothing: for $p = 5$ the curves $y^5 - y = x^{28} + cx^{12}$, $c \neq 0$, carry $\mathbb{Z}/5 \rtimes \mathbb{Z}/16$ with χ surjective, have genus 54, and move in a family. The eigen-equation (2) below forces only $\text{supp}(f) \subset m + (p - 1)^2\mathbb{Z}$, a single monomial exactly when $m < (p - 1)^2$. The square matters for the same reason: with a complement of order $p - 1$ the minimal stratum is a $(p - 3)$ -dimensional family, not a point.

Remark 3.3 (minimal field of definition). Since $p^n \equiv 1 + n(p - 1) \pmod{(p - 1)^2}$, one has $\text{ord}_{(p-1)^2}(p) = p - 1$: the smallest field containing $\mu_{(p-1)^2}$ is $\mathbb{F}_{p^{p-1}}$, and by Lemma 3.4 the tangent character embeds $\mathbb{Z}/(p - 1)^2$ into $k^\times$. The base field of this paper is therefore forced by G' ; it is, coherently, the residue field of the maximal unramified subfield of the division algebra of invariant $1/(p - 1)$.

3.1. Two Characters, One Break. Two characters are in play, for unrelated reasons. The **conjugation character** $\chi: H \rightarrow \mathbb{F}_p^\times$, $c\sigma c^{-1} = \sigma^{\chi(c)}$, is pure group theory, canonical because $\text{Aut}(\mathbb{Z}/p) = \mathbb{F}_p^\times$ independently of a generator. The **tangent character** $\theta: H \rightarrow k^\times$, the scalar by which H acts on a cotangent line $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, is pure linear algebra. The break is the bridge, and the shape of the bridge is what lets a group-theoretic hypothesis pin a ramification invariant: the tame part is fully visible on the tangent line, while the wild part is invisible there ($\theta(\sigma) = 1$ always) and shows its first datum only in degree $m + 1$. Invariantly, H acts on $R_m := \text{Hom}(\mathfrak{m}_Q/\mathfrak{m}_Q^2, \mathfrak{m}_Q^{m+1}/\mathfrak{m}_Q^{m+2})$ by θ^m , and $\sigma^j \mapsto [t \mapsto (\sigma^j)^*t - t]$ is an injective H -equivariant map $P \hookrightarrow R_m$.

Lemma 3.4 (two characters, one break). *Let $H \simeq C_{(p-1)^2}$ be the tame complement of Convention 1 (not its central subgroup C_{p-1} , on which χ is trivial), $b \in B$ with $Q \in X$ above it, and $H_b \subseteq H$ the stabilizer. Then H_b acts on the tangent line at Q by an injective character θ , and*

$$\chi|_{H_b} = \theta^{-m_b}, \quad \text{hence} \quad |\chi(H_b)| = \frac{|H_b|}{\gcd(|H_b|, m_b)}.$$

Proof. $P \trianglelefteq G'$ forces every element of G' to preserve the fiber over b , so H_b acts on $\mathfrak{m}_Q/\mathfrak{m}_Q^2$. Since $p \nmid |H_b|$ we may average: this produces a uniformizer t with $c^*t = \theta(c)t$ exactly. If $\theta(c) = 1$ then c fixes a topological generator of $\widehat{\mathcal{O}}_{X,Q}$ and is trivial, so θ is injective. Now $\sigma^*t = t + at^{m_b+1} + \dots$ with $a \neq 0$, and $(\sigma^j)^*t = t + jat^{m_b+1} + \dots$: the coefficient of t^{m_b+1} reads off the exponent. Conjugating,

$$(c\sigma c^{-1})^*t = t + a\theta(c)^{-m_b}t^{m_b+1} + \dots,$$

which must be $\sigma^{\chi(c)}$. The index formula follows since $\theta(H_b)$ is cyclic. $\square$

This lemma is the curve-side avatar of $\pi^{p-1} = -p$: at $m = p - 1$ it reads $\chi = \theta^{-(p-1)}$, the mod- p shadow of the identity in $\mathbb{Z}_p[\zeta_p]$, with tangent character $\leftrightarrow$ Lubin–Tate character, conjugation character $\leftrightarrow$ cyclotomic character, and break $\leftrightarrow e(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$. We return to this in Remark 3.11 once the break has been forced.

For the proof of uniqueness, we will need a size estimate on tame cyclic automorphism groups.

Lemma 3.5 (Wiman–Harvey). *Let H be cyclic of order N acting faithfully on a smooth projective curve of genus $g_Y \geq 1$ over $k = \bar{k}$, with $p \nmid N$. Then $N \leq 4g_Y + 2$.*

Reference. The proof of Wiman’s bound (Wim95, Har66) minimizes the orbifold Euler characteristic $\mathcal{E} = 2g_{Y/H} - 2 + \sum_j (1 - 1/n_j)$ subject to $2g_Y - 2 = N\mathcal{E}$ and uses only tame Riemann–Hurwitz, so it holds verbatim in characteristic p when $p \nmid N$, with no lifting argument. $\square$

3.2. Proof of Uniqueness. The proof has three steps, each with its own intuition, so we separate them: (A) minimality forces $g_Y = 0$; (B) it forces a single branch point with break $p - 1$; (C) the eigen-equation then writes the equation. Let’s go.

Lemma 3.6 (Step A). *If $g_Y \geq 1$ then $g_X > T$.*

The idea: H is a large cyclic group, of order $\sim p^2$. A curve of positive genus has no room for it, so g_Y must itself be of order p^2 , and (1) multiplies that expense by p , overshooting $T \sim p^2/2$ by a factor of p . There is exactly one configuration in which the amplification fails — $g_Y = 1$ with q unramified — and it is excluded by structure rather than size.

Proof. Case $g_Y \geq 2$. By Lemma 3.5, $g_Y \geq ((p - 1)^2 - 2)/4$. Discarding the non-negative ramification sum in (1),

$$2g - 2 \geq 2p(g_Y - 1) \geq \frac{p}{2}((p - 1)^2 - 6),$$

and $\frac{p}{2}((p - 1)^2 - 6) > p(p - 3) \iff p^2 - 4p + 1 > 0$, true for $p \geq 4$. For $p = 3$ directly: $2g - 2 \geq 2p(g_Y - 1) \geq 6 > 0 = 2T - 2$.

Case $g_Y = 1$, $r \geq 1$. Every point stabilizer in H has order at most 6 (Lemma 3.5 at $g_Y = 1$), so every H -orbit in Y has at least $(p - 1)^2/6$ points; B is H -stable and nonempty, so $r \geq (p - 1)^2/6$ and

$$2g - 2 \geq -2p + 2(p - 1) \cdot \frac{(p-1)^2}{6} = -2p + \frac{(p-1)^3}{3},$$

which exceeds $p(p - 3)$ for $p \geq 5$ since $(p - 1)^3 \geq 3p(p - 3) + 6p$ there. For $p = 3$, (1) with $r \geq 1$ gives $2g - 2 \geq 0 + 2 \cdot 3 = 6 > 0$.

Case $g_Y = 1$, $r = 0$. Now (1) gives $g - 1 = p(g_Y - 1) = 0$, so $X \cong Y$ is an elliptic curve and P acts freely, by translation t_a by a point a of exact order p ; in particular X is ordinary. H has a fixed point (the ramification points of $X \rightarrow X/H$,

nonempty since $g_{X/H} = 0$), so H injects into $\text{Aut}(X, O)$ for a suitable origin. For $\varphi \in \text{Aut}(X, O)$, $\varphi t_a \varphi^{-1} = t_{\varphi(a)}$, so χ factors through the action on $X[p] \cong \mathbb{Z}/p$ and $|\chi(H)| \leq |\text{Aut}(X, O)|$. For $p \geq 5$: $|H| = (p-1)^2 \leq |\text{Aut}(X, O)| \leq 6$ is impossible. For $p = 3$: we need $H \cong \mathbb{Z}/4$ inside $\text{Aut}(X, O)$, but X is ordinary, and in characteristic 3 ordinary means $j \neq 0$, where $\text{Aut}(X, O) = \{\pm 1\}$. Contradiction in both cases. $\square$

Remark 3.7. This is the only lemma of the section whose proof genuinely differs at $p = 3$, and not by accident: at $p = 3$ the target genus is 1, so genus-one competitors cannot be excluded by size and must be excluded by structure — the supersingularity dichotomy. For $p \geq 5$ the argument is purely numerical.

Lemma 3.8 (Step B). *The minimum of g is T , attained exactly when $r = 1$ and $m = p - 1$.*

Proof. By Lemma 3.6, $Y \cong \mathbb{P}^1$, and $H = \mu_{(p-1)^2}$ acts as $z \mapsto \alpha z$: two fixed points, every other orbit free of size $(p-1)^2$.

$r = 0$ is impossible, since $\pi_1^{\text{ét}}(\mathbb{P}^1) = 1$.

B meets a free orbit. Then $r \geq (p-1)^2$, and (1) gives $2g - 2 \geq -2p + 2(p-1)^3 > p(p-3)$ for all $p \geq 3$ (at $p = 3$: $16 - 6 = 10 > 0$).

$r = 1$. The stabilizer of the branch point is all of H and χ is surjective, so Lemma 3.4 gives

$$p-1 = \frac{(p-1)^2}{\gcd((p-1)^2, m)} \iff \gcd((p-1)^2, m) = p-1.$$

Writing $m = (p-1)\kappa$ this says $\gcd(p-1, \kappa) = 1$; with $p \nmid m$ the least admissible value is $\kappa = 1$:

$$\boxed{m = p-1}, \quad \text{and then} \quad 2g-2 = -2p + p(p-1) = p(p-3), \quad g = T.$$

$r = 2$, i.e. B is the pair of fixed points. Both stabilizers equal H , so each break is $\geq p-1$ by the previous paragraph, and $2g-2 \geq -2p + 2p(p-1) = 2p(p-2) > p(p-3)$. $\square$

Lemma 3.9 (Step C). *On the minimal stratum, $X \cong \{y^p - y = z^{p-1}\}$.*

Proof. Since $Y \cong \mathbb{P}^1$, Artin–Schreier theory gives $k(X) = k(z)(y)$ with $y^p - y = f \in k(z)$, and f may be taken *reduced*: if $v_{P_0}(f) = -pn$ with uniformizer u and leading coefficient c , subtracting $\varphi(c^{1/p}u^{-n})$ lowers the pole order (k is perfect). For reduced f the branch points are the poles and the break at a pole is its order. By Lemma 3.8 there is one pole, of order $p-1$; move it to $z = \infty$, so $f \in k[z]$ has degree $p-1$. Reducedness forbids only the constant term, which lies in $\varphi(k) = k$ and may be deleted. Thus $f = \sum_{i=1}^{p-1} a_i z^i$ with $a_{p-1} \neq 0$.

Since P is normal, H acts on $k(X)^P = k(z)$, faithfully because $H \cap P = 1$, fixing $z = \infty$; so $H \subseteq \mathbb{G}_a \rtimes \mathbb{G}_m$. The projection $H \rightarrow \mathbb{G}_m$ is injective, its kernel lying in the p -torsion group $\mathbb{G}_a$ while $p \nmid |H|$. A generator $z \mapsto \alpha_0 z + \beta_0$ has $\alpha_0 \neq 1$, hence a fixed point $z_0 = \beta_0/(1-\alpha_0)$; translating $z \mapsto z - z_0$ (which fixes ∞ and preserves the shape of f , deleting the new constant term) we obtain $H = \{z \mapsto \alpha z : \alpha \in \mu_{(p-1)^2}\}$.

Let $c \in H$ with $c^*z = \alpha z$. Then $\varphi(c^*y) = f(\alpha z)$ and $k(z)(c^*y) = k(X)$, so by the Artin–Schreier correspondence the classes of $f(\alpha z)$ and f generate the same subgroup of $k(z)/\varphi k(z)$: there are $\lambda \in \mathbb{F}_p^\times$ and $h \in k(z)$ with

$$f(\alpha z) = \lambda f(z) + \varphi(h). \quad (2)$$

A pole of h of order $s \geq 1$ would give $\wp(h)$ a pole of order $ps \geq p$; but $\wp(h)$ is a polynomial of degree $\leq p-1$ by (2), so $h \in k$ and $\wp(h)$ is constant. Comparing coefficients of z^i , $i \geq 1$: $a_i \alpha^i = \lambda a_i$. Taking $i = p-1$ gives $\lambda = \alpha^{p-1}$. If $a_i \neq 0$ for some $1 \leq i \leq p-2$ then $\alpha^{i-(p-1)} = 1$ for every $\alpha \in \mu_{(p-1)^2}$, so $(p-1)^2 \mid (p-1) - i$ with $0 < (p-1) - i \leq p-2 < (p-1)^2$: contradiction. Hence $f = a_{p-1} z^{p-1}$, and rescaling z by a $(p-1)$ -st root of a_{p-1} gives $f = z^{p-1}$. $\square$

Proof of Theorem A. Lemma 3.6 is (1), Lemma 3.8 is (2) and the value of the minimal genus, and Lemma 3.9 is the equation together with uniqueness over $\bar{k}$. For (3), Deuring–Shafarevich (Sub75) for the $\mathbb{Z}/p$ -cover totally ramified over one point gives $\gamma_X - 1 = p(0-1) + (p-1) = -1$, so the p -rank is 0. For (4): if $p \geq 5$, $\text{Aut}(X) = G'$ by Valentini–Madan (VM80) (the big-action regime is $m \equiv 1 \pmod{p}$; here $m \equiv -1$); if $p = 3$ the curve is the elliptic curve $y^3 - y = z^2$, supersingular by (3), and characteristic 3 has a unique supersingular j -invariant, $j = 0$, with $\text{Aut}(X, O)$ the dicyclic group of order 12, which is $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$. $\square$

Remark 3.10 (more general k). Steps A and B need only $\mu_{(p-1)^2} \subset k^\times$; geometric connectedness excludes the constant-field-extension covers of $\mathbb{P}_k^1$, and the unique branch point is Galois-stable, hence rational, so “genus 0” does mean $\mathbb{P}_k^1$. Step C uses $\bar{k}$ twice, to delete the constant term and to normalize a_{p-1} ; over general perfect $k \supseteq \mathbb{F}_{p^{p-1}}$ the minimal stratum is the set of forms

$$y^p - y = a z^{p-1} + c, \quad (a, c) \in k^\times / (k^\times)^{p-1} \times k / \wp(k),$$

all isomorphic over an extension of degree dividing $p(p-1)$.

Remark 3.11 (the arithmetic shadow: the many faces of $p-1$). The break is now forced, and Lemma 3.4 at $m = p-1$ reads $\chi = \theta^{-(p-1)}$: the characteristic- p avatar of $\pi^{p-1} = -p$ in $\mathbb{Z}_p[\zeta_p]$, with $p-1$ the ramification index of the level-one Lubin–Tate extension of $\mathbb{Q}_p$. The same $p-1$ appears as the break of q , the exponent of the semidirect product, the degree of the field of definition (Remark 3.3, the same statement counted multiplicatively), and, in the intended application, the height — for one reason. Geometrically: an Artin–Schreier break of $p-1$ is what one sees in the special fiber when a μ_p -torsor of p branch points collides along $\pi^{p-1} = -p$; this is the last row of the table in Subsection 0.4, and Part 3 is where the collision is deformed.

4. THE ROOF AND ITS RAMIFICATION

The curve of Theorem A comes with a roof of Galois covers

$$\mathbb{P}^1 \xleftarrow{C_{p-1}} X \xrightarrow{C_p} \mathbb{P}^1,$$

and we now write its automorphisms and ramification explicitly. Theorem A(4) identifies the automorphism group abstractly; the following lemma exhibits the generators we compute with for the rest of the paper.

Lemma 4.1. *The curve $X: y^p - y = x^{p-1}$ over $\mathbb{F}_{p^{p-1}}$ has automorphism group $G' = \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$ generated by*

$$\sigma: (x, y) \mapsto (x, y+1), \quad \tau: (x, y) \mapsto (\zeta^p x, \zeta^{p-1} y),$$

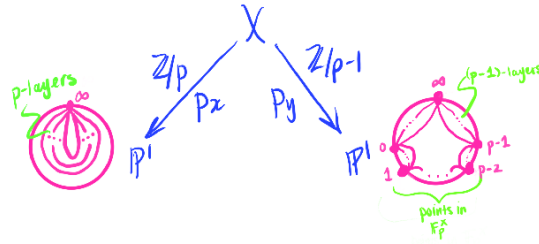
with ζ a primitive $(p-1)^2$ -th root of unity.

Proof. σ is an automorphism because $y^p - y = \prod_{i \in \mathbb{F}_p} (y - i)$ is translation-invariant in characteristic p ; the element $\eta = \zeta^{p-1}$ of order $p-1$ acts by $x \mapsto \eta x$ fixing y , so X is both an Artin–Schreier and a Kummer cover and $\text{Aut}(X) \supseteq C_p \times C_{p-1}$. The interesting part is τ : applying it to the equation, τ is an automorphism iff $\zeta^{p(p-1)} = \zeta^{p-1}$, i.e. iff $p(p-1) \equiv p-1 \pmod{(p-1)^2}$, which holds since $p(p-1) - (p-1) = (p-1)^2$. Its interaction with both x and y is what makes $\text{Aut}(X)$ a semidirect product rather than a product: $\tau\sigma\tau^{-1} = \sigma^{\zeta^{p-1}}$, i.e. $\chi(\tau) = \eta$, which is $\theta^{-(p-1)}$ for the tangent character $\theta(\tau) = \zeta^{-p}$ — Lemma 3.4 in coordinates. That these generate all of $\text{Aut}(X)$ is Theorem A(4). $\square$

Lemma 4.2. *Let $X: y^p - y = x^{p-1}$ over $k = \mathbb{F}_{p^2}$, with projections $p_x, p_y: X \rightarrow \mathbb{P}^1$.*

- p_x is $\mathbb{Z}/p$ -Galois and totally wildly ramified at the single point $[1:0:0] = \infty$, with break $p-1$: $\text{Ram}(p_x) = p[\infty]$.
- p_y is $\mathbb{Z}/(p-1)$ -Galois and tamely totally ramified over $\mathbb{F}_p \cup \{\infty\} \subset \mathbb{P}^1$:

$$\text{Ram}(p_y) = \sum_{i \in \mathbb{F}_p} (p-2) [0:i:1] + (p-2) [1:0:0] = \sum_{P \in \mathbb{F}_p \cup \{\infty\}} (p-2) [P].$$



Proof. C_p acts by $y \mapsto y + 1$ fixing x , so p_x is a C_p -cover. It is the projection from $[0:1:0] \notin X$, hence of degree p by Lemma 2.5; the point at infinity $[1:0:0]$ of $X \subset \mathbb{P}^2$ is fixed by C_p , and it is the unique branch point with break $p-1$ by Theorem A(2). C_{p-1} acts by $x \mapsto \eta x$ fixing y , so p_y is a C_{p-1} -cover; it is the projection from $[1:0:0] \in X$, hence of degree $p-1$. Its ramification points are the fixed points of $x \mapsto \eta x$, namely $x = 0$ — the points $[0:i:1]$, $i \in \mathbb{F}_p$, since $y^p - y = \prod (y - i)$ — and $[1:0:0]$; the stabilizer at each is all of C_{p-1} , and for a tame map the ramification coefficient is $|G_x| - 1 = p - 2$. $\square$

5. REPRESENTATIONS OF THE SYMMETRIC GROUP VIA RAMIFIED COVERS

Riemann’s existence theorem tells us that degree- d covers of Y are equivalent to representations $\pi_1(Y \setminus \{y_1, \dots, y_n\}) \rightarrow \Sigma_d$: a cover is captured by its local monodromy. Here we use the tame G -cover $f: X \rightarrow X/G$, $G = C_{p-1}$, to encode the remaining degree- p Artin–Schreier cover through the action of G'/G on the branch points $(y_1, \dots, y_n)$ of f — the mechanism promised in the Overview.

Definition 5.1. *For a curve X with a G -action and quotient $f: X \rightarrow X/G$, the restricted automorphism group is the quotient $\text{Aut}^f(X)$ of $\text{Aut}(X)$ by $\text{Aut}(f: X \rightarrow X/G)$:*

$$\text{Aut}(f: X \rightarrow X/G) \hookrightarrow \text{Aut}(X) \twoheadrightarrow \text{Aut}^f(X).$$

Example 5.2. *The subgroup $C_{p-1} \hookrightarrow C_{(p-1)^2}$, $\zeta \mapsto \zeta^{p-1}$, is exactly $\text{Aut}(f: X \rightarrow \mathbb{P}^1)$: it is the central C_{p-1} of G' .*

$$\mathrm{Aut}(f: X \rightarrow X/G) \hookrightarrow \mathrm{Aut}(X) \twoheadrightarrow \mathrm{Aut}^f(X)$$

$$C_{p-1} \hookrightarrow C_p \rtimes C_{(p-1)^2} \twoheadrightarrow C_p \rtimes C_{p-1}$$

This is the group-action clarification of the Conventions: the acting group of the paper is G' , its action on the branch configuration factors through $C_p \rtimes \mathrm{Aut}(C_p)$, and the central C_{p-1} acts on configurations trivially — it will act only through the grading.

For the rest of the section let B_{aff} be the branch points of $f: X \rightarrow \mathbb{P}^1$ other than the point at infinity.

Lemma 5.3. *$\mathrm{Aut}(X)$ preserves the set B_{aff} , and an element of $\mathrm{Aut}^f(X)$ fixing B_{aff} pointwise is the identity.*

Proof. By Lemma 4.2, $B_{\mathrm{aff}} = \{[i : 1] : i \in \mathbb{F}_p\}$. The generator of $\mathbb{Z}/p$ acts by $[i : 1] \mapsto [i + 1 : 1]$; the generator τ of $\mathbb{Z}/(p-1)^2$ acts by $[i : 1] \mapsto [\eta i : 1]$ with η a primitive $(p-1)$ -st root of unity in $\mathbb{F}_p$, fixing $[0 : 1]$ and permuting the rest in a $(p-1)$ -cycle. Every nontrivial element of $\mathrm{Aut}^f(X) \simeq C_p \rtimes C_{p-1}$ therefore moves some point of B_{aff} . $\square$

Example 5.4. *Abbreviate $[i : 1] \in \mathbb{P}_{\mathbb{F}_5}^1$ as $[i]$; 2 is a primitive 4th root of unity in $\mathbb{F}_5$, and acting by it sends the ordered branch points $\{[0], [1], [2], [3], [4]\}$ to $\{[0], [2], [4], [1], [3]\}$ — this is a 4-cycle in Σ_5 fixing $[0]$.*

Corollary 5.5 (symmetric subtlety). *With $\Sigma_{B_{\mathrm{aff}}} \simeq \Sigma_p$ the symmetric group on the p affine branch points, there is an injective homomorphism*

$$\mathrm{Aut}^f(X) \hookrightarrow \Sigma_{B_{\mathrm{aff}}} = \Sigma_p.$$

Proof. This follows from Lemma 5.3: a homomorphism with trivial kernel is injective. $\square$

Remark 5.6. The proof was inspired by a theorem in complex geometry: for $W(X)$ the Weierstrass points of X , there is a faithful map $\mathrm{Aut}(X) \rightarrow \Sigma_{W(X)}$. Gorgeous stuff. Keeping the point at infinity would give the weaker embedding into Σ_{p+1} .

This concludes the analysis of the curve X built from G' , with its natural decoration by G . In Part 3 we deform it.

Part 2. Stacky Background

6. RECAP: GEOMETRIC MODELLING

Definition 6.1. *Given pre-stacks $\mathcal{M}$ and $\mathcal{N}$, consider a natural transformation $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$. Consider an object $X \in \mathcal{M}(k)$ and the corresponding object $\mathcal{F}(X)$ in $\mathcal{N}(k)$, this induces a functor*

$$\mathcal{F}^\wedge : \mathrm{Def}_X^{\mathcal{M}} \rightarrow \mathrm{Def}_{\mathcal{F}(X)}^{\mathcal{N}}.$$

*If $\mathcal{F}^\wedge$ is a G' -equivariant equivalence, then we say the following tuple is a **geometric model** for $\mathrm{Def}_{\mathcal{F}(X)}^{\mathcal{N}} \in \mathrm{Stk}^{G'}$ as a G' -stack*

$$(X \in \mathcal{M}(k), \mathrm{Def}_X^{\mathcal{M}} \in \mathrm{Stk}^{G'}, \mathcal{F}^\wedge : \mathrm{Def}_X^{\mathcal{M}} \rightarrow \mathrm{Def}_{\mathcal{F}(X)}^{\mathcal{N}}).$$

Remark 6.2. It is also sufficient to have the tuple with $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$ such that $\mathcal{F}^\wedge$ is an equivalence. This is strictly stronger.

We have constructed (X, G) in Section 1. We will define $\mathcal{M} := \text{Curve}_G$ in Section 7 as well as its corresponding deformation stack $\text{Def}_{(X, G)}^{\text{Curve}_G}$. We consider $\text{Def}_{(X, G)}^{\text{Curve}_G}$ as a G' -stack by explicitly identifying $\text{Aut}(X)$ with a subgroup of Σ_p acting on a configuration space of p points, we do this by defining uniformizing global stacks in Part 9.1 and Subsection 9.3.

We construct the functor (Defn 10.3), demonstrate its G' -equivariant equivalence to show its a geometric model. We consider deformations of both ungraded and graded formal groups, which we discuss in the expository prelude (Ray25). The G' -equivariant equivalence is Theorem C for the ungraded case, and C(b) for the graded case.

After this, we can conclude, dear readers, that we have ourselves a geometric model!

Theorem 6.3. (Ray25) *Having a geometric model*

$$(X \in \mathcal{M}(k), \text{Def}_X^{\mathcal{M}} \in \text{Stk}^{G'}, \mathcal{F} : \mathcal{M} \rightarrow \mathcal{N})$$

of $\text{Def}_{\mathcal{F}(X)}^{\mathcal{N}}$ as a G' -stack constructs an equivalence of stacks

$$\text{Def}_X^{\mathcal{M}} // G' \simeq \text{Def}_{\mathcal{F}(X)}^{\mathcal{N}} // G'.$$

Corollary 6.4. (Ray25) (group time) *If $\text{Def}_X^* \simeq \text{Def}_{\mathcal{F}X}^*$ is an equivalence, and $\mathcal{F}$ is G' -equivariant, then*

$$H^*(G', \mathcal{O}(\text{Def}_X^*)) \simeq H^*(G', \mathcal{O}(\text{Def}_{\mathcal{F}X}^*)).$$

The general theorems 6.3 and 6.4 in our example give, with a bit of massaging, our Theorem D.

7. MODULI STACKS OF G -CURVES

7.1. Deformation Functors. Let k be a char p field. Let $\widehat{\mathbf{Art}}_k$ be the category of complete local algebras with a finitely generated maximal ideal and specified map to the residue field k . The literature often calls this $\text{Pro}(\text{CAlg}_k^{\text{aug}})$. In the rest of this document, we denote the base change of an object $\mathfrak{X} \times_{\text{Spec } R} \text{Spec } k$ as $\mathfrak{X}|_k$.

We define a deformation moduli problem in a prestack $\mathcal{F}$ where we allow morphisms to reduce to a subset of automorphisms of X .

Definition 7.1. *Let $\mathcal{F}$ be a functor $\mathcal{F} : \widehat{\mathbf{Art}}_k \rightarrow \text{Grpd}$ and $X \in \mathcal{F}(k)$, we consider the functor $\text{Def}_X^G : \widehat{\mathbf{Art}}_k \rightarrow \text{Grpd}$. Given $G \subseteq \text{Aut}(X)$, the groupoid $\text{Def}_X^G(R)$ has*

- as objects tuples

$$\{\mathfrak{X} \in \mathcal{F}(R), \iota : \mathfrak{X}|_k \simeq X\},$$

- as morphisms: maps $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$ such that there exists $g \in G$ for which the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}|_k & \xrightarrow{\phi|_k} & \mathfrak{X}'|_k \\ \downarrow \iota & & \downarrow \iota' \\ X & \xrightarrow{g} & X \end{array}$$

We will denote $\text{Def}_X := \text{Def}_X^{\text{Aut}(X)}$. When the prestack $\mathcal{F}$ we are deforming in is unclear, we will specify it by writing $\text{Def}_X^{\mathcal{F}}$.

Historically, morphisms which reduce to the identity on the residue field are referred to as star-isomorphisms. As a notational convention, we will refer to the special case of Def_X^{id} as Def_X^* .

The main action of interest arises as the action of automorphisms of an object Y on the deformation stack of the object Y . We define this action now.

Definition 7.2. *The group $\text{Aut}(X)$ acts on Def_X^G , as follows, for $g' \in \text{Aut}(X)$:*

- on objects, it sends $(\mathfrak{X}, \mathfrak{X}|_k \xrightarrow{\iota} X)$ to the object $(\mathfrak{X}, \mathfrak{X}|_k \xrightarrow{\iota} X \xrightarrow{g'} X)$,
- on morphisms, it sends morphisms to themselves on $\mathfrak{X} \xrightarrow{\phi} \mathfrak{X}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}|_k & \xrightarrow{\phi|_k} & \mathfrak{X}'|_k \\ \textcolor{red}{g' \circ \iota} \downarrow & & \downarrow \textcolor{red}{g' \circ \iota'} \\ X & \xrightarrow{g} & X \end{array}$$

Definition 7.3. *Let $q : Y \rightarrow X$ be a morphism of categories fibered in groupoids over some base category. The group $\text{Aut}_Y(X)$ of automorphisms of Y over X consists of equivalence classes of pairs (f, ψ) , where $f : Y \rightarrow Y$ is a 1-morphism of groupoids and $\psi : q \rightarrow qf$ is a 2-morphism. Two such pairs (f, ψ) and (f', ψ') are equivalent if there is a 2-morphism $\phi : f \rightarrow f'$ so that $\psi'\phi = \psi$. The composition law reads*

$$(g, \psi)(f, \phi) = (gf, (f^*\psi)\phi).$$

Definition 7.4. *There is a homomorphism $\text{Aut}(Y) \rightarrow \text{Aut}_{\text{Def}_Y}(\text{Def}_Y^*)$ sending σ to f_σ , where f_σ is the transformation $(G, i) \rightarrow (G, i\sigma)$, as in Defn 7.2.*

Lemma 7.5. *(halo) Given a pro-representable moduli problem Def_Y^* , pro-represented by a ring R , then the homomorphism in Defn 7.4 induces an isomorphism of automorphism groups $\text{Aut}(Y) \simeq \text{Aut}_{\text{Def}_Y}(\text{Def}_Y^*)$.*

Proof. By being pro-representable, the deformation problem is discrete. The claim follows from the argument given in Lemma 7.18 in (Goe08), which uses only Lemma 7.13, both of which work word for word in more general discrete deformation moduli problems. $\square$

7.2. G -Stacks of Curves.

Definition 7.6. *We define a curve as a smooth proper scheme of relative dimension one over a base scheme.*

Definition 7.7. *We define the prestack*

$$\text{Curve}_G : \text{CRing} \rightarrow \text{Grpd},$$

such that $\text{Curve}_G(R)$ is the groupoid with

- objects: tuples $(\mathfrak{X}/_R, G)$ consisting of a curve $\mathfrak{X}$ over R with a G -action $G \rightarrow \text{Aut}_R(\mathfrak{X})$.
- morphisms: G -equivariant maps between curves.

Remark 7.8. If $f : R \rightarrow S$, and C is a curve over $\mathrm{Spec}(R)$, then f^*C is a curve over $\mathrm{Spec}(S)$. Since $f^*(G \times_{\mathrm{Spec}(R)} C) = G \times_{\mathrm{Spec}(S)} f^*C$, the pull-back curve has G -action. Thus, we have a prestack. In the previous sentence, G is shorthand for $\mathrm{Spec}(\mathrm{map}(G, R))$ or $\mathrm{Spec}(\mathrm{map}(G, S))$ respectively.

Definition 7.9. *Deformations over this prestack are $\mathrm{Def}_{(X,G)}(\Lambda)$, this is the prestack with*

- *objects: triples $\xi := (\mathcal{X}, \rho, \iota)$: a smooth proper curve $\mathcal{X}/\Lambda$, a lift $\rho: G \rightarrow \mathrm{Aut}_\Lambda(\mathcal{X})$ of the action $\rho_0: G \hookrightarrow \mathrm{Aut}_k(X)$, and a G -equivariant isomorphism $\iota: \mathcal{X}_s \xrightarrow{\sim} X$ of the special fiber,*

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{Aut}_\Lambda(\mathcal{X}) \\ & \searrow \rho_0 & \downarrow \mathrm{res} \\ & & \mathrm{Aut}_k(X). \end{array}$$

- *morphisms: $\xi \rightarrow \xi'$ is a G -equivariant isomorphism $\phi: \mathcal{X} \rightarrow \mathcal{X}'$ over Λ with $\iota' \circ \phi_s = \iota$.*

Example 7.10. (of Defn 7.2) *We will be interested in the action of $\mathrm{Aut}(X, k, G)$ on $\mathrm{Def}_{(X,G)}^*$. Let's see how it changes as G changes.*

- *If $G \simeq \mathrm{Aut}(X, k)$, then $\mathrm{Aut}(X, k, \mathrm{Aut}(X, k)) \simeq Z(\mathrm{Aut}(X, k))$, where Z denotes the center of the group.*
- *If $G \simeq \mathrm{id}$, then $\mathrm{Aut}(X, k, \mathrm{id}) \simeq \mathrm{Aut}(X, k)$.*
- *If $G \subseteq Z(\mathrm{Aut}(X, k))$, then $\mathrm{Aut}(X, k, G) \simeq \mathrm{Aut}(X, k)$, since $g \in G$ commutes with every element of $\mathrm{Aut}(X, k)$.*

Thus, there is an $\mathrm{Aut}(X, k)$ action on $\mathrm{Def}_{(X,G)}$ when G is in the center of $\mathrm{Aut}(X, k)$. This will come in handy for us.

7.3. Local and Marked Moduli Problems. As usual for the source of our formal moduli problems, we work with $\widehat{\mathbf{Art}}_k$, the category of complete k -algebras with given reduction to the residue field $(R, \iota: R/\mathfrak{m} \rightarrow k)$.

We define the moduli problem of divisors of a curve.

Definition 7.11. *Given a curve X and with a divisor $D \in \mathrm{Div}(X)$, define the prestack*

$$\mathrm{Def}_{(X;D)} : \widehat{\mathbf{Art}}_k \rightarrow \mathrm{Grpd},$$

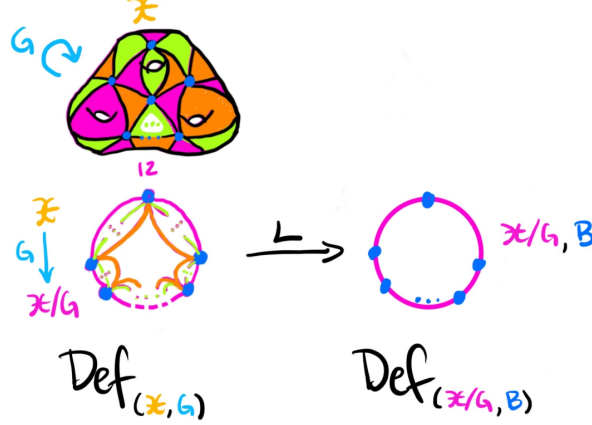
such that $\mathrm{Def}_{(X;D)}(R)$ is the groupoid with

- *objects : tuples $(\mathfrak{X}, \mathfrak{D})$ consisting of*
 - *a curve $\mathfrak{X}$ over R*
 - *a divisor $\mathfrak{D} \in \mathrm{Div}(\mathfrak{X})$*
 - *a map $\iota: \mathfrak{X}|_k \simeq X$ witnessing the lift of the curve which induces $\iota^*: D \simeq \mathfrak{D}|_k$ witnessing the lift of the divisor.*
- *morphisms are pairs $(\mathfrak{X}, \mathfrak{D}) \rightarrow (\mathfrak{X}', \mathfrak{D}')$ consisting of a morphism $f: \mathfrak{X} \rightarrow \mathfrak{X}'$ such that the induced morphism on divisors $f^*: \mathrm{Div}(\mathfrak{X}') \rightarrow \mathrm{Div}(\mathfrak{X})$ sends $\mathfrak{D}' \mapsto \mathfrak{D}$, and such that $f|_k$ reduces to the identity on X .*

Definition 7.12. *The functor*

$$L : \mathrm{Def}_{(X,G)} \rightarrow \mathrm{Def}_{(X/G;B)}$$

is described as follows. Given a G -equivariant deformation of X , $(\mathfrak{X}, G)$, consider its corresponding algebraic quotient $q : \mathfrak{X} \rightarrow \mathfrak{X}/G$. Let B be the reduced branch points of the map q , i.e., the reduced branch divisor. Regard the pair $(\mathfrak{X}/G; B)$ of the quotient curve and its branch points.



Next, we define a moduli problem of G -representations of discs in char p .

Definition 7.13. Fix a finite subgroup G of $\text{Aut}(k[[T]])$, this inclusion can be considered as a representation

$$\rho_G : G \hookrightarrow \text{Aut}(k[[T]]).$$

Consider the deformation problem

$$\text{Def}_{\rho_G} : \widehat{\text{Art}}_k \rightarrow \text{Grpd},$$

such that $\text{Def}_{\rho_G}(A)$ has as

- objects : representations

$$\tilde{\rho}_G : G \rightarrow \text{Aut}(A[[T]])$$

in $\text{Rep}(A[[T]])$, such that the composition of $\tilde{\rho}_G$ with $\text{Aut}(A[[T]]) \rightarrow \text{Aut}(k[[T]])$ is the representation ρ_G .

- morphisms : isomorphisms of representations.

Construction 2. Given a curve X over k with an action of G , consider every point x with a nontrivial stabilizer group G_x . For each such point, the restriction of the action to the disc around x gives a map

$$G_x \rightarrow \text{Aut}(\mathcal{O}_{X,x}) \simeq \text{Aut}(k[[T]]).$$

We define the associated composite local deformation problem as

$$\text{Def}_{(X,G)}^{\text{loc}} := \prod_i \text{Def}_{\rho_{G_{x_i}}}$$

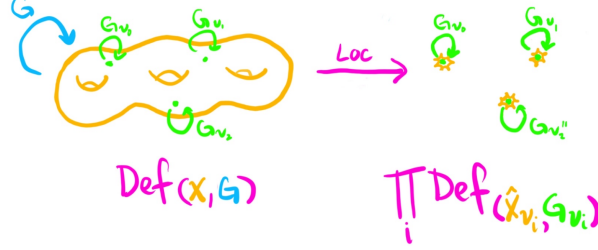
Definition 7.14. The localization functor

$$\text{Loc} : \text{Def}_{(X,G)} \rightarrow \text{Def}_{(X,G)}^{\text{loc}}.$$

is defined by considering for every deformation $(\mathfrak{X}/A, G \subseteq \text{Aut}_A(\mathfrak{X}), \iota)$, all points $\mathfrak{x} \in X$ with nontrivial stabilizer. For each such point, the restriction of the action to the disc around $\mathfrak{x}$ gives a map

$$\rho_{G_{\mathfrak{x}}} : G_{\mathfrak{x}} \rightarrow \text{Aut}(A[[T]]),$$

where the lift of G and the information of ι ensure we have the appropriate reduction property to a corresponding representation $\rho_{G_x} : G_x \hookrightarrow \text{Aut}(k[[T]])$ for a point $x \in X$ with nontrivial stabilizer $G_x \simeq G_{\bar{x}}$. Taking the product of all points with nontrivial stabilizer gives us an element of $\text{Def}_{(X,G)}^{\text{loc}}$ as desired.



Lemma 7.15. ((BM00) Theorem 3.3.5) ((Hui18) Lemma 2.9) *Loc is formally smooth.*

7.4. Graded Version. To access $\pi_*(E_{(p-1)})$ rather than just $\pi_0(E_{(p-1)})$, we will work with deformations of formal groups with a Lie algebra structure. We mention the corresponding stack on the curve side.

Let G be a finite group. Given a G -action on a curve $\mathfrak{X}$, this will induce a G -action on $\mathfrak{g} := H^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^1)$, and thus an idempotent decomposition

$$\mathfrak{g} \simeq \bigoplus_{\chi \in \text{Hom}(G, \mathbb{G}_m)} \mathfrak{g}^{\chi^i}.$$

The notion of idempotent decomposition is discussed in more detail in section 9.4.

Definition 7.16. *We define the prestack*

$$\text{Curve}_G^{\text{Lie}\chi^i} : \text{CRing} \rightarrow \text{Grpd},$$

such that $\text{Curve}_G^{\text{Lie}\chi^i}(R)$ is the groupoid with

- *objects:* tuples $(\mathfrak{X}_R, \mathfrak{g}^{\chi^i}, G)$ consisting of a curve $\mathfrak{X}$ over R , $\mathfrak{g}^{\chi^i}$, and a G -action $G \rightarrow \text{Aut}(R, \mathfrak{X})$
- *morphisms:* G -equivariant maps between curves.

Part 3. Moduli Stacks of G -Curves as G' -Representations

OVERVIEW

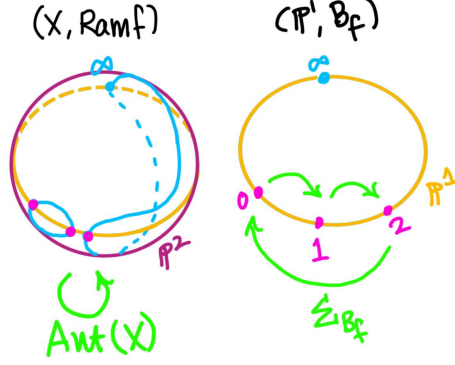
The aim of this section is to understand the $\text{Aut}(X, G)$ -action on $\text{Def}_{(X,G)}$. We will do this in two main stages.

We show in Lemma 8.20 that deformations of any curve X with a tame totally ramified action of G are equivalent to deformations of its quotient X/G and the branch points of $q : X \rightarrow X/G$. Further, the equivalence is $\text{Aut}(X, G)/G$ equivariant. In our example, the G'/G -action on our deformation space $\text{Def}_{(X,G)}$ can be understood as a restriction of the permutation action of the group G' on the branch points of the map $q : X \rightarrow X/G$.

Theorem B.1. *There is a G'/G -equivariant isomorphism of moduli problems*

$$L : \text{Def}_{(X, C_{p-1})} \simeq \text{Def}_{(\mathbb{P}^1, p+1)} \simeq \text{Def}_{(\mathbb{A}^1, p)}.$$

The second phase is to understand the action of G'/G on $\text{Def}_{(\mathbb{A}^1, p)}$, which we do by passing to a global moduli of configuration spaces of n points on $\mathbb{A}^1$. We also connect this analysis to past work on other global stacks related to $\text{Def}_{(X, G)}$ in Subsection 9.3.



Theorem B.2. *The stack $\text{Conf}(\mathbb{A}^1, n)$ is representable as a moduli problem, and the ring representing it is as a Σ_n -module*

$$R \simeq \text{Sym}(\lambda)[N^{-1}] // \mathbb{G}_m,$$

where λ is the standard $n - 1$ -dimensional representation of Σ_n .

Combining Theorems B.1 and B.2, we get the G'/G -action on $\text{Def}_{(X, G)}$ stated fully in Theorem B.

8. THEOREM B.1: MODULI STACKS OF G -CURVES AS CONFIGURATION SPACES OF RAMIFICATION DATA

Before we get started on showing Theorem B.1, let us fully justify the choice of C_{p-1} in our deformation problem.

Inverse Galois theory behaves well with respect to deformation problems. Recall that G -Galois covers of curves with source X are equivalent to G -actions on X . Under this equivalence, ramification points correspond to points with nontrivial stabilizer group. Our curve X in characteristic p defined solely by having automorphism group G' has a convenient tame scapegoat, $G := C_{p-1}$. The points of X with nontrivial stabilizer under the action of G' and under the action of its subgroup $G = C_{p-1}$ coincide. We use this to our advantage, work with $p_y : X \rightarrow X/C_{p-1}$ and encode the remaining G'/G action through its action on the p affine branch points

$$G'/G \hookrightarrow \Sigma_{B_{\text{aff}}} = \Sigma_p.$$

Thus recording of the action on branch points exactly encodes the corresponding C_p -cover.

8.1. Our Tangent Complexes. We will only work with smooth objects X with an action of a finite group G in this paper, in which case the equivariant cotangent complex collapses to $\pi_0(\mathcal{L}_{X/k}^G) \simeq \Omega_X$, where the latter is considered as an $G - \mathcal{O}_X$ -module equipped with the G -action inherited from X . We refer the reader to Illusie (Ill72) and Adeel Khan's notes on the general equivariant cotangent complex (Kha23).

Definition 8.1. An equivariant extension of $\mathcal{O}_Y$ by $\mathcal{I}$ is a short exact sequence of sheaves

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Y \rightarrow 0$$

where we consider $\mathcal{I}$ to be sheaf of ideals with square zero and a subsheaf of $\mathcal{E}$.

These G -equivariant extensions allow us to classify deformations of G -equivariant schemes. We write Hom_G as short-hand for $\mathrm{Hom}_{G-\mathcal{O}_X}$ internal to G -equivariant quasi-coherent sheaves on X .

Corollary 8.2. (Cor 2.3 (Wew05)) (i) The group of automorphisms of any fixed equivariant extension of Y by $\mathcal{I}$ is canonically isomorphic to $\mathrm{Hom}_G(\Omega_{Y/S}, \mathcal{I})$.

(ii) The group of isomorphism classes of equivariant extensions of Y by $\mathcal{I}$ is canonically $\mathrm{Ext}_G^1(\mathcal{L}_{Y/S}, \mathcal{I})$.

(iii) (Theorem 3.3 (Wew05)) The obstruction to deforming Y equivariantly by $\mathcal{I}$ is canonically $\mathrm{Ext}_G^2(\mathcal{L}_{Y/S}, \mathcal{I})$.

Definition 8.3. We define the *tangent sheaf* T_X as $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$, and the *equivariant tangent sheaf* as $\mathrm{Hom}_G(\Omega_X, \mathcal{O}_X)$.

As stated in Lemma 8.2, G -equivariant extensions of the morphism $X \rightarrow S$ by a quasicoherent G - $\mathcal{O}_Y$ -module $\mathcal{F}$ are classified by the group $\mathrm{Ext}_G^1(\mathcal{L}_{X/S}, \mathcal{F})$. This is the basis for all results on equivariant deformations of $X \rightarrow S$.

Theorem 8.4. (i) (Lemma 2.5, 2.6) (Hui18) $(\mathrm{Def}_{(X,G)}^{\mathrm{Curve}_G})^*$ is discrete and pro-representable when $g \geq 2$ and X is smooth.

(ii) (Prop 2.15) (Hui18) $\mathrm{Def}_{(X,G,Q_1)}^*$ is discrete and pro-representable when X is a smooth genus one curve with marked point Q_1 , i.e., (X, Q_1) is an elliptic curve.

Theorem 8.5 (Prop 3.2.1 (BM00)). If X is non-singular, there is a canonical identification of k -vector spaces

$$T(\mathrm{Def}_{(X,G)}) \simeq R^1\Gamma^G(X, T_X) \simeq \mathrm{Ext}_G^1(\Omega_X, \mathcal{O}_X).$$

Lemma 8.6. [Prop 3.3.2] (BM00) Let R be the points in X with nontrivial stabilizer group under the action of a finite group G . Let $\widehat{T}_{X,x} \simeq k[[T]] \frac{d}{dT}$ be the tangent sheaf completed at a point x , where T is the uniformizer at that point. The tangent space of $\mathrm{Def}_{(X,G)}^{\mathrm{loc}}$ is

$$T(\mathrm{Def}_{(X,G)}^{\mathrm{loc}}) \simeq \bigoplus_{x \in R} H^1(G_x, \widehat{T}_{X,x}).$$

8.2. Equivariant Sheaves and Equivariant Sheaf Cohomology. In order to define the tangent complexes of our deformation problems, we first must introduce equivariant sheaves and their cohomology.

Definition 8.7. Given an action $a : G \times_S X \rightarrow X$ of a group scheme G on a scheme X an equivariant sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$ -modules together with an isomorphism of $\mathcal{O}_{G \times X}$ modules

$$a^* \mathcal{F} \simeq p_2^* \mathcal{F}.$$

Definition 8.8. We define the G -equivariant global sections as the right adjoint of the functor which constructs trivial G -sheaves.

$$\begin{array}{ccc} \mathrm{QCoh}(*) & \begin{array}{c} \xrightarrow{\mathrm{triv}} \\ \perp \\ \xleftarrow{\Gamma^G} \end{array} & \mathrm{QCoh}^G(X) \end{array}$$

The right adjoint of the forgetful functor from $\mathrm{QCoh}^G(X) \rightarrow \mathrm{QCoh}(X)$ forgetting the G -action is the coinduced module of the trivial action, $\widetilde{(-)}: \mathcal{F} \mapsto a_* p_2^* \mathcal{F} =: \widetilde{\mathcal{F}}$ (this coincides with the induced module since G is finite).

$$\begin{array}{ccc} & \xrightarrow{\widetilde{(-)}} & \\ \mathrm{QCoh}(X) & \perp & \mathrm{QCoh}^G(X) \\ & \xleftarrow{u} & \end{array}$$

We denote the algebraic quotient of X by a finite G action as $Y := X/G$. The stacky quotient we refer to as $X // G$.

Construction 3. Given a finite G group scheme acting on X , the quotient map $q : X \rightarrow X/G$ factors through $X // G$ uniquely by the universal property that all maps from X to objects with trivial G action must factor uniquely through $X // G$.

$$\begin{array}{ccc} X & & \\ \downarrow f & \searrow g & \\ & X // G & \\ & \swarrow h & \\ X/G & & \end{array}$$

Equivariant sheaves on X are sheaves on the stacky quotient of X by G , that is, $\mathrm{QCoh}(X // G) \simeq \mathrm{QCoh}^G(X)$. These maps induce via pushforward a map from G -sheaves on X to sheaves on $Y := X/G$.

$$\mathrm{QCoh}^G(X) \xrightarrow{\simeq} \mathrm{QCoh}(X // G) \xrightarrow{q_*^G} \mathrm{QCoh}(X/G)$$

Given a G -Galois covering $f : X \rightarrow Y$, and $\mathcal{F}$ a G - $\mathcal{O}_X$ module, we may construct a sheaf of $\mathcal{O}_Y$ modules $q_*^G \mathcal{F}$.

Remark 8.9. If $X \rightarrow X/G$ is an étale map, then $\mathrm{QCoh}(X // G) \simeq \mathrm{QCoh}(X/G)$.

$$\begin{array}{ccccc} & & \Gamma^G & & \\ & \searrow & \xrightarrow{\Gamma^G} & \searrow & \\ \mathrm{QCoh}^G(X) & \xrightarrow{q_*^G} & \mathrm{QCoh}(X/G) & \xrightarrow{\Gamma} & \mathrm{QCoh}(*) \end{array}$$

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\quad} & q_*^G \mathcal{F} & \xrightarrow{\quad} & \Gamma(X/G, q_*^G \mathcal{F}) \\ \parallel & & & & \downarrow \simeq \\ \mathcal{F} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \Gamma^G(X, \mathcal{F}) \end{array}$$

The composite map is the constant sheaf on X with trivial action, and its right adjoint is equivariant global sections Γ^G . Right adjoints are commutative, so we may consider the right adjoint to the map f , which gives us a sheaf on Y whose global sections are the equivariant global sections.

Definition 8.10. We define $R^q \Gamma^G(X, \mathcal{F})$ to be the equivariant cohomology of $\mathcal{F} \in \mathrm{QCoh}^G(X)$. Sometimes we denote this by $H^q(G, \mathcal{F})$.

Since

$$\Gamma^G(X, \mathcal{F}) \simeq \Gamma(X/G, q_*^G(\mathcal{F})),$$

i.e., $\Gamma_X^G \simeq \Gamma_{X/G} \circ q_*^G$ we may use the Grothendieck spectral sequence for compositions of functors. We denote $Y := X/G$. The spectral sequence collapses to give us the following short exact sequence:

$$H^1(Y, q_*^G(\mathcal{F})) \rightarrow R^1\Gamma_X^G(\mathcal{F}) \rightarrow H^0(Y, R^1(q_*^G\mathcal{F}))$$

8.3. Analysis of Pushforwards of Stacky Curves. Let X be a curve with an action of a finite group G . We will quickly set up some foundational lemmas about stacky curves to work with the map $q_*^G : \mathrm{QCoh}^G(X) \rightarrow \mathrm{QCoh}(X/G)$ induced by the map $\pi : X // G \rightarrow X/G$. For a thorough foundational treatment, please consult Section 4 of (Kob21).

Remark 8.11. The floor functions in Kobin (Kob21) and Bertin-Mezard (BM00) Section 4 should be replaced by ceiling functions, as noted in Remark 1.6.2 of (CK03).

Let $\mathcal{Y}$ be a stacky curve, and Y be its coarse counterpoint, in our case, $\mathcal{Y} := \mathcal{X} // G$, and $Y := X/G$. Consider the map $h : \mathcal{Y} \rightarrow Y$.

Definition 8.12. *The degree of a divisor $D = \sum_P n_P [P]$ on a stacky curve is the formal sum $\deg(D) = \sum_P \frac{n_P}{|G_P|}$ where $|G_P|$ is the degree of the stabilizer group at P .*

Lemma 8.13. *(Lemma 4.10 (Kob21)) Given a stacky curve $\mathcal{X}$ with coarse space morphism $\pi : \mathcal{X} \rightarrow X$, there's an isomorphism*

$$\pi_* \mathcal{O}_{\mathcal{X}}(D) \rightarrow \mathcal{O}_X(\lceil D \rceil).$$

The following result is easily extended to wild cases and cases not totally ramified, we discuss the tame totally ramified case for cleanliness of exposition.

Lemma 8.14. *(diamond beneath) For X a totally ramified curve with a tame action of G , let $\mathfrak{R}$ be the unreduced ramification divisor, let B be the reduced branch divisor (i.e., image of the points with nontrivial stabilizer groups G_x under the quotient map $q : X \rightarrow X/G$). Then,*

$$q_*^G \mathcal{O}_X(-\mathfrak{R}) \simeq \mathcal{O}_Y(-B).$$

Proof. The pushforward $q_*^G \mathcal{O}_X(-\mathfrak{R})$ is considering $\mathcal{O}_X(-\mathfrak{R})$ as a G -module, which is equivalent to considering $\mathcal{O}_{X/G}(-\mathfrak{R})$ under the equivalence $\mathrm{QCoh}^G(X) \simeq \mathrm{QCoh}(X // G)$. Let β_P denote the local different at P (for a tame curve $\beta_P = |G_P| - 1$)

By Lemma 8.13, $\deg \beta_P [P] = \frac{\beta_P}{|G_P|}$, thus

$$q_*^G \mathcal{O}_X(-\mathfrak{R}) \simeq \mathcal{O}_Y(-\sum_P \lceil \frac{\beta_P}{|G_P|} \rceil q(P)) \simeq \mathcal{O}_Y(-\sum_P \lceil \frac{|G_P| - 1}{|G_P|} \rceil q(P)).$$

The conclusion follows, as $\sum_P q(P)$ is the reduced branch divisor B by definition. □

Remark 8.15. The statement of 8.14 is usually written in the literature as $q_*^G \mathcal{O}_X(-\mathfrak{R}) \simeq \mathcal{O}_{X/G} \cap q_*(\mathcal{O}_X(-\mathfrak{R}))$, where the ceiling function is listed as the next step in the evaluation of the right hand side (see (CK03) Prop 1.6, and (BM00) Prop 5.3.2). They are implicitly using stacky gerbes to calculate this cap product as a ceiling function.

Lemma 8.16. *(drop weight) For X a curve with a totally ramified tame action of G , let B be the reduced branch divisor (i.e., image of the points with nontrivial stabilizer groups G_x under the quotient map $q : X \rightarrow X/G$). Then,*

$$q_*^G T_X \simeq T_{X/G}(-B)$$

Proof. By taking the dual of Riemann-Hurwitz, Lemma 2.4,

$$T_X \simeq q^* T_Y \otimes \mathcal{O}_X(-\mathfrak{R})$$

where $\mathfrak{R}$ is the (unreduced) ramification divisor. Applying q_*^G , we get

$$q_*^G T_X \simeq T_Y \otimes q_*^G \mathcal{O}_X(-\mathfrak{R}),$$

so our analysis is reduced to considering $q_*^G \mathcal{O}_X(-\mathfrak{R})$. By lemma 8.14, $q_*^G \mathcal{O}_X(-\mathfrak{R}) \simeq \mathcal{O}_Y(-B)$ and the result follows. $\square$

Corollary 8.17. *(kernel sanders) Let $Y := X/G$, and let B be the reduced ramification divisor of the map $q : X \rightarrow X/H$, then*

$$H^1(Y, q_*^G T_X) \simeq H^1(Y, T_Y(-B))$$

and $\dim H^1(Y, q_*^G T_X) = 3g_Y - 3 + |B|$.

Proof. The equivalence $H^1(Y, q_*^G T_X) \simeq H^1(Y, T_Y(-B))$ is an immediate corollary of Lemma 8.16 which states that $q_*^G T_X \simeq T_Y(-B)$. The dimension count follows from Riemann-Roch applied to $H^1(Y, T_Y(-B))$. $\square$

Lemma 8.18. *(mood stabilizer) Let X be a curve over a field k carrying a G -action. Let R be the set of points in $x \in X$ with nontrivial stabilizer group G_x . Let $\widehat{T}_{X,x}$ denote the tangent sheaf of X completed at the point x , this is a module of the form $\widehat{T}_{X,x} \simeq k[[T]] \frac{d}{dT}$ where T is the uniformizer at the point x . Then, (i)*

$$R^q q_*^G T_X \simeq \bigoplus_{x \in R} H^q(G_x, \widehat{T}_{X,x}).$$

(ii) For $|G_x|$ invertible in k , $H^q(G_x, \widehat{T}_{X,x})$ is zero for $q > 0$.

Proof. (i) Consider an affine V on X , and $f(V)$ on X/G , then restrict ourselves to this affine set, then consider the collapse of this spectral sequence from earlier. Given a subset $\iota : V \hookrightarrow X$, and a sheaf $\mathcal{G} \in \text{QCoh}(X)$, then by adjunction we have a canonical map $\mathcal{G} \rightarrow \iota_* \iota^* \mathcal{G} \simeq \iota_* \mathcal{G}|_V$.

Note that since $\text{Hom}_G(\Omega_X, \mathcal{O}_X) \simeq \text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)^G$, we have another spectral sequence converging to $R^i \Gamma_X^G(T_X)$, since $\Gamma_X^G \simeq \Gamma_{X/G} \circ q_*^G T_X \simeq \Gamma_X \circ (-)^G$. Considering these two spectral sequences, we find a collapse:

$$\begin{aligned} H^p(X/G|_{f(V)}, R^q q_*^G T_X|_{f(V)}) &= 0 \text{ if } p > 0 \\ H^q(G, H^0(X|_V, T_X|_V)) &= 0 \text{ if } q > 0 \end{aligned}$$

This collapse of spectral sequences gives us an equivalence connecting the two:

$$R\Gamma_X^G(T_X|_V) \simeq H^0(X/G|_{f(V)}, R^q q_*^G T_X|_{f(V)}) \simeq H^q(G, \Gamma(X|_V, T_X|_V))$$

Consider the closed subset $\iota : V \hookrightarrow X$ of points on which G acts nonfreely, and consider the complement $V^c := X - V$. At discs around points y such that $f^{-1}(y)$ lay in V^c (that is, points y which are not branch points), $(q_*^G \mathcal{F})_y \simeq \bigoplus_{x \in f^{-1}(y)} \mathcal{F}$ is

a free summand of $f^{-1}(y)$ copies of the sheaf $\mathcal{F}$, permuted cyclically according to the global action of G . Free constructions are exact, thus for $q \geq 0$,

$$R^q q_*^G(\mathcal{F}|_{V^c}) \simeq 0.$$

Thus, $R^q q_*^G(\mathcal{F})$ is a skyscraper sheaf with values on $y \in f(V)$ such that G has nontrivial stabilizer on $f^{-1}(y)$. Let x be a point on X in which G acts with nontrivial stabilizer, then $\Gamma(X|_x, T_X|_x) \simeq \widehat{T}_{X,x}$, where the latter is a G_x -module.

The cohomology group of interest at a point $y_i \in f(V)$ is of the form

$$H^0(X/G|_{y_i}, R^q q_*^G T_X|_{y_i}) \simeq H^q(G, \prod_{x \mapsto y_i} \widehat{T}_{X,x}).$$

Note that for any $x' \in \text{Orb}_G(x)$

$$\text{Ind}_{G_{x'}}^G \widehat{T}_{X,x'} \simeq \text{Ind}_{G_x}^G \widehat{T}_{x,x},$$

by the orbit-stabilizer theorem thus it matters not which point from the orbit we choose. Further,

$$\text{Ind}_{G_x}^G \widehat{T}_{X,x} \simeq \bigoplus_{x \in \text{Orb}_G} \widehat{T}_{X,x}.$$

Shapiro's lemma further tells us

$$H^q(G, \text{Ind}_{G_x}^G \widehat{T}_{X,x}) \simeq H^q(G_x, \widehat{T}_{X,x}).$$

The clean formulation claimed follows from

$$\bigoplus_{x \in R} H^q(G_x, \widehat{T}_{X,x}) \simeq \bigoplus_{y \in \text{supp}(B_f)} \bigoplus_{x_k \in \text{Orb}_G(f^{-1}(y))} H^q(G_{x_k}, \widehat{T}_{X,x_k}).$$

(ii) Consider a cyclic group C_m acting on an R -module, the group cohomology $H^i(C_m, R)$ will vanish when $i > 0$ if m is invertible in R as the transfer map is invertible. $\square$

8.4. Equivalence of Deformation Problems.

Theorem 8.19. *(break it up) For X a smooth curve over a field k with an action of G , if $n = \dim \text{Ext}_G^1(\Omega_X, \mathcal{O}_X)$ is finite. Then, $\text{Def}_{(X,G)}$ is representable by a ring R which carries an $\text{Aut}(X, G)$ -action by Defn 7.2.*

(i) without further assumptions,

$$R \simeq W(k)[[t_1, \dots, t_n]]/(f_1, \dots, f_i)$$

(ii) Let R_σ be the ring representing $\text{Def}_{(X,G)}^{\text{loc}}$, and $m = \dim H^0(X/G, R^1 q_*^G T_X)$ then

$$R \simeq R_\sigma[[t_1, \dots, t_m]]$$

(iii) If $\text{Ext}_G^2(T_X, \mathcal{O}_X) = 0$ then $\text{Def}_{(X,G)}$ is formally smooth and represented by a ring

$$R \simeq W(k)[[t_1, \dots, t_n]].$$

Proof. (i) and (iii) follow immediately from Lemma 8.2. (ii) This proof takes the strategy of identifying the map between the cotangent complexes of these formal moduli problems $\text{Loc} : \text{Def}_{(X,G)} \rightarrow \text{Def}_{(X,G)}^{\text{loc}}$ with the surjective map arising from the short exact sequence computing $R^1 \Gamma^G(T_X)$ given by Grothendieck spectral sequence by $\Gamma_X^G \simeq \Gamma_{X/G} \circ q_*^G$.

$$H^1(Y, q_*^G T_X) \rightarrow \text{Ext}_G^1(\Omega_X, \mathcal{O}_X) \rightarrow H^0(Y, R^1 q_*^G T_X).$$

Lemma 8.18 shows that $H^0(X/G, R^1 q_*^G T_X) \simeq \bigoplus_{x \in R} H^1(G_x, T_{X,x})$, the latter being the tangent space of $\text{Def}_{(X,G)}^{\text{loc}}$ by Lemma 8.6. The map $\delta(\text{Loc})$ is the surjective map above by Lemma 3.3.2 (BM00). We summarize the result in the diagram below.

$$\begin{array}{ccccc} T(\text{Def}_{(X,G)}) & \xrightarrow{\delta(\text{Loc})} & T(\text{Def}_{(X,G)}^{\text{loc}}) & & \\ \downarrow \simeq & & \downarrow \simeq & & \\ & & \bigoplus_{x \in R} H^1(G_x, T_{X,x}) & & \\ \downarrow \simeq & & \downarrow \simeq & & \\ H^1(X/G, h_* T_X) & \hookrightarrow & R^1 \Gamma_X^G(T_X) & \twoheadrightarrow & H^0(X, R^1 h_* T_X) \end{array}$$

□

The totally ramified assumptions in the lemma below can be removed by considering the more general case of Lemma 8.14 and weighting B accordingly.

Theorem 8.20. (*group to branch*) For X a smooth curve over a field k with a tame action by a group G which is totally ramified, and $\text{Ext}_G^2(\Omega_X, \mathcal{O}_X) \simeq 0$, let B be the set of branch points of the quotient map $q_* : X \rightarrow X/G$. Then, the functor L is an isomorphism of formal moduli problems

$$L : \text{Def}_{(X,G)} \simeq \text{Def}_{(X/G;B)}$$

Proof. This reduces to showing that the map L defined in Defn 7.12 induces an isomorphism on tangent spaces of deformations. When G is a totally ramified tame action, $H^0(Y, R^1 q_*^G T_X) \simeq 0$ by Lemma 8.18, thus

$$H^1(Y, q_*^G T_X) \simeq \text{Ext}_G^1(\Omega_X, \mathcal{O}_X).$$

It remains to show that the map $\text{Ext}_G^1(\Omega_X, \mathcal{O}_X) \rightarrow H^1(Y, q_*^G T_X)$ induced by the isomorphism of the exact sequence is the map on tangent spaces $\delta(L)$. This follows from combining (OP06) Theorem 5.5 and (Wew05) Prop 4.10. □

8.5. Artin-Schreier Case.

Theorem 8.21. For X the Artin Schreier curve and $G = C_{p-1}$ described in Section 1, $\text{Def}_{(X,G)}$ is representable and represented by a ring $R \simeq W(k)[[t_1, \dots, t_{p-2}]]$ with $\text{Aut}(X)$ -action induced by Defn 7.2.

Proof. This is a special case of Theorem 8.19. As before, we are putting together

$$H^1(Y, q_*^G T_X) \rightarrow \text{Ext}_G^1(\Omega_X, \mathcal{O}_X) \rightarrow H^0(Y, R^1 q_*^G T_X).$$

Let's start with the image. Since $R^1 q_*^G T_X$ is a skyscraper sheaf $H^0(Y, R^1 q_*^G T_X) \simeq R^1 q_*^G T_X$. By Theorem 8.18, $H^0(Y, R^1 q_*^G T_X) \simeq 0$. Thus, $R_\sigma \simeq W(k)$.

By Theorem 8.17, $\dim H^1(Y, q_*^G T_X) = 3g_{\mathbb{P}^1} - 3 + (p+1) = p-2$, since $|B| = p+1$ by Lemma 4.2.

The local-to-global spectral sequence also shows ((BM00) 3.3.3, (Wew05) 3.3) that

$$\text{Ext}_G^2(\mathcal{L}_{X/k}, \mathcal{O}_X) \simeq H^1(Y, \mathcal{E}xt_G^1(\mathcal{L}_{X/k}, \mathcal{O}_X)).$$

However, $\mathcal{E}xt_G^1(\mathcal{L}_{X/k}, \mathcal{O}_X)$ is a skyscraper sheaf by 8.18, whose cohomology vanishes above H^0 . Hence, $\mathcal{E}xt_G^2(\mathcal{L}_{X/k}, \mathcal{O}_X)$ vanishes and $\text{Def}_{(X,G)}$ is unobstructed.

Since G is in the center of $\text{Aut}(X)$, $\text{Aut}(X, G) \simeq \text{Aut}(X)$ as shown in Example 7.10. Thus, the automatic action of $\text{Aut}(X, G)$ given by Defn 7.2 is in fact an action of $\text{Aut}(X)$. $\square$

Theorem B.1. *For X the Artin-Schreier curve over a field k with a tame action by a group $C = C_{p-1}$ described in section 1, let B be the set of branch points of the quotient map $q_* : X \rightarrow X/G$ described in Lemma 4.2. Then, there is an $\text{Aut}(X)/G$ -equivariant equivalence of formal moduli problems*

$$\text{Def}_{(X,G)} \simeq \text{Def}_{(\mathbb{P}^1, p+1)} \simeq \text{Def}_{(\mathbb{A}^1, p)}$$

Proof. The isomorphism of formal moduli problems follows as an example of Theorem 8.20, since $(X/G, B) \simeq (\mathbb{P}^1, p+1)$ by Lemma 4.2. The isomorphism $\text{Def}_{(\mathbb{P}^1, p+1)} \simeq \text{Def}_{(\mathbb{A}^1, p)}$ follows by fixing one of the points to be the point at infinity, thus restricting to the affine branch points $|B_{\text{aff}}| = p$.

We now justify its enhancement to an equivariant equivalence. Our deformation problems are discrete by Lemma 8.4, which implies we have an equivalence of automorphisms of the universal object and automorphisms of the object being deformed by Lemma 7.5.

The action $\text{Aut}(X, G) \simeq \text{Aut}(X)$ since G is in the center of $\text{Aut}(X)$. To check that $L : \text{Def}_{(X,G)} \rightarrow \text{Def}_{(X/G, b_j)}$, it suffices to check that $\text{Aut}(X, G)/G \rightarrow \Sigma_{|B_{\text{aff}}|}$ is an injection. This is shown in Lemma 5.5. Since it's an injection, it induces an isomorphism. $\square$

9. THEOREM B.2: REPRESENTATIONS OF GLOBAL CONFIGURATION SPACES

It's time to analyze the symmetric group action on configuration stacks of points on $\mathbb{P}^1$.

Definition 9.1. *We define the moduli stack $\text{Conf}(X, k)$ for an object X with a topology as*

$$\text{Conf}(X, k) := \{(x_1, \dots, x_n) \in X^k \mid x_i \neq x_j, i \neq j\}.$$

This is an open space of X^k obtained removing the closed diagonal subspaces

$$\Delta_{i,j} := \{(x_1, \dots, x_n) \in X^k \mid x_i = x_j\}.$$

Each tuple is called an ordered configuration, and each entry is a point of the configuration.

Definition 9.2. *The symmetric group of k -letters acts on $\text{Conf}(X, k)$ by permuting the points of the configuration. If $\sigma \in \Sigma_k$ and $(x_1, \dots, x_k) \in \text{Conf}_k(X)$ then $\sigma(x_1, \dots, x_k) = (x_{\sigma(1)}, \dots, x_{\sigma(k)})$. The orbit space is denoted by*

$$C_k(X) = \{\{x_1, \dots, x_k\} \mid x_i \in X, x_i \neq x_j\}.$$

The symmetric group action is free and properly discontinuous, so the quotient map $\text{Conf}_k(X) \rightarrow C_k(X)$ is a regular covering space of degree $k! = \Sigma_k$.

The affine group $\mathbb{G}_a \rtimes \mathbb{G}_m \simeq \text{Stab}_\infty(\mathbb{P}^1) \simeq \text{Aut}(\mathbb{A}^1)$ is the subgroup of $\text{Aut}(\mathbb{P}^1)$ fixing infinity (thus acting only 2-transitively), thus,

$$M_{0,n+1} := \text{Conf}(\mathbb{P}^1, n+1) // \text{Aut}(\mathbb{P}^1) \simeq \text{Conf}(\mathbb{A}^1, n) // \text{Aut}(\mathbb{A}^1).$$

Considering the action of Σ_n by permuting marked points, an n -marked stack represented by a ring becomes a Σ_n -representation. Let's quickly review a representation we will be using intensively.

Definition 9.3. *The group Σ_n acts on R^n by permuting basis vectors; this defines a representation called the permutation representation. This representation has a 1-dimensional invariant subspace, spanned by the vector $e_1 + e_2 + \cdots + e_n$ which is the trivial representation. A complementary subspace to this is*

$$V = \{a_1 e_1 + \cdots + a_n e_n \mid a_1 + \cdots + a_n = 0\}.$$

*This is called the **standard representation** or irreducible $n-1$ dimensional representation λ of Σ_n .*

9.1. Configuration Space as a Σ_n Representation. Let's consider the action of Σ_n on the group ring $R[C_n]$ considered as a module $V := R\{g_1, \dots, g_n\}$ with basis $B := \{g_1, \dots, g_n\}$. Applying $\text{Sym}(V)$ gives a polynomial ring with generators our basis B .

Theorem 9.4. *(simple gods) (i) Let Δ be the fat diagonal, that is, all 2-planes defined by $x_i = x_j$ for $i \neq j$.*

$$\text{Conf}(\mathbb{A}^1, n) \simeq (\text{Sym}(\mathbb{A}^1) - \Delta) // (\mathbb{G}_a \rtimes \mathbb{G}_m)$$

(ii) $\text{Conf}(\mathbb{A}^1, n)$ is represented by the ring $\text{Sym}(\lambda)[N^{-1}] // \mathbb{G}_m$ as a Σ_n -representation, where λ is the standard representation $\lambda := W(k)\{y_0, \dots, y_{n-1}\}/(y_1 + \cdots + y_{n-1})$ with $|y_i| = -2$, and $N = \prod_{i=0}^{n-1} \sigma^i(y_0)$, where σ generates C_n .

Remark 9.5. Specifying the $\mathbb{G}_m$ action is equivalent to specifying a grading on $\text{Sym}(\lambda)[N^{-1}]$. We could consider $\text{Conf}(\mathbb{A}^1, n)$ with a different grading say, $\mathbb{G}_m$ acts by μ^{-k} rather than μ itself. In this case, we'd have $\text{Sym}(\lambda)$ is a graded ring where $|x_i| = -k$.

Proof. (i) Choosing n points on $\mathbb{A}^1$ is equivalent to an n -tuple, that is a point of $\mathbb{A}^n \simeq \text{Sym}(\mathbb{A}^1)$. To make sure points do not coincide, we remove the fat diagonal, which is the removal of all 2-planes defined by $x_i = x_j$ for $i \neq j$. We wish to consider our states up to automorphisms of $\mathbb{A}^1$ which are $\text{Aut}(\mathbb{A}^1) \simeq \mathbb{G}_a \rtimes \mathbb{G}_m$, where $\mathbb{G}_a$ is the translation action, and $\mathbb{G}_m$ is the scaling action, which acts on $\mathbb{A}^n$ by scaling all coordinates equally.

(ii) The ring representing $\text{Sym}(\mathbb{A}^1) - \Delta$ is $Y := R[x_1, \dots, x_n][N^{-1}]$, where $N := \prod_{i \neq j} (x_i - x_j)$. When we quotient by $\mathbb{G}_a$ on the stack, this translates to forcing translation invariance on functions. On coordinates this gives us

$$\tilde{Y} := Y^{\mathbb{G}_a} := (R[x_1, \dots, x_n][N^{-1}])^{\mathbb{G}_a} \simeq R[x_2 - x_1, \dots, x_n - x_1][N^{-1}].$$

Note that the latter is a basis of the reduced permutation representation.

By choosing the additive basis $x_i - x_{i+1}$ for the standard representation σ , this gives a map between the two rings,

$$\text{Sym}(\lambda)[N^{-1}] \rightarrow \tilde{Y}.$$

After change of basis, $N^{-1} := \prod \sigma^i(y_0)$. It remains to endow the left with an action of $\mathbb{G}_m$ and show that the map is $\mathbb{G}_m$ -equivariant. The scaling action $x_i \mapsto \mu x_i$, coincides with the action on the right. This concludes the demonstration of their equivalence. $\square$

Remark 9.6. For reference, we include the projective version. For $n \geq 3$,

$$\mathrm{Conf}(\mathbb{P}^1, n) \simeq (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \Delta,$$

where $\Delta = \{(q_i)_i : \exists i \neq j \text{ with } q_i = q_j\}$ is the fat diagonal. The meat of why this holds comes from the isomorphism $(\mathbb{P}^1)^3 - \Delta \simeq \mathrm{PGL}_2$.

Definition 9.7. *The reduced regular representation of C_n is defined as $\bar{\rho} := \ker(R[C_n] \rightarrow R)$. This inherits an action of $\mathrm{Aut}(C_n)$, we also call the $\bar{\rho}$ corresponding representation of $C_n \rtimes \mathrm{Aut}(C_n)$.*

Remark 9.8. The restriction of the standard Σ_n -representation λ to $\mathbb{Z}/n \rtimes \mathbb{Z}/(n-1)$ is $\bar{\rho}$.

Corollary 9.9. *Given a set of points $P := \{p_1, \dots, p_n\}$ in $\mathbb{A}^1$,*

$$\mathrm{Def}(\mathbb{A}^1; n) \simeq \mathrm{Conf}(\mathbb{A}^1, n)_P^\wedge$$

is representable by the graded ring

$$\mathrm{Sym}(\lambda)[N^{-1}]_I^\wedge // \mathbb{G}_m$$

as a Σ_n -representation, where $I = (x_1 - p_1, \dots, x_n - p_n)$.

Theorem B.2. *In char $k = p$, consider $\mathbb{F}_p$ in $\mathbb{A}^1 \simeq \mathrm{Spec} k[t]$, that is, the set of points $P := \{g^i(p)\}$ where g is the generator of $\mathbb{Z}/p$ which acts by $t \mapsto t + 1$, then*

$$\mathrm{Def}(\mathbb{A}^1; n) \simeq \mathrm{Conf}(\mathbb{A}^1, n)_P^\wedge$$

is representable by the graded ring

$$R \simeq \mathrm{Sym}(\lambda)[N^{-1}]_I^\wedge // \mathbb{G}_m$$

as a Σ_n -representation. Here λ is the standard representation $\lambda := W(k)\{y_0, \dots, y_{n-1}\}/(y_1 + \dots + y_{n-1})$ with $|y_i| = -2$, and $N = \prod_{i=0}^{n-1} \sigma^i(y_0)$, $I = (p, I := (1 - g^j(y_i)))$, where σ generates C_n .

Proof. This is immediate from Cor 9.9 of Theorem 9.4. $\square$

Theorem B. *Let R be as in Theorem B.2, $\mathrm{Def}_{(X,G)}^*$ is represented by R as a $\mathrm{Aut}(X)/G$ -representation.*

Proof. This immediately follows combining Theorem B.2 with Theorem B.1, the latter of which states that

$$\mathrm{Def}_{(X,G)} \simeq \mathrm{Def}_{(\mathbb{A}^1, p)}$$

and this is an equivariant equivalence with respect to $\mathrm{Aut}(X)/G \hookrightarrow \Sigma_p$. $\square$

Remark 9.10. Once we have R as an $\mathrm{Aut}(X)/G$ representation, we also get it as a $\mathrm{Aut}(X)$ representation by induction, as shown in Lemma 10.36.

9.2. Symmetric Group Action on Cohomology of Configuration Spaces.

We include this section for when we later discuss grading choices for the graded Lubin-Tate action in Theorem [C\(b\)](#).

Considering the action of Σ_n by permuting marked points, the cohomology of any of stack of n -marked objects becomes a Σ_n -representation. Let us consider the cohomology of $\text{Conf}(\mathbb{A}^1, n)$ as a Σ_n -representation.

Lemma 9.11.

$$H^*(\text{Conf}(\mathbb{A}^1, n)(R)) \simeq \begin{cases} \text{triv} & * = 0 \\ \lambda & * = -2 \\ 0 & \text{else} \end{cases}.$$

where λ is the irreducible dimension $n - 1$ representation from definition [9.3](#).

Proof. Recall, the affine group $G := \mathbb{G}_a \rtimes \mathbb{G}_m$ is the subgroup of $\text{Aut}(\mathbb{P}^1)$ fixing the point $\infty \in \mathbb{P}^1$. As stated earlier, it follows that the quotient $\text{Conf}(\mathbb{A}^1, n) // G$ is equal to the moduli space $M_{0,n+1}$ of $(n + 1)$ points on the projective line modulo symmetries. In other words,

$$M_{0,n+1} := \text{Conf}(\mathbb{P}^1, n + 1) // \text{Aut}(\mathbb{P}^1) \simeq \text{Conf}(\mathbb{A}^1, n) // \text{Aut}(\mathbb{A}^1).$$

This makes $\text{Conf}(\mathbb{A}^1, n)$ homotopic to a trivial $\mathbb{G}_a \rtimes \mathbb{G}_m$ -bundle over $M_{0,n+1}$ (in char 0, this would be a trivial circle bundle since $\mathbb{G}_a$ is contractible, in general this is a $\mathbb{G}_m$ bundle), and so

$$H^\bullet(\text{Conf}(\mathbb{A}^1, n)) \cong H^\bullet(M_{0,n+1}) \oplus H^{\bullet-1}(M_{0,n+1}).$$

Only needs to construct such an Σ_n -equivariant factorization for the cohomology $H^\bullet(M_{0,n+1})$. There is a locally trivial Σ_n -equivariant fiber bundle

$$M_{0,n+1} \rightarrow M_{0,n}$$

by forgetting the last marking such that each fiber F is $\mathbb{P}^1$ minus n points ([FN62](#))(Theorem 3). This is because the fiber over a point $Q_n \in M_{0,n}$ is exactly the choices of an $n + 1$ th point distinct from Q_n , which is $\mathbb{P}^1 - Q_n$. By excision $H^0(F) \cong \text{triv}$, and $H^1(F) \cong \lambda$ as Σ_n -representations. The Leray-Serre spectral sequence for $M_{0,n+1}$ degenerates shifting degree, and this gives the claimed result. $\square$

Remark 9.12. In the notation of ([FN62](#)), $F_{0,n}(\mathbb{A}^1) // \text{Aut}(\mathbb{A}^1) \simeq F_{2,n-2}(\mathbb{A}^1 - Q_2)$. Note that $F_{2,n-2}(\mathbb{A}^1 - Q_2)$, that is, the configuration space of $\mathbb{A}^1$ minus 2 points is exactly the j line.

9.3. Comparison with Global Stacks: Marked Points as Level Structure.

The picture of $\text{Aut}(X)$ acting on $\text{Def}_{(X,G)}$ through the branch points has a global form: a moduli stack of $(p - 1)$ -elliptic curves with ordered marked points, the analogue of the Legendre family $x^2 = y(y - 1)(y - \lambda)$, i.e. of $X(2)$, the moduli of elliptic curves with level 2 structure ([Sto12](#), Prop. 7.1). For $p > 3$ the curves carry no group structure, and the ordered branch points take the place of the level structure.

Definition 9.13. (*ford truck*) Let F^{ord} be the moduli functor on $\mathbb{Z}[\frac{1}{p-1}]$ -schemes whose objects over R are smooth families $x^{p-1} = (y - y_0) \cdots (y - y_{p-1})$ with $y_0, \dots, y_{p-1} \in R$ pairwise distinct, marked by the ordered points $P_i = [0 : y_i : 1]$, and whose morphisms are the coordinate changes $x \mapsto \mu^{-p}x$, $y \mapsto \mu^{-(p-1)}y + c$ preserving this form.

Theorem 9.14. *The stack $\mathcal{M}_{g,C_{p-1}}$ of smooth curves of genus $g = (p-1)(p-2)/2$ with a C_{p-1} -cover of $\mathbb{P}^1$ is equivalent to F^{ord} , and F^{ord} is represented by $(\mathbb{A}^p - \Delta) // (\mathbb{G}_m \rtimes \mathbb{G}_a)$, where $\mathbb{G}_a$ translates the y_i and $\mathbb{G}_m$ acts by μ^{p-1} ; in the coordinates $r_i = y_i - y_0$ this is $(\mathbb{A}^{p-1} - \{0\}) // \mathbb{G}_m \simeq \mathbb{P}^{p-2}$.*

Proof. As in (Sto12, Prop. 7.1). A C_{p-1} -cover of $\mathbb{P}^1$ is a Kummer extension $x^{p-1} = f(y)$ (Lemma 2.2), and Riemann–Hurwitz forces $\deg f = p$ with $p+1$ branch points, one at infinity; over an étale extension in which f splits the curve has the form of Definition 9.13, and conversely every such curve carries the action $x \mapsto \zeta x$ with the P_i as its fixed points. The morphisms of F^{ord} are exactly the changes of variables of $\mathbb{P}^2$ preserving the split form, which is $\mathbb{G}_m \rtimes \mathbb{G}_a$, and the discriminant is inverted because the y_i are distinct. Quotienting by $\mathbb{G}_a$ first, $y_i \mapsto r_i = y_i - y_0$, leaves $\mathbb{A}^{p-1} - \{0\}$ with $\mathbb{G}_m$ acting by μ^{p-1} . $\square$

Remark 9.15. The Hopf algebroid of $(p-1)$ -elliptic curves without the ordering is the one used by Hill (Hil06) to compute $\pi_* EO_{p-1}$. The difference from $\text{Conf}(\mathbb{A}^1, p)$ of Theorem 9.4 is only the $\mathbb{G}_m$ -action, a stacky memory of the C_{p-1} -action on the branch locus. The stack $\mathcal{M}_{g,C_{p-1}}$ is also the C_{p-1} -fixed locus of the moduli of curves with level structure on their Jacobians (BM00, Thm. 5.2, 5.4), and its completion at X is $\text{Def}_{(X,G)}$ (BM00, Lemma 5.2.1).

Lemma 9.16. *(past and future) Let R be the set of points of X with nontrivial stabilizer under $G = C_{p-1}$. Then $(F^{\text{ord}})_{(X,R)}^\wedge \simeq \text{Def}_{(X,G)}$.*

Proof. Send $(\mathfrak{X}, G)$ to $(\mathfrak{X}, \mathcal{R})$, $\mathcal{R}$ its points with nontrivial stabilizer. All curves in $\text{Def}_{(X,G)}$ have the genus of X (Chevalley–Weil, (BR11, Prop. 3.1)), so essential surjectivity and full faithfulness follow from Theorem 9.14. $\square$

Remark 9.17. These global problems sit in the smooth locus of a Hurwitz stack, whose compactification is the natural home for a connective eo_{p-1} ; we leave that to another place. The interested reader can find this and other related global stacks in v1 of this paper. Most of the comparison was removed in this version as suggested by a reviewer.

Part 4. Controlling the p -Divisible Group of the Jacobian of our Curve

OVERVIEW

We consider the curve X with affine equation $y^p - y = x^{p-1}$ over a characteristic p field k which contains $\mathbb{F}_{p^{p-1}}$. Given this curve X , we devote this section to understanding the formal group of its Jacobian. We shall see that the group $G' := \text{Aut}(X)$, which determines the curve X , also determines its eigenvalues and its center $G := C_{p-1}$ determines its idempotent decomposition.

9.4. Background: Idempotent Decompositions. Our analysis of the formal group law $\text{Jac}(X)_e^\wedge$ relies on constructing a splitting of this formal group law into smaller dimensional pieces. This originates from a more general construction which we briefly outline.

Construction 4. *Any p -complete additive category Y with a G -action splits due to a splitting of the source of the following map,*

$$k[G] \hookrightarrow \text{End}(Y),$$

For $S_1, \dots, S_n$ representatives of the isomorphism classes of irreps of G , there is an isomorphism of rings.

$$k[G] \simeq \text{End}_k(S_1) \times \cdots \times \text{End}_k(S_n)$$

If we take G to be coprime to k , thus the representatives are characters of G . Then for $i = 1, \dots, n$ the element $e_i = f^{-1}(\text{id}_{\text{End}_k(S_i)})$ is a central idempotent (projector) in $k[G]$. Further, we can write e_i as

$$e_i = \frac{1}{|G|} \sum_{g \in G} \chi_{S_i}(g) g^{-1}.$$

In other words,

$$k[G] \simeq \bigoplus_{i=1}^n e_i Z_p[G].$$

This induces a splitting on Y ,

$$Y \simeq \bigoplus_{i=1}^n e_i Y.$$

Remark 9.18. This also works for an object Y in an idempotently complete category. Let's go Maschke's theorem.

Lemma 9.19. *Consider a group G' such that $G \subseteq Z(G')$ lies in its center. Let $\mathcal{C}$ be an additive category with a G' action.*

Let $\mathcal{I}^i$ be the functor defined by projecting onto the i th idempotent component of a G -equivariant idempotent decomposition of $\mathcal{C}$, as in Construction 4.

$$\mathcal{I}^i: \mathcal{C} \rightarrow e_i \mathcal{C}$$

Then, $\mathcal{I}^i$ is G' -equivariant.

Proof. This is because G sits inside of the center of G' , which means that for any $g \in G'$, $g(e_i \mathcal{C}) = e_i g(\mathcal{C})$. $\square$

9.5. Main Theorems and Proof Outline. Fix X to be the Artin-Schreier curve over $k = \mathbb{F}_{p^{p-1}}$ with affine form $y^p - y = x^{p-1}$.

Theorem C.1. *(splitty dooda) Let A be the p -divisible group of $\text{Jac}_k(X)$, then A has an C_{p-1} idempotent integral decomposition*

$$A \simeq \bigoplus_{\chi \in \text{Hom}(C_{p-1}, \mathbb{G}_m)} A^{\chi^i}.$$

into $p-2$ summands of dimension $1, 2, \dots, p-1$, all of height $p-1$, where A^{χ^i} is of dimension i . There is no component corresponding to the trivial character.

Corollary C.1'. *(h-splitting) The p -divisible group A^{χ^1} is connective and corresponds to a formal group of dimension 1 and height $p-1$.*

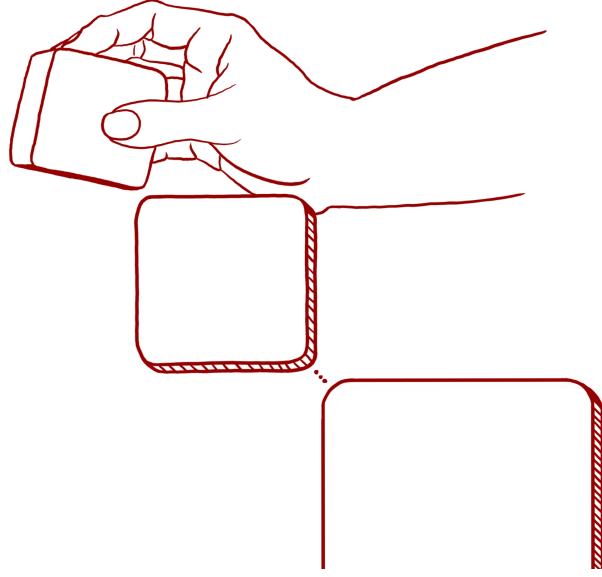


FIGURE 2. These cubes are the Jordan blocks of the decomposition of our p -divisible group into lower dimensional summands in Theorem C.1 and Corollary C.1'. Our functor selects the one dimensional summand off the top (Defn 10.3)

We abbreviate $\text{Jac}(X)$ as $\mathcal{J}$. The plan: the central $G = C_{p-1}$ splits $\mathcal{J}[p^\infty]$ by its idempotents (Section 9.4); the dimension of each summand is read off the G -action on $H^0(X, \Omega_X^1) = T_e^* \mathcal{J}$ (Proposition 9.20); and the height of each summand is read off the slopes of Frobenius, which Manin's formula computes from the automorphisms of X (Theorem 9.22). Theorem C.1 assembles the two through the Hodge filtration on $H_{\text{cris}}^1/p = H_{\text{dR}}^1$, and Corollary C.1' names the one-dimensional summand.

9.6. Dimensions of the Split Formal Group from Decomposition of Holomorphic Differentials, H_{dR}^1 . We fix the following notation for the rest of this section. Let $\text{Jac}(X)$ be the Jacobian of a projective curve X with affine form $y^p - y = x^{p-1}$ over $R \in \mathbf{Art}_k$, and let $\text{Jac}(X)_e^\wedge$ be its formal group.

Convention 2 (group element names). $G = C_{p-1} = \langle \tau^{p-1} \rangle$ is the central tame subgroup, acting by $x \mapsto \eta x$, $y \mapsto y$, $\eta = \zeta^{p-1}$ where ζ is a $(p-1)^2$ root of unity.

Proposition 9.20. (decomposition)

- (1) The holomorphic differentials $\Omega := \Omega_X^1$ have the basis

$$\{x^i y^j \omega : 0 \leq i + j \leq d - 3\}$$

where $\omega := \frac{dy}{P_x} = \frac{dx}{P_y}$ where P_x, P_y are the derivatives of P with respect to x and y respectively.

- (2) Let us consider characters of G , $\chi \in \text{Hom}(G, \mathbb{G}_m)$. The group G acts on Ω by $\chi^{-(i+1)}$,

$$\dim H^0(\Omega^1)^{\chi^j} = j \quad (1 \leq j \leq p-2), \quad H^0(\Omega^1)^{\chi^0} = 0,$$

- (3) The χ^1 -block is the line $k\omega_1$, $\omega_1 = dy/x$. The generator τ of $C_{(p-1)^2}$ acts on ω_1 by ζ^{-1} , a primitive $(p-1)^2$ -th root of unity, and σ fixes ω_1 .

Proof. (1) A genus of smooth projective plane curve of degree d is $(d-1)(d-2)/2$, so we need to construct that many linearly independent holomorphic differentials. For a smooth plane curve X given by equation $P(x, y) = 0$ we have $P_x dx + P_y dy = 0$ (by differentiating $P = 0$), where P_x, P_y are the derivatives of P with respect to x and y respectively. Since X is smooth the vector (P_x, P_y) doesn't vanish anywhere on X . Hence we can define a holomorphic nowhere (on X) vanishing 1-form $\omega = \frac{dy}{P_x} = -\frac{dx}{P_y}$. We may also choose any non-zero scalar multiple of ω as our ω .

Let us examine pairs (i, j) such that $0 \leq i + j \leq d - 3$. This is choosing ordered pairs of numbers from the set $\{0, 1, \dots, d - 2\}$, in other words, from a set of $d - 1$ numbers. There are then $\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$ of such (i, j) .

We thus get a set of precisely $(d-1)(d-2)/2$ holomorphic forms $x^i y^j \omega$ on X , and they are linearly independent, hence a basis: the canonical sheaf of a smooth plane curve of degree d is $\mathcal{O}_X(d-3)$ (Har77, II, Example 8.20.3), so $H^0(X, \Omega_X^1)$ is the restriction of the forms of degree $d-3$ on $\mathbb{P}^2$, and the monomials $x^i y^j$ with $i + j \leq d - 3$ restrict injectively because X lies on no curve of degree less than d .

(2) Under $x \mapsto \eta x$, using the notation of convention 2, $\omega := \frac{dy}{x^{p-2}}$ is of weight $\chi^{-(p-2)}(\eta) = \eta^{-(p-2)}$, and the weight of $x^i y^j \omega$ is $\eta^{i-(p-2)}$. Now we do some combinatorics to show the dimension breakup. There are $d-2$ pairs such that $i = 0$, i.e., $(0, 1), (0, 2), \dots, (0, d-2)$. Further, there are $d-3$ pairs such that $i = 1$, and in general, $(d-2) - k$ pairs such that $i = k$. Ending with 1 pair such that $i = d-3$, i.e., $(d-3, d-2)$. Let's set $a = i - (p-2)$. Thus, according to the pairs computation we just did, $\dim(\chi^a) = a$ for $1 \leq a \leq p-2$, and nothing for $a = 0$.

(3) This is a direct consequence of (2). There is only one element such that $i = p-3$ ($a = 1$), which is $x^{p-3} \frac{dy}{x^{p-2}} = \frac{dy}{x}$. Finally, $\tau^*(dy/x) = \frac{\zeta^{(p-1)} dy}{\zeta^p x} = \zeta^{-1} \frac{dy}{x}$. The action of σ is trivial, since it has order p and $k^\times$ has no nontrivial p -th roots of unity. Hence, σ fixes x and dy . $\square$

Remark 9.21. This is the Lubin–Tate character row of the table in Subsection 0.4: the wild group acts on the block through $\mathbb{Z}_p[\zeta_p]$, and $\pi = \zeta_p - 1$ is its uniformizer, with

$$\pi^{p-1} = -p \cdot (\text{unit}),$$

because $[p](T)/T = ((1+T)^p - 1)/T$ is Eisenstein of degree $p-1$. This is the identity of Subsection 0.4 in its third costume: the height of F is $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p-1$ because the ramification index of $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ is the exponent in $\pi^{p-1} = -p$, and the slope $1/(p-1)$ of Manin's formula (Lemma 9.22) is $1/e$.

9.7. Isogeny Decomposition Using Eigenvalues of Frobenius. We now describe its slope decomposition, which is a description of H_{cris}^1 up to isogeny, i.e. $H_{cris}^1[1/p]$. These are the eigenvalues of Frobenius.

Let us consider χ to be a multiplicative character, and ϕ to be an additive character. An additive character is a character ϕ of the additive group $\mathbb{F}_q$ so $\phi(a+b) = \phi(a)\phi(b)$ for all a and b . A multiplicative character is a character χ of the unit group $\mathbb{F}_q^\times$ extended to be 0 on nonunits in order to be defined on the whole ring, while preserving the rule $\chi(a)\chi(b) = \chi(ab)$. For a multiplicative character χ on $\mathbb{F}_q$, its Gauss sum is $G(\chi) = \sum_{a \in \mathbb{F}_q} \chi(a) \zeta^{\text{tr}(a)}$ where $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the trace mapping, and ζ is a p th root of unity.

Let $t \in \mathbb{F}_p^\times$ and let's choose a lift $\tilde{t}$ to $W(k)$. We define $\phi_i(t) = \zeta^{\tilde{t}^i}$, and $\chi_j(t) = -\tilde{t}^{-j}$.

Theorem 9.22. (*Thm 4.1 (Man63), (Kat81)*) *The eigenvalues of Frobenius on the curve $X : y^p - y = x^{p-1}$ are sums of the following form.*

$$\tau(\phi_i, \chi_j) = \sum_{t \in \mathbb{F}_p^\times} \phi_i(t) \chi_j(t).$$

where $1 \leq i \leq p - 1$ and $1 \leq j \leq p - 2$.

Remark 9.23. This is originally a theorem of Davenport-Hasse. A beautiful proof of this by Coleman (Col88) constructs a correspondence associated to $\text{Aut}(X)$ which is equivalent to Frobenius as a correspondence. This determines the eigenvalues of Frobenius as an operator acting on $H_{\text{ét}}^1(X)$, which give the slopes of our p -divisible group and the height decomposition of our formal group up to isogeny. In this way the automorphisms of the curve X entirely determine its eigenvalues.

Corollary 9.24. *The slopes of the p -divisible group associated to the curve X are $\{1/(p-1), 2/(p-1), \dots, (p-2)/(p-1)\}$.*

Proof. We use the eigenvalues above. The key observation of Stickelberger is that for $\lambda = 1 - \zeta$,

$$\tau(\phi_i, \chi_j) = -j^{-1} \lambda^j \pmod{\lambda^{j+1}}.$$

Since $v_p(\lambda) = 1/(p-1)$, this means that $v_p(\tau(\phi_i, \chi_j)) = \frac{j}{p-1}$. So we get $(p-1)$ copies of each $1 \leq j \leq (p-2)$. Each of these eigenvalues has multiplicity $p-1$. $\square$

We now show how to go from this slope decomposition up to isogeny to an integral decomposition.

9.8. Decomposition of $H_{\text{cris}}^1(X)$ Gives Formal Group Law. We have collected our lemmas. We are now ready to prove Theorem C.1 and most importantly Corollary C.1'.

Theorem C.1. (*splitty dooda*) *Let A be the p -divisible group of $\text{Jac}_k(X)$, then A has an C_{p-1} idempotent integral decomposition*

$$A \simeq \bigoplus_{\chi \in \text{Hom}(C_{p-1}, \mathbb{G}_m)} A^{\chi^i}.$$

into $p-2$ summands of dimension $1, 2, \dots, p-1$, all of height $p-1$, where A^{χ^i} is of dimension i . There is no component corresponding to the trivial character.

Proof. Theorem A(3) says that X has p -rank 0, hence $\text{Jac}(X)[p]$ (and thus $\text{Jac}(X)[p^\infty]$) are connected. Thus, all summands of $\text{Jac}(X)[p^\infty]$ are also connected. Let χ be a multiplicative character of $\mathbb{Z}/(p-1)$. We use Theorem 9.20, which tells us that $H^0(X, \Omega_X^1) \simeq \bigoplus_{j \in \mathbb{Z}/(p-1)} H^0(X, \Omega_X^1)^{\chi^j}$ breaks up, where the components acted on by χ^j are j -dimensional.

Further, by Serre duality,

$$H^0(X, \Omega_X^1) \simeq H^1(X, \mathcal{O}_X)^\vee.$$

Because of the dual in this isomorphism, the summand acted on by χ^j is sent to the summand χ^{-j} , this is crucial for what follows. In particular, the summand of $H^1(X, \mathcal{O}_X)^{\chi^j}$ is $(p-1) - j$ -dimensional.

$$\begin{aligned} H^0(X, \Omega_X^1)^{\chi^j} &\rightarrow H^1(X, \mathcal{O}_X)^{\chi^{-j}} \\ H^0(X, \Omega_X^1)^{\chi^{-j}} &\rightarrow H^1(X, \mathcal{O}_X)^{\chi^j} \end{aligned}$$

Since $H_{dR}^1(X)$ is an extension of $H^1(X, \mathcal{O}_X)$ by $H^0(X, \Omega_X^1)$ as a $\mathbb{Z}/p-1$ -representation, the summand $H_{dR}^1(X)^{\chi^j}$ has dimension $j + p - 1 - j = p - 1$ for every $1 \leq j \leq p - 2$. We use the following short exact sequence.

$$H^0(X, \Omega_X^1) \rightarrow H_{dR}^1(X) \rightarrow H^1(X, \mathcal{O}_X)^\vee.$$

It follows that in the decomposition $H_{cris}^1 = \bigoplus (H_{cris}^1)^{\chi^j}$ every summand has rank $p - 1$ as a $W(k)$ -module. By Manin's slope formulas (Lemma 9.22), slope $1/(p - 1)$ appears at least once in H_{cris}^1 . Therefore it appears in some $(H_{cris}^1)^{\chi^j}$, which is to say that $(H_{cris}^1)^{\chi^j}[1/p]$ has $D_{1/(p-1)}$ as a direct summand.

Since the dimension of $(H_{cris}^1)^{\chi^j}[1/p]$ is $(p - 1)$ and not larger, we get that $D_{1/(p-1)}$ is actually all of $(H_{cris}^1)^{\chi^j}[1/p]$. So $(H_{cris}^1)^{\chi^j}$ is the integral Dieudonné module of some p -divisible group, and the sum of all slopes appearing is $(p - 1) * 1/(p - 1) = 1$ so the dimension of this p -divisible group is 1. The same argument works for all $(p - 1) * j/(p - 1) = j$ where $0 < j \leq p - 1$, giving a p -divisible group of dimension j and height $p - 1$. $\square$

We explicitly draw attention to the dimension one formal group, the spoils of our labor.

Corollary C.1'. *The formal group $A^{\chi^1} =: F$ has dimension 1 and height $p - 1$, with Lie algebra dual to the Hodge line $k\omega_1$*

In Dieudonné terms: Given the C_{p-1} -decomposition

$$H_{cris}^1(X, \Omega_X^1) \simeq \bigoplus_{j \in \text{Hom}(C_{p-1}, \mathbb{G}_m)} H_{cris}^1(X, \Omega_X^1)^{\chi^j},$$

the Dieudonné module of the formal group of dimension 1 and height h is the summand of $H_{cris}^1(X, \Omega_X^1)$ corresponding to $j = 1$.

Proof. We can conclude that $j = 1$, from the proof of Lemma 9.20. This is because a 1-dimensional p -divisible group has to have a 1-dimensional Lie algebra, and its Lie algebra is dual to F_{Hodge}^1 on $(H_{cris}^1)^{\chi^j}/p = (H_{dR}^1)^{\chi^j}$ but by Lemma 9.20 this F_{Hodge}^1 has dimension j . $\square$

Definition 9.25. *Let us consider a finite G cover $f : X \rightarrow Y := X/G$ of curves over a field k . This corresponds to a field theory extension $K(X)/K(X)^G$. The field theory trace is*

$$\text{Tr}_{X/Y} : K(Y) \rightarrow K(X)$$

defined by

$$\text{Tr}_{X/Y}(y) = \sum_{\sigma \in G} \sigma(y).$$

9.9. Wild Action on the Jacobian. To show injectivity of the functor we will later define, we need to also understand the action of C_p on our Jacobian.

Proposition 9.26 (the one-dimensional block). *Let $F := e_1 \operatorname{Jac}(X)[p^\infty]$, the ψ -eigenpart of the p -divisible group of the Jacobian. Let ζ_p be a p th root of unity. Then, the action of $\mathbb{Z}_p[C_p]$ on F factors through $\mathbb{Z}_p[C_p]/(N) \cong \mathbb{Z}_p[\zeta_p] =: \mathcal{O}_L$, $\sigma \mapsto \zeta_p$, and the Dieudonné module of F is free of rank one over $\mathcal{O}_{\tilde{L}} := \mathcal{O}_L \otimes W(\bar{k})$.*

Proof. We begin by considering the norm element $N := \sum_i \sigma^i$ in the group ring $\mathbb{Z}_p[C_p]$. The norm N is the trace map of the cover $X \rightarrow X/C_p$ as defined in Defn 9.25, since $K(X)/K(X)^{C_p} \simeq K(X/C_p)$ in the sense of an algebraic quotient defined in Def 2.1. Thus, the action of N on $\operatorname{Jac}(X)$ factors through the quotient $\operatorname{Jac}(X/C_p) \simeq \operatorname{Jac}(\mathbb{P}^1)$,

$$\operatorname{Jac}(X) \rightarrow \operatorname{Jac}(X/C_p) \rightarrow \operatorname{Jac}(X).$$

The abelian variety $\operatorname{Jac}(\mathbb{P}^1)$ has trivial Jacobian, as genus 0 curves have no non-trivial line bundles. Thus, N acts as zero on $\operatorname{Jac}(X)$, and thus N acts as zero on $\operatorname{Jac}(X)[p^\infty]$. Hence, the action of $\mathbb{Z}_p[C_p]$ on $\operatorname{Jac}(X)[p^\infty]$ factors through the quotient $\mathbb{Z}_p[C_p]/(N)$. This quotient ring, $\mathbb{Z}_p[C_p]/(N) \simeq \mathbb{Z}_p[\zeta_p] =: \mathcal{O}_L$. This isomorphism sends $\sigma \mapsto \zeta_p$. The action of $\mathbb{Z}_p[C_p]$ on $\operatorname{Jac}(X)[p^\infty]$ commutes with the idempotent e_1 because G is central in G' , thus G acts trivially on $\mathbb{Z}_p[G]$. Thus, $\mathcal{O}_L$ acts on $F = e_1 \operatorname{Jac}(X)[p^\infty]$. We have shown that the height of F is $p - 1$ in Corollary C.1'. Since the height of a formal group F' over a field k of characteristic p is the dimension of the Dieudonné module $D(F')$ over $W(k)$, this means that the Dieudonné module of F is a module of rank 1 over $\mathcal{O}_{\tilde{L}} := \mathcal{O}_L \otimes W(k)$, which is the ring of integers of the completion of the unramified extension L of $\mathbb{Q}_p$ such that $[L : \mathbb{Q}_p] = p - 1$. $\square$

Part 5. Theorem C: Functors and Lubin-Tate Representations

10. THEOREM C VIA KODAIRA-SPENCER

This is the heart¹. The proof of Theorem C is a coordinate-free Kodaira–Spencer argument: it needs no equation for the universal curve, and it is the proof that generalizes. It has exactly three parts, and every subsection of this section serves one of them.

(i) *The functor.* The central $G = C_{p-1}$ acts on the Jacobian of every deformation of (X, G) , and since $p - 1$ is a unit its idempotents e_j are integral. Set

$$\widehat{\mathcal{F}}^1(\mathcal{X}, \rho, \iota) := e_1 \operatorname{Pic}^0(\mathcal{X})[p^\infty],$$

the ψ -eigenpart of the p -divisible group, for ψ the character of G on the extremal differential $\omega_1 = dy/x$. Its special fiber is a one-dimensional formal group F of

¹An earlier draft of this paper proved the equivalence of stacks the other way round: it first proved that the formal group of the universal curve is universal, by writing down a p -typically adapted presentation of the universal curve and applying a recognition principle to its invariant differential, and only then deduced the equivalence. That draft contained the remark: “The author strongly believes there is an alternative way to phrase the recognition principle which neither uses coordinates nor requires an explicit universal curve equation to check. Most likely by being clever with the Kodaira–Spencer map. If the reader sees a way to do it, please email the author. It would soothe my soul.” This section satisfies that desire. Universality is now a corollary of the equivalence rather than its engine, the explicit presentation is not used anywhere in the paper, and the soul was soothed in-house.

height $p - 1$ (Corollary C.1'): the wild C_p acts on $\text{Jac}(X)$ with $N = \sum \sigma^i = 0$ because $X/C_p = \mathbb{P}^1$, so it acts on the block through $\mathbb{Z}_p[C_p]/(N) = \mathbb{Z}_p[\zeta_p]$, and the Dieudonné module of F is free of rank one over $W(\bar{k})[\zeta_p]$ (Proposition 9.26) — its height $p - 1 = [\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p]$ is the exponent of $\pi^{p-1} = -p$. The functor is G' -equivariant because G is central (Lemma 10.5), and G' embeds in $\text{Aut}(F)$ with surjective conjugation character (Lemma 10.9).

(ii) *The derivative.* By Grothendieck–Messing, a first-order deformation of F is a displacement of its Hodge line inside the Dieudonné module, and by Katz–Oda that displacement is cup product with the Kodaira–Spencer class. Hence the derivative of $\widehat{\mathcal{F}}^1$ at the closed point is

$$\mu: H^1(X, T_X)^G \longrightarrow \text{Hom}(k\omega_1, H^1(X, \mathcal{O}_X)^\psi), \quad \vartheta \longmapsto (\omega_1 \mapsto \vartheta \cup \omega_1)$$

(Lemma 10.11): cup product with the Hodge line. This is a map between $(p - 2)$ -dimensional spaces, diagonal in the characters of G , and the sequence $0 \rightarrow T_X \xrightarrow{\omega_1} \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$ for $D = \text{div}(\omega_1)$ shows it is surjective, with kernel $H^0(\mathcal{O}_D)^\psi$ (Lemma 10.12). The whole proof is therefore one character count on the divisor of dy/x : that divisor is $(p - 3) \sum_{i \in \mathbb{F}_p} R_i$ on the tame fixed points, the characters occurring in $H^0(\mathcal{O}_D)$ are $\chi^0, \chi^{-1}, \dots, \chi^{-(p-4)}$, and the multiplicity $p - 3$ is small enough — by exactly two — to miss ψ (Lemmas 10.13, 10.14). So μ is an isomorphism (Proposition 10.15). The three facts consumed — the support of $\text{div}(\omega_1)$, its multiplicity $(2g - 2)/p = p - 3$, and the extremality of ψ — are each consequences of minimality (Remark 10.16): *injectivity of the Kodaira–Spencer map is the minimal-genus theorem seen from the formal group.*

(iii) *From tangent isomorphism to equivalence.* Both sides are pro-represented by formally smooth $W(k)$ -algebras of dimension $p - 2$ (Lemma 10.18: unobstructedness of tame equivariant deformations of curves; Lubin–Tate), so a tangent isomorphism is an isomorphism of representing rings and $\widehat{\mathcal{F}}^1$ is an equivalence. The transported G' -action is *the* Lubin–Tate action because G' sits in $\mathbb{G}_{p-1}$ in a single conjugacy class (Bujard, Lemma 0.1; reproved in Appendix A.1, where $\pi^{p-1} = -p$ is the equation of the uniformizer $\pi = \omega_0 \Pi$ of the maximal totally ramified subfield). The graded refinement, Theorem C(b), is a torsor statement over this one: the χ -line of the Hodge bundle of the family is canonically the module of invariant differentials of the block (Lemma 10.22), so the grading is carried along with nothing further to compute, and the action of G' on the periodicity generator u is read off the Hodge line (Corollary 10.24).

Convention 3 (characters). $G = C_{p-1} = \langle \tau^{p-1} \rangle$ is the central tame subgroup, acting by $x \mapsto \eta x$, $y \mapsto y$, $\eta = \zeta^{p-1}$. Let $\chi: G \rightarrow k^\times$ be the character by which G acts on the differential² $\omega_1 := dy/x = x^{p-3}dx$, i.e. $\chi(\eta) = \eta^{-1}$. This is the convention of Part C.1's splitting (Lemma 9.20): the χ^j -block of $H^0(\Omega^1)$ has dimension j . Against Part 1: $\chi = \theta^{-1}$ for the tangent character θ of Lemma 3.4, and $\chi^{p-2} = \chi^{-1} = \theta$. We write $e_j := \frac{1}{p-1} \sum_{g \in G} \chi^{-j}(g) g$ for the idempotents of $\mathbb{Z}_p[G]$ (integral since $p - 1$ is a unit), and $\psi := \chi$ for the character of the one-dimensional block.

10.1. Part (i): the functor, its target, and its equivariance. We begin by making notations and the group actions as crystal clear as possible for the reader by recounting notation.

²Not the conjugation character $\chi: H \rightarrow \mathbb{F}_p^\times$ of Part 1, which is trivial on the central G ; the two are reconciled at the end of this convention.

Convention 4 (upper star, lower star, and the other star). *Throughout, $f: Y \rightarrow Z$ is an isomorphism of curves (g' on X , $\rho(g)$ on $\mathcal{X}$, $\iota: \mathcal{X}_s \rightarrow X$, or $\phi: \mathcal{X} \rightarrow \mathcal{X}'$).*

- f^* is pullback: on functions, differentials, cohomology, line bundles ($\text{Pic}^0(Z) \rightarrow \text{Pic}^0(Y)$), and vector fields ($f^*\vartheta = (df)^{-1}\vartheta f$, compatible with contraction). It reverses composition. Every character superscript is a pullback eigenspace, $V^\chi = \{v : g^*v = \chi(g)v\}$.
- $f_* = (f^{-1})^*$ is pushforward, used only on Pic^0 and its p -divisible group. It preserves composition, so $g' \mapsto g'_*$ is a left action of the nonabelian G' ; that is why Jacobians carry f_* , not f^* . When f_* commutes with G , $e_1(f_*)$ is its restriction to the e_1 -summands.
- The $\star$ of $\text{Def}^\star$ means rigidified (Section 7); a subscript s means special fiber.

Since $\text{AJ} \circ g = g_* \circ \text{AJ}$ up to translation, g_* acts as g^* on $\omega_{\text{Pic}^0} = H^0(\Omega^1)$ and H_{dR}^1 , and as $(g^{-1})^*$ on $\text{Lie Pic}^0 = H^1(\mathcal{O}_X)$. So the block F , where every g_* is the scalar $[\psi(g)]$, has

$$\omega_F = H^0(\Omega^1)^\psi = k\omega_1, \quad \text{Lie}(F) = H^1(\mathcal{O}_X)^{\psi^{-1}}, \quad \mathbb{D}(F)/p = H_{\text{dR}}^1(X)^\psi,$$

with Hodge quotient $H^1(\mathcal{O}_X)^\psi$ of dimension $p-2$ (Proposition 9.20(2), Serre duality) — the Lie algebra of $F^\vee$, not of F .

Definition 10.1. *Given the prestack of p -divisible groups, consider the deformation prestack $\text{Def}_F^\star$ associated a fixed p -divisible group F over k (in the sense of Defn 7.1). An object of $\text{Def}_F^\star(\Lambda)$ is a pair (F', α) : a p -divisible group F'/Λ and an isomorphism $\alpha: F'_s \xrightarrow{\sim} F$; a morphism is an isomorphism $F' \rightarrow F''$ compatible with α, α' .*

Remark 10.2. Note that no group action is part of the datum in this prestack (as opposed to the G carried around in $\text{Def}_{(X,G)}$).

- Convention 5** (objects, and who acts on what). (a) The lift ρ , a datum, and an action of G only (as in Definition 7.1). An element of $G' \setminus G$ acts on X but in general not on $\mathcal{X}$: it would have to preserve the branch locus of $\mathcal{X} \rightarrow \mathcal{X}/G$, which a general deformation moves off the G'/G -orbit (Theorem 8.20). So there is no “ g'_* ” on $\text{Pic}^0(\mathcal{X})$.
- (b) The action of G' on the source $(\mathcal{X}, \rho, \iota) \in \text{Def}_{X,G}^\star(\Lambda)$ (Definition 7.2): $g' \cdot (\mathcal{X}, \rho, \iota) := (\mathcal{X}, \rho, g' \circ \iota)$, which moves the marking, not the curve. Since G is central, $g' \circ \iota$ is again G -equivariant for the same ρ ; so this is a well-defined left action, and no conjugation of ρ enters.
- (c) The action of $\text{Aut}(F)$ on the target $(F', \alpha) \in \text{Def}_F^\star(\Lambda)$: $a \cdot (F', \alpha) := (F', a \circ \alpha)$, and G' acts through the homomorphism $\iota_{\text{aut}}: G' \rightarrow \text{Aut}(F)$ of Lemma 10.4, which lives on the special fiber.

Construction 5. Let Jac be the functor that takes a curve $\mathfrak{X}$ to $\text{Jac}(\mathfrak{X})$ (in AbVar), and consider its formal group $\text{Jac}(\mathfrak{X})_e^\wedge$. We define the following functor by taking the Jacobian and then its p -divisible group, the p -power torsion points of the Jacobian:

$$\mathcal{F}^{\text{all}}: \text{Curve}_G \rightarrow \text{AbVar}_G \rightarrow \text{FG}_G$$

$$(\mathfrak{X}, \rho) \mapsto (\text{Jac}(\mathfrak{X}), \text{Jac}(\rho)) \mapsto (\text{Jac}(\mathfrak{X})_e^\wedge, \text{Jac}(\rho)[p^\infty]).$$

Definition 10.3 (the functor). For $\xi = (\mathcal{X}, \rho, \iota) \in \text{Def}_{(X,G)}^\star(\Lambda)$ set

$$\widehat{\mathcal{F}}^1(\mathcal{X}, \rho, \iota) := (e_1 \text{Pic}^0(\mathcal{X})[p^\infty], e_1(\iota_*)) \in \text{Def}_F^\star(\Lambda),$$

where:

- G acts on $\mathrm{Pic}^0(\mathcal{X})$ through ρ by pushforward, $g \mapsto \rho(g)_* = (\rho(g)^{-1})^*$ (Convention 4), and $e_1 \in \mathbb{Z}_p[G]$ is the idempotent of ψ (Convention 3), integral because $p-1$ is a unit;
- $\iota_*: \mathrm{Pic}^0(\mathcal{X}_s) \xrightarrow{\sim} \mathrm{Pic}^0(X)$ is induced by ι ; it commutes with G because ι is G -equivariant, so it restricts to $e_1(\iota_*): e_1 \mathrm{Pic}^0(\mathcal{X}_s)[p^\infty] \xrightarrow{\sim} e_1 \mathrm{Pic}^0(X)[p^\infty] = F$.

A morphism ϕ is sent to $e_1(\phi_*)$. The special fiber is F by Theorem C.1; the construction commutes with base change in Λ ; and the value is a connected p -divisible group of dimension one because connectedness and dimension are insensitive to nilpotent thickenings. In the language of Construction 5, $\widehat{\mathcal{F}}^1$ is the functor induced on deformation problems (Definition 7.1) by $\mathcal{F}^{all}$, followed by the projection $\mathcal{I}^1$ of Lemma 9.19.

Lemma 10.4 (the action on the special fiber). *For $g' \in G' \simeq \mathrm{Aut}(X)$ the pushforward $g'_* = (g'^{-1})^*$ on $\mathrm{Pic}^0(X)$ (Convention 4) commutes with e_1 , and*

$$\iota_{\mathrm{aut}}: G' \longrightarrow \mathrm{Aut}(F), \quad g' \longmapsto e_1(g'_*) = g'_*|_F,$$

is a homomorphism. On the central subgroup it is the Teichmüller scalar,

$$\iota_{\mathrm{aut}}(g) = [\psi(g)] \in \mu_{p-1}(\mathbb{Z}_p) \subset \mathbb{Z}_p^\times = Z(\mathrm{Aut}(F)) \quad (g \in G).$$

Proof. G' acts on $\mathrm{Pic}^0(X)$ by functoriality, and $g'_*h'_* = (g'h')_*$. Since G is central in G' , g'_* commutes with every $\rho_0(g)_*$ and hence with $e_1 \in \mathbb{Z}_p[G]$ (Lemma 9.19); so g'_* preserves $F = e_1 \mathrm{Pic}^0(X)[p^\infty]$. For $g \in G$, $g e_1 = \psi(g) e_1$ in $\mathbb{Z}_p[G]$ with $\psi(g)$ read in $\mu_{p-1}(\mathbb{Z}_p)$, so g_* acts on F as multiplication by that scalar; and the centre of $\mathrm{Aut}(F) = \mathcal{O}_D^\times$ is $\mathbb{Z}_p^\times$. $\square$

To rephrase using nomenclature of Lemma 7.5, action (b) is the group of deck transformations of the rigidified problem over the unrigidified one, and so is action (c):

$$G' \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Def}_{(X,G)}}(\mathrm{Def}_{(X,G)}^*), \quad g' \mapsto f_{g'},$$

$$\mathrm{Aut}(F) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Def}_F}(\mathrm{Def}_F^*), \quad a \mapsto f_a,$$

with $f_{g'}(\mathcal{X}, \rho, \iota) = (\mathcal{X}, \rho, g' \circ \iota)$ and $f_a(F', \alpha) = (F', a \circ \alpha)$.

Lemma 10.5 (equivariance: $\widehat{\mathcal{F}}^1$ lies over ι_{aut}). *The functor $\widehat{\mathcal{F}}^1$ carries deck transformations to deck transformations, and the homomorphism it induces between these two groups is the ι_{aut} of Lemma 10.4: for every $g' \in G'$ there is an equality of functors*

$$\begin{array}{ccc} \widehat{\mathcal{F}}^1 \circ f_{g'} & = & f_{\iota_{\mathrm{aut}}(g')} \circ \widehat{\mathcal{F}}^1, \\ \mathrm{Def}_{(X,G)}^* & \xrightarrow{\widehat{\mathcal{F}}^1} & \mathrm{Def}_F^* \\ f_{g'} \downarrow & & \downarrow f_{\iota_{\mathrm{aut}}(g')} \\ \mathrm{Def}_{(X,G)}^* & \xrightarrow{\widehat{\mathcal{F}}^1} & \mathrm{Def}_F^* \end{array}$$

compatibly with base change in Λ and covering the unrigidified functor $\mathrm{Def}_{(X,G)} \rightarrow \mathrm{Def}_F$.

Remark 10.6. In other words, $\widehat{\mathcal{F}}^1$ is a morphism from the G' -torsor $\mathrm{Def}_{(X,G)}^* \rightarrow \mathrm{Def}_{(X,G)}$ to the $\mathrm{Aut}(F)$ -torsor $\mathrm{Def}_F^* \rightarrow \mathrm{Def}_F$ lying over the group homomorphism $\iota_{\mathrm{aut}}: G' \rightarrow \mathrm{Aut}(F)$. This is what “ G' -equivariant” means in Theorem C; on objects it reads $\widehat{\mathcal{F}}^1(g' \cdot \xi) = \iota_{\mathrm{aut}}(g') \cdot \widehat{\mathcal{F}}^1(\xi)$.

Proof. On objects. The objects ξ and $f_{g'}(\xi)$ have the same $(\mathcal{X}, \rho)$, hence the same idempotent and the same p -divisible group $e_1 \text{Pic}^0(\mathcal{X})[p^\infty]$; only the rigidification differs. On the special fiber,

$$e_1((g' \circ \iota)_*) = e_1(g'_* \circ \iota_*) = e_1(g'_*) \circ e_1(\iota_*) = \iota_{\text{aut}}(g') \circ e_1(\iota_*),$$

the first equality by functoriality of Pic^0 , the second because both g'_* and ι_* commute with G (Lemma 10.4; ι is equivariant). That is $f_{\iota_{\text{aut}}(g')}$ applied to $\widehat{\mathcal{F}}^1(\xi)$. Every map in this computation is a map of special fibers: nothing has been asked of g' on $\mathcal{X}$.

On morphisms, and over the unrigidified problems. Both $f_{g'}$ and f_a are the identity on morphisms (Definition 7.2), and $\widehat{\mathcal{F}}^1(\phi) = e_1(\phi_*)$ does not involve the rigidifications, so the two composites agree on morphisms as well; the equality is one of functors, not merely of isomorphism classes. For the same reason $\widehat{\mathcal{F}}^1$ commutes strictly with forgetting rigidifications, so in the notation of Definition 7.4 it sends the pair $(f_{g'}, \text{id})$ to the pair $(f_{\iota_{\text{aut}}(g')}, \text{id})$. That $g' \mapsto \iota_{\text{aut}}(g')$ is a homomorphism is Lemma 10.4. $\square$

This lemma will only be used later in the cohomology section, but we will prove it here where the notation is best established.

Lemma 10.7 (the central G acts trivially on both sides). *For $g \in G$ and $\xi := (\mathcal{X}, \rho, \iota)$, the automorphism $\rho(g)^{-1}$ of $\mathcal{X}$ is an isomorphism $\xi \xrightarrow{\sim} g \cdot \xi$ in $\text{Def}_{(X,G)}^*(\Lambda)$, and $\widehat{\mathcal{F}}^1$ carries it to the isomorphism $[\psi(g)]^{-1}: \widehat{\mathcal{F}}^1(\xi) \xrightarrow{\sim} \iota_{\text{aut}}(g) \cdot \widehat{\mathcal{F}}^1(\xi)$.*

Remark 10.8. In other words, G acts trivially on isomorphism classes on both sides, the G' -actions on the two pro-representing rings factor through $G'/G \cong \mathbb{F}_p^\times \rtimes \mathbb{F}_p^\times$, and $\widehat{\mathcal{F}}^1$ is compatible with these trivializations.

Proof. $\rho(g)^{-1}$ is G -equivariant (G is abelian) and $(\rho_0(g) \circ \iota) \circ \rho(g)_s^{-1} = \rho_0(g) \circ \rho_0(g)^{-1} \circ \iota = \iota$, which is the condition to be a morphism $\xi \rightarrow g \cdot \xi$. On the target, $[\psi(g)] \in \mathbb{Z}_p^\times$ is an automorphism of every p -divisible group F' , and $(\iota_{\text{aut}}(g) \circ \alpha) \circ [\psi(g)]_s^{-1} = \alpha$ by Lemma 10.4. Finally $e_1((\rho(g)^{-1})_*) = [\psi(g)]^{-1}$ on $e_1 \text{Pic}^0(\mathcal{X})[p^\infty]$, by the definition of e_1 . $\square$

10.1.1. *Faithfulness on isotropy.* Intuitively, $\widehat{\mathcal{F}}^1$ extracts the formal group that sees the wild C_p -action, and we now check that it sees *all* of G' , not a quotient. We do not need this for equivariance of $\widehat{\mathcal{F}}$. We have yet to show that ι_{aut} is injective. A morphism of torsors lies over a homomorphism of groups whether or not that homomorphism is injective, and the proof of Lemma 10.5 used nothing about ι_{aut} beyond its definition: the square commutes for any homomorphism. But we need this injectivity afterward for two reasons.

First, the injectivity is what makes the G' on the Lubin–Tate side a *subgroup* of $\mathbb{G}_{p-1}$. Forgetting rigidifications, the unrigidified functor

$$\text{Def}_{(X,G)}^* \simeq \text{Def}_{(X,G)}^* // G' \longrightarrow \text{Def}_F \simeq \text{Def}_F^* // \text{Aut}(F)$$

induces ι_{aut} on the automorphism group of the closed point, so ι_{aut} is injective exactly when this functor is faithful there. If ι_{aut} had a kernel K , Theorem 6.3 would still produce an equivalence of quotient stacks, but the G' acting on Def_F^* would act through G'/K : the identification with the maximal finite subgroup (Proposition A.2) would fail, and Theorem D, with its Tate companion, would compute the cohomology of the wrong group.

Second, it is not already known. The automorphism group of a curve of genus at least two acts faithfully on its Jacobian, but ι_{aut} is that action followed by projection to the one summand e_1 , and projection to a summand can kill elements: on the block e_j the central G acts through χ^j , which has a kernel as soon as $\gcd(j, p-1) > 1$. Nor can injectivity be read off the deformation space: by Lemma 10.7 the map $G' \rightarrow \text{Aut}(R)$ has kernel G (exactly G , by Lemma 5.3 and Theorem B). The central G survives only as a deck transformation — the pair (f_g, id) of Lemma 7.5 is not equivalent to the identity, because the comparison 2-morphism $\rho(g)^{-1}$ is a nontrivial automorphism once the rigidification is forgotten — and as the character ψ on ω_F . So injectivity on G has to come from the Hodge line, and injectivity on σ from the Dieudonné module; these are the two steps of the proof.

Lemma 10.9 (faithfulness on isotropy: injectivity of ι_{aut}). *The homomorphism $\iota_{\text{aut}}: G' \rightarrow \text{Aut}(F)$ is injective; equivalently, the unrigidified functor $\text{Def}_{(X,G)} \rightarrow \text{Def}_F$ is injective on the automorphism group of the closed point. The conjugation character of its tame part on its Sylow p -subgroup is surjective onto $\mathbb{F}_p^\times$.*

Proof. Write an element of the kernel of ι_{aut} as $\sigma^a \tau^b$.

We will show first that τ^b acting trivially implies that $(p-1)^2 \mid b$. Consider the action of the tame part on the Hodge line. An automorphism of F acts on the invariant differentials $\omega_F = k\omega_1$, where $\iota_{\text{aut}}(\tau)$ acts by ζ^{-1} (Prop 9.20(3)). A kernel element acts trivially on ω_F , so $\zeta^{-b} = 1$, and $(p-1)^2 \mid b$ and $\tau^b = 1$.

Next, we show that σ^a acting trivially implies that $p \mid a$. We show this by contradiction, by showing that $\sigma^a \neq 1$ in $\text{Aut}(F)$ for $p \nmid a$. Consider the action of the wild part on the Dieudonné module $D(F)$, Dieudonné theory over the perfect field k is fully faithful, so it suffices to see that $\iota_{\text{aut}}(\sigma^a)$ acts nontrivially on its Dieudonné module.

By Proposition 9.26 the action of $\mathbb{Z}_p[C_p]$ on F factors through $\mathbb{Z}_p[C_p]/(N) \cong \mathbb{Z}_p[\zeta_p] = \mathcal{O}_L$, the norm $N = \sum_i \sigma^i$ vanishes and $\mathbb{D}(F)$ is a free $\mathcal{O}_L$ -module of rank one. Under any identification $\mathbb{Z}_p[C_p]/(N) \cong \mathcal{O}_L$, $\sigma^a - 1$ maps to $\zeta_p^a - 1$, which is a nonzero element of the domain $\mathcal{O}_L$ when $p \nmid a$, and a nonzero element of a domain acts injectively on a nonzero free module. Hence σ^a is nontrivial on $\mathbb{D}(F)$, hence on F .

We have shown that a kernel element $\sigma^a \tau^b$ has $(p-1)^2 \mid b$ by the tame step and then $p \mid a$ by the wild step; so ι_{aut} is injective. Surjectivity of the conjugation character χ onto $\mathbb{F}_p^\times$ is a property of the abstract group G' (Lemma 4.1) and transports along any injection. $\square$

Remark 10.10. The proof locates the image in $\text{Aut}(F) = \mathcal{O}_D^\times$: the tame part maps isomorphically onto the subgroup of order $(p-1)^2$ of $\mathbb{F}_{p^{p-1}}^\times$ under the reduction $\mathcal{O}_D^\times \rightarrow \mathbb{F}_{p^{p-1}}^\times$ (note $(p-1)^2 \mid p^{p-1} - 1$), its subgroup G landing in the center $\mathbb{Z}_p^\times$ (Lemma 10.4), and the wild part lies in the kernel $1 + \mathfrak{m}_D = S_{p-1}$; this is the subgroup of order p of Ravenel's Theorem 14 (Rav78), read off the curve.

10.2. Part (ii): the derivative is an isomorphism. Write $\mu := d\widehat{\mathcal{F}}^1$ for the derivative at the closed point of a map of tangent spaces $T\text{Def}_{(X,G)}^* \rightarrow T\text{Def}_F^*$.

Lemma 10.11 (the derivative is cup product with the Hodge line). *Under the identifications $T\text{Def}_{(X,G)}^* = H^1(X, T_X)^G$ and $T\text{Def}_F^* = \text{Hom}(k\omega_1, H^1(X, \mathcal{O}_X)^\psi)$,*

$$\mu(\vartheta) = (\omega_1 \mapsto \vartheta \cup \omega_1).$$

Proof. For $\vartheta \in H^1(T_X)^G$ with first-order deformation $\mathcal{X}_\vartheta/k[\epsilon]$, the crystalline nature of H^1 trivializes $H^1_{dR}(\mathcal{X}_\vartheta/k[\epsilon]) \cong H^1_{dR}(X) \otimes k[\epsilon]$; by Grothendieck–Messing (Mes72) the deformation of F is the displacement of its Hodge line inside this trivialization, and by Katz–Oda (KO68) that displacement is cup product with the Kodaira–Spencer class ϑ . Apply e_1 : the Hodge line of F is $k\omega_1$ and the Hodge quotient $\mathbb{D}(F)/p/\omega_F$ is $H^1(\mathcal{O})^\psi$ (Corollary C.1'; Convention 4). $\square$

Lemma 10.12 (clean baby). *μ is surjective. Let $D := \text{div}(\omega_1)$; then μ is injective if and only if $H^0(\mathcal{O}_D)^\psi = 0$.*

Proof. The differential ω_1 gives an exact sequence $0 \rightarrow T_X \xrightarrow{\omega_1} \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$ of G -sheaves, whose long exact sequence reads

$$H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_D) \rightarrow H^1(T_X) \xrightarrow{\mu} H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_D) = 0,$$

the last term vanishing because $\mathcal{O}_D$ is a skyscraper. The map $H^1(T_X) \rightarrow H^1(\mathcal{O}_X)$ is cup product with ω_1 , i.e. μ after passing to G -eigenspaces; taking G -invariants of a sequence of G -modules is exact since $p \nmid |G|$. So μ is surjective, and

$$H^0(\mathcal{O}_X)^\psi \rightarrow H^0(\mathcal{O}_D)^\psi \rightarrow H^1(T_X)^G \xrightarrow{\mu} H^1(\mathcal{O}_X)^\psi \rightarrow 0$$

with $H^0(\mathcal{O}_X) = k$ in the trivial character and $\psi \neq 1$: injectivity of μ is exactly $H^0(\mathcal{O}_D)^\psi = 0$. $\square$

Now we will spend some time to understand our new object of interest to make sure it dies.

Lemma 10.13 (the divisor of ω_1). *$D = \text{div}(dy/x) = (p-3) \sum_{i \in \mathbb{F}_p} R_i$, where $R_i = (0, i)$ are the tame ramification points of p_y (Lemma 4.2); in particular D is supported on the G -fixed points, avoids the wild point Q , and $\deg D = p(p-3) = 2g-2$. At each R_i the function x is a uniformizer and a G -eigenvector, with τ^{p-1} acting by η .*

Proof. At Q : x and y have poles only at Q , of orders p and $p-1$ (Lemma 4.2), so $v_Q(x) = -p$ and, as $p \nmid p-1$, $v_Q(dy) = -p$; hence $v_Q(dy/x) = 0$. At $x = 0$ the fiber is $\{R_i\}$, since $y^p - y = \prod (y - j)$ and only the factor $y - i$ vanishes at R_i ; p_y is totally ramified at R_i with index $p-1$, so $v_{R_i}(y - i) = p-1$ and $v_{R_i}(x) = 1$. Writing $y - i = ux^{p-1}$ with $u(R_i) = \prod_{j \neq i} (i - j)^{-1} \neq 0$, $dy = x^{p-2}((p-1)u + xu')dx$ has $v_{R_i}(dy) = p-2$, hence $v_{R_i}(dy/x) = p-3$. The degree check $p(p-3) = 2g-2$ confirms no other zeros. G fixes each R_i and $x \mapsto \eta x$. $\square$

Lemma 10.14 (the character count). *As a G -module,*

$$H^0(\mathcal{O}_D) = \bigoplus_{i \in \mathbb{F}_p} \mathcal{O}_{X, R_i} / \mathfrak{m}_{R_i}^{p-3} = \bigoplus_{i \in \mathbb{F}_p} \bigoplus_{a=0}^{p-4} k \cdot x^a, \quad (\tau^{p-1})^* x^a = \eta^a x^a,$$

so the characters occurring are χ^{-a} for $0 \leq a \leq p-4$, each with multiplicity p , and the two characters $\chi^{-(p-3)} = \chi^2$ and $\chi^{-(p-2)} = \chi^1 = \psi$ do not occur. In particular $H^0(\mathcal{O}_D)^\psi = 0$.

Proof. By Lemma 10.13, $\mathcal{O}_D = \bigoplus_i \mathcal{O}_{X, R_i} / \mathfrak{m}^{p-3}$ with x simultaneously a uniformizer and an eigenvector at each R_i , so no averaging is needed and the action is diagonal on the nose. The $p-1$ characters of G are $\chi^0, \dots, \chi^{p-2}$; the powers x^a , $0 \leq a \leq p-4$, realize $\chi^0, \chi^{-1}, \dots, \chi^{-(p-4)}$, which misses exactly $\chi^{-(p-3)}$ and $\chi^{-(p-2)}$, and $\chi^{-(p-2)} = \chi$ since $\chi^{p-1} = 1$. $\square$

Proposition 10.15 (Kodaira–Spencer). $\mu: T \operatorname{Def}_{(X,G)}^* \rightarrow T \operatorname{Def}_F^*$ is an isomorphism, and both sides have dimension $p - 2$.

Proof. Surjectivity is Lemma 10.12; injectivity is Lemma 10.12 combined with Lemma 10.14. The dimension of the source is $p - 2$ by Theorem B (or directly: $h^1(T_X)^G = h^0(\Omega^{\otimes 2})^{\chi^{-1}\text{-twist}}$, the configuration count), and of the target by Lubin–Tate, $h - 1 = p - 2$. $\square$

Remark 10.16 (three inputs, all from minimality). The count in Lemma 10.14 used three facts, and none is a free choice: (a) $\operatorname{div}(\omega_1)$ is supported on the G -fixed points of the tame cover; (b) its multiplicity is $p - 3 = (2g - 2)/p$, small enough to miss two characters; (c) ψ is *extremal* — ω_1 lives in the block of the last character rather than a middle one. Each is a consequence of Theorem A. Change the curve and any of them breaks: for $p = 5$ and the non-minimal $y^5 - y = x^{28} + cx^{12}$ of genus 54, one still gets a surjection, but $2g - 2 = 106$ is not divisible in the same way, the eigendifferential’s divisor is not concentrated with small multiplicity on a single G -fixed orbit, and the vanishing $H^0(\mathcal{O}_D)^\psi = 0$ fails. For injectivity to fail here, ω_1 would have to vanish to order at least $p - 1$ at some R_i , i.e. $\deg D \geq p(p - 1)$; but $\deg D = 2g - 2 = p(p - 3)$ is forced. Injectivity of μ is the minimal-genus theorem seen from the formal group.

Remark 10.17 (what $\ker \mu$ would have been). Deformation-theoretically, $\ker \mu$ consists of first-order equivariant deformations of X that the height- $(p - 1)$ formal group cannot see: directions in which the curve moves but the ψ -part of the Hodge filtration does not. Its vanishing says the block sees every equivariant direction — the statement that Serre–Tate, which we lost at height > 2 , has been recovered on the nose for this one summand.

10.3. Part (iii): from tangent isomorphism to equivalence.

Lemma 10.18 (both sides are smooth of the same dimension). $\operatorname{Def}_{(X,G)}^*$ and Def_F^* are pro-represented by formally smooth complete local $W(k)$ -algebras of relative dimension $p - 2$: $R \simeq W(k)[[e_1, \dots, e_{p-2}]]$ (Theorem B) and $A_{\text{LT}} \simeq W(k)[[u_1, \dots, u_{p-2}]]$ (Lubin–Tate).

Proof. The source is Theorem B: unobstructedness of curves plus exactness of G -invariants (tame), with tangent $H^1(T_X)^G$ of dimension $p - 2$. The target is (LT66b). $\square$

Before stating the theorem we set up the graded problem, so that the ungraded and graded forms can be stated and proved together: all of the content is in the ungraded form, and the graded one is a torsor statement over it.

10.4. The grading: the Hodge line is the invariant differential. We are ultimately interested in computing the cohomology of the graded Lubin–Tate ring R_* , as this is the one converging to the local homotopy groups of spheres. We remind the reader of the graded formal group setting. An excellent discussion of invariant differentials of formal groups is given Section 4 in Goerss (Goe08).

Definition 10.19. The moduli of one dimensional formal groups with trivialized Lie algebra $\mathcal{M}_{\text{fg}_1}^{\text{Lie} \simeq \text{triv}}$ is the étale sheaf that associates to any ring R the groupoid of pairs (G, ϕ) where $G \rightarrow \operatorname{Spec} R$ is a formal group and $\phi: \omega_G \simeq R$ is the trivialization of its sheaf of invariant differentials.

Lemma 10.20. (*Goe08*) (*Ray25*) (*graded lazard*) *Fixing a formal group $F \in \mathcal{M}_{\text{fg1}}^{\text{Lie} \simeq \text{triv}}(k)$ of height h , the formal moduli problem Def_F^* is represented by a graded ring R_* which is noncanonically isomorphic to*

$$R_* \simeq W(k)[[u_1, \dots, u_{h-1}]] [u^\pm],$$

where $|u_i| = 0$ and $|u| = -2$.

The curve counterpart of the graded formal group moduli problem is Definition 7.16; we first make its trivialization datum explicit, and identify it with the invariant differential of the block (Lemma 10.22), so that no coordinate and no explicit universal curve is needed.

Definition 10.21 (the graded curve problem, made explicit). *For $(\mathcal{X}, \rho, \iota) \in \text{Def}_{(X,G)}^*(\Lambda)$ let*

$$\mathcal{L}(\mathcal{X}) := (\pi_* \Omega_{\mathcal{X}/\Lambda}^1)^\chi,$$

the χ -isotypic part for the pullback action of G through ρ . By Proposition 9.20(2) and cohomology-and-base-change it is a line bundle on $\text{Spec } \Lambda$, and ι identifies its special fibre with $k\omega_1$. Set

$$\text{Def}_{(X,G)}^{\text{Lie}}(\Lambda) := \{(\mathcal{X}, \rho, \iota, \omega) : \omega \text{ a generator of } \mathcal{L}(\mathcal{X})\},$$

so that the forgetful map $\text{Def}_{(X,G)}^{\text{Lie}} \rightarrow \text{Def}_{(X,G)}^*$ is a $\mathbb{G}_m$ -torsor (scaling ω); the fibre over $\omega \equiv \omega_1$ is a torsor under $1 + \mathfrak{m}_\Lambda$, and the grading of Lemma 10.20 is the $\mathbb{G}_m$ -weight. This is Definition 7.16 at $i = 1$, with $\mathfrak{g}^{\chi^1} := \omega$. (For $i \geq 2$ the χ^i -part has rank i by Proposition 9.20(2), so “trivialization” would have to mean of its determinant; only $i = 1$ is used in this paper.)

Lemma 10.22 (the Hodge line is the invariant differential, in families). *There is a canonical isomorphism of line bundles on $\text{Def}_{(X,G)}^*$, natural in $(\mathcal{X}, \rho, \iota)$ and G' -equivariant,*

$$\omega_{F_{\mathcal{X}}} \xrightarrow{\sim} \mathcal{L}(\mathcal{X}), \quad F_{\mathcal{X}} := e_1 \text{Pic}^0(\mathcal{X})[p^\infty] = \widehat{\mathcal{F}}^1(\mathcal{X}, \rho, \iota),$$

where ω_F is the module of invariant differentials of the formal group $F_{\mathcal{X}}$ over Λ . On the special fibre it is the identification $\omega_F = k\omega_1$ of Corollary C.1'.

Proof. The Lie algebra of the Picard scheme is $\text{Lie Pic}_{\mathcal{X}/\Lambda}^0 = R^1 \pi_* \mathcal{O}_{\mathcal{X}}$, compatibly with base change (a deformation of the trivial line bundle over $\Lambda[\epsilon]$ is a class in $H^1(\mathcal{X}, \mathcal{O}) \otimes \epsilon\Lambda$), so $\omega_{\text{Pic}^0} = (R^1 \pi_* \mathcal{O}_{\mathcal{X}})^\vee \cong \pi_* \Omega_{\mathcal{X}/\Lambda}^1$ by Grothendieck–Serre duality for the smooth proper curve $\mathcal{X}/\Lambda$, again compatibly with base change and with automorphisms of $\mathcal{X}$. An automorphism g of $\mathcal{X}$ acts on Pic^0 by $(g^{-1})^*$, hence on Lie by the contragredient of g^* , hence on ω_{Pic^0} by g^* : this is why the χ -part for the pullback action appears, matching Convention 3, and why the block’s cotangent line is $k\omega_1$ (dimension 1) and not the χ^{-1} -block (dimension $p - 2$). Passing to the p -divisible group and to its connected part changes neither Lie nor ω , and the idempotent e_1 commutes with everything since G acts by automorphisms of $\mathcal{X}$; so $\omega_{F_{\mathcal{X}}} = e_1 \omega_{\text{Pic}^0} = (\pi_* \Omega_{\mathcal{X}/\Lambda}^1)^\chi$. Naturality in $(\mathcal{X}, \rho, \iota)$ and G' -equivariance are the functoriality of Pic^0 and of duality; on the special fibre the statement is Corollary C.1'. $\square$

Definition 10.23 (the graded functor). $\widehat{\mathcal{F}}_{\text{Lie}}^1(\mathcal{X}, \rho, \iota, \omega) := (F_{\mathcal{X}}, e_1(\iota_*), \omega)$, with ω read as a generator of $\omega_{F_{\mathcal{X}}}$ through Lemma 10.22. It covers $\widehat{\mathcal{F}}^1$ and is $\mathbb{G}_m$ -equivariant: the square

$$\begin{array}{ccc} \text{Def}_{(X,G)}^{\text{Lie}} & \xrightarrow{\widehat{\mathcal{F}}_{\text{Lie}}^1} & \text{Def}_F^{\text{Lie}} \\ \downarrow & & \downarrow \\ \text{Def}_{(X,G)}^* & \xrightarrow{\widehat{\mathcal{F}}^1} & \text{Def}_F^* \end{array}$$

commutes, and the vertical maps are $\mathbb{G}_m$ -torsors.

10.5. Theorem C.

Theorem C. (*Equivalent Exchange*)

(a) *The functor*

$$\widehat{\mathcal{F}}^1: \text{Def}_{(X,G)}^* \longrightarrow (\text{Def}_{\widehat{\mathcal{F}}^1(X,G)}^{FG \dim 1})^*$$

is a G' -equivariant equivalence of formal moduli problems, where $\widehat{\mathcal{F}}^1(X, G) = F$ is a one-dimensional formal group of height $p - 1$, and the induced G' -action on the target is the canonical Lubin–Tate action.

(b) (*Graded Equivalent Exchange*) *The functor*

$$\widehat{\mathcal{F}}_{\text{Lie}}^1: \text{Def}_{(X,G)}^{\text{Curve}_G^{\text{Lie}\chi^1}} \longrightarrow \text{Def}_F^{\mathcal{M}_{\text{fg}1}^{\text{Lie}\simeq\text{triv}}}$$

of Definition 10.23 is a G' -equivariant equivalence of formal moduli problems, a map of $\mathbb{G}_m$ -torsors over (a), where the G' -action on the target is the graded Lubin–Tate action.

Proof. (a): *equivalence.* Let $\varphi^\#: A_{\text{LT}} \rightarrow R$ classify the p -divisible group $\widehat{\mathcal{F}}^1(\mathcal{U})$ of the universal curve. By Proposition 10.15, $\varphi^\#$ induces an isomorphism $\mathfrak{m}_A/(\mathfrak{m}_A^2 + pA) \rightarrow \mathfrak{m}_R/(\mathfrak{m}_R^2 + pR)$; since $p \in \varphi^\#(\mathfrak{m}_A)$, Nakayama and completeness give surjectivity; a surjection between regular local domains of the same dimension $p - 1$ (Lemma 10.18) is an isomorphism. By Yoneda, $\widehat{\mathcal{F}}^1$ is an equivalence. (a : *equivariance.* Lemma 10.5: $\widehat{\mathcal{F}}^1$ lies over the homomorphism $\iota_{\text{aut}}: G' \rightarrow \text{Aut}(F)$).

(a): *identification of the action.* By Lemma 10.9, ι_{aut} embeds G' into $\text{Aut}(F) = \mathbb{G}_{p-1}$ with surjective conjugation character. By Bujard (Buj12) (Theorem 0.1), the maximal finite subgroups of $\mathbb{G}_{p-1}$ with nontrivial p -torsion form a single conjugacy class, and any embedded copy of G' with surjective χ is maximal; so $\iota_{\text{aut}}(G')$ is the canonical subgroup up to conjugation, and conjugation by an element of $\mathbb{G}_{p-1}$ acts on Def_F^* by an automorphism intertwining the two actions. Hence the transported action is the Lubin–Tate action.

($a \Rightarrow b$): *equivalence.* By Lemma 10.22 and Definition 10.23, $\widehat{\mathcal{F}}_{\text{Lie}}^1$ is well defined, covers $\widehat{\mathcal{F}}^1$, and is a $\mathbb{G}_m$ -equivariant morphism of $\mathbb{G}_m$ -torsors. Such a morphism covering an equivalence is an equivalence: both problems are discrete, and over each $\xi \in \text{Def}_{(X,G)}^*(\Lambda)$ it is a $\Lambda^\times$ -equivariant map from the $\Lambda^\times$ -torsor of generators of $\mathcal{L}(\mathcal{X})$ to the $\Lambda^\times$ -torsor of generators of $\omega_{F_{\mathcal{X}}}$, hence a bijection. No tangent space, smoothness or dimension count is needed beyond those of (a).

($a \Rightarrow b$): *equivariance and the action.* As in Convention 5, G' acts on the graded problems by changing the rigidification and nothing else,

$$g' \cdot (\mathcal{X}, \rho, \iota, \omega) = (\mathcal{X}, \rho, g' \circ \iota, \omega), \quad a \cdot (F', \alpha, \phi) = (F', a \circ \alpha, \phi),$$

the second being the graded Lubin–Tate action ((Ray25), Lemma 10.20). Neither $\mathcal{L}(\mathcal{X})$, nor ω_{F_X} , nor the isomorphism between them in Lemma 10.22 involves ι . So by Lemma 10.5, $\widehat{\mathcal{F}}_{\Sigma^c}^1(g' \cdot (\xi, \omega)) = \iota_{\text{aut}}(g') \cdot \widehat{\mathcal{F}}_{\Sigma^c}^1(\xi, \omega)$, with the same ι_{aut} , and the identification of the action is the conjugacy argument of (a). The action of G' on the grading generator u is not visible at this level: it appears on the universal family, through the comparison isomorphisms, and is computed in Corollary 10.24. $\square$

Convention 6. *Let u be the generator of the invariant differential over the universal deformation $R_* = R[u^{\pm}]$ corresponding to the universal differential $\omega \in \mathcal{L}(\mathcal{X}^{\text{univ}})$. For $g' \in G'$ let $\gamma_{g'}$ be the automorphism of the base by which g' acts, and $\theta_{g'}: \mathcal{X}^{\text{univ}} \rightarrow \gamma_{g'}^* \mathcal{X}^{\text{univ}}$ the unique isomorphism of deformations from $g' \cdot (\mathcal{X}^{\text{univ}}, \rho, \iota)$ to $\gamma_{g'}^*(\mathcal{X}^{\text{univ}}, \rho, \iota)$ (Lemma 7.5); on the special fiber $\theta_{g'}$ is g' itself. This, and not the lift ρ (which exists only for G), is the sense in which G' acts on the universal curve.*

Corollary 10.24 (the action on the grading, canonically). *The action of G' on the universal curve induces an action of G' the generator u of the invariant differential of the universal formal group law: $g' \cdot u = c(g') u$ with $c(g') \in R^\times$ the unit defined by $\theta_{g'}^* \omega = c(g') \omega$, by the naturality in Lemma 10.22, and its reduction is the character of G' on the Hodge line:*

$$c(\sigma) \equiv 1, \quad c(\tau) \equiv \zeta^{-1} \pmod{\mathfrak{m}_R}$$

.

Proof. This is a restatement of (Proposition 9.20(3)). $\square$

Remark 10.25. This is the $g'(0)$ of Gorbounov–Mahowald (GM00), obtained without a coordinate: it is the ratio of the pulled-back invariant differential of the universal curve to itself.

Corollary C.2. *(golden fish) Let $\mathcal{U}$ be the universal object over $\text{Def}_{(X,G)}^*$ and R the ring representing $\text{Def}_{(X,G)}^*$. Then the formal group $\widehat{\mathcal{F}}^1(\mathcal{U})$ over R is a universal deformation of F ; in particular it is a universal one-dimensional formal group of height $p - 1$ over $W(k)$.*

Proof. Theorem C(a) identifies R with A_{LT} so that $\widehat{\mathcal{F}}^1(\mathcal{U})$ is the universal deformation. $\square$

Remark 10.26 (what the proof used). Parts (i)–(iii) used the equation of the curve only through three facts: the isotypic decomposition of $H^0(\Omega^1)$ (Proposition 9.20), the divisor of the extremal differential (Lemma 10.13), and $N = 0$ on the Jacobian (Proposition 9.26). Each is a consequence of minimality (Theorem A) and of the tame quotient being $\mathbb{P}^1$. This is why the argument, unlike the coordinate proof of the earlier version, is the one that generalizes to higher levels.

Remark 10.27 (other gradings). Hopkins, and Hill (Hil06), present R_* on the generators $(uu_1, \dots, uu_{h-1}, u^{\pm 1})$, all in degree -2 ; the cohomology of $\text{Conf}(\mathbb{A}^1, p)$ in Subsection 9.2, with the standard representation in degree -2 , which shows that $\text{Def}_{(X,G)} \simeq \text{Def}_{(\mathbb{A}^1, p)}$ uses this convention implicitly. Gorbounov–Mahowald (GM00) ignore the grading, act by a power series g on $(u_1, \dots, u_{h-1})$ in degree 0, and recover the action on u separately as $g'(0)$ (this methodology is discussed in detail in the introduction of (GS98)); Corollary 10.24 is that $g'(0)$ without a coordinate. Hovey–Strickland (HS99) take a similar approach to Hopkins, working with $(vv_1, \dots, vv_{h-1})$,

but they assume the grading of the variables comprising the graded ring representing the moduli problem is $|vv_i| = -(p^i - 1)$ rather than -2 , which can be understood as them working with the cohomology of tensor powers of the line bundle defined by the universal differential $\omega^{\otimes p^i - 1}$.

Part 6. Theorem D and E: Cohomological Fruits of Our Labor

Remark 10.28. The reader may recognize the ring of Theorem [E](#) from the Hopkins–Miller computation of the E_2 -term at height $p - 1$, unpublished but recorded by Nave ([Nav10](#)) and in Hill’s thesis ([Hil06](#)). We recover the ring by a different route, and with the action of the full $C_{(p-1)^2}$ on the grading derived rather than assumed ([Lemma 10.34](#)), and include the full computation here.

SETUP

We establish some notation and conventions which we will use to refer to representations in our explicit calculation to come.

Notation. We call $\mathbb{W} := W(\mathbb{F}_{p^{p-1}})$.

Definition 10.29. *The restricted standard representation of the symmetric group on p letters Σ_p ,*

$$\lambda := \mathbb{W}\{y_0, \dots, y_{p-1}\} / (y_0 + \dots + y_{p-1})$$

with $|y_i| = -2$. This acts by permuting the basis y_i . We define $\bar{\rho}$ to be the graded restricted representation of λ to $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$ generated by σ and τ , this acts on $\sigma(y_i) = y_{\sigma(i)}$ and $\tau(y_i) = y_{\tau(i)}$.

Remark 10.30. An equivalent definition of the action disregarding grading is as follows. Let R be a $\mathbb{Z}_p$ -algebra. There is an action of $G = C_p \rtimes C_{p-1} \simeq (\mathbb{F}_p, +) \rtimes (\mathbb{F}_p^*, \times)$ on the group ring $R[\mathbb{F}_p] \simeq R[t]/(t^p - 1)$ by $(c, m) \cdot t^x \mapsto t^{c+mx}$. Let $\sigma := (1, 1)$ be the element of order p in G . Let $a \in \mathbb{F}_p^\times$ be a primitive element, then, let $\tau := (0, a)$ be the corresponding element of order $p - 1$.

Theorem D. *Let p be any prime and let $\mathbb{W} := \mathbb{W}(\mathbb{F}_{p^{p-1}})$. Let $\bar{\rho}$ be the graded $C_p \rtimes C_{p-1}$ representation [Defn 10.29](#), where the generators of $\mathbb{W}[C_p]$ have degree -2 .*

Then there is a $C_p \rtimes C_{(p-1)^2}$ -equivariant $\mathbb{W}[C_p]$ -module equivariant isomorphism

$$\pi_*(E_{(p-1)}) \simeq (\text{Sym}(\bar{\rho})[d^{-1}])_I^\wedge.$$

If x_0 denotes $[1] \in \bar{\rho}$ and σ generates C_p , then $d = \prod_{i=0}^{p-1} \sigma^i(x_0) = x_0 \cdots x_{p-1}$. The ideal I is the $(C_p \rtimes C_{(p-1)^2})$ -equivariant ideal $I := (p, y_i)$ where $y_i := (1 - \sigma^i)(x_0)$.

Remark 10.31. One would like to complete with respect to (p, x_i) , but that isn’t C_p -equivariant, which is why we instead complete with respect to this cookie looking ideal I . In other words, we complete with respect to the ideal $(p, x_0, \dots, x_{p-1})^{C_p}$.

Proof. The equivalence as $C_p \rtimes C_{p-1}$ representations follows immediately by combining [Theorem C](#) which states that $\tilde{\mathcal{F}}^1$ is an $\text{Aut}(X)$ -equivariant equivalence, where $\text{Aut}(X) \simeq C_p \rtimes C_{(p-1)^2}$, with [Theorem B](#) which describes the $\text{Aut}(X)/C_{p-1}$ action on $\text{Def}_{(X,G)}$. The action of the full $C_{(p-1)^2}$ on the graded ring — which [Theorem B](#) sees only through its quotient C_{p-1} , because the central G acts trivially on $\text{Def}_{(X,G)}$

— is pinned down, without a coordinate, by Lemma 10.34 below: it is the configuration action on the degree-0 part together with the character $c(\tau) \equiv \zeta^{-1}$ on the grading generator (Corollary 10.24). $\square$

10.6. C_p -Cohomology.

Notation. We establish some quick notation. We call $\rho := \mathbb{W}[C_p]$ the regular representation of C_p , and $\bar{\rho}$ the reduced regular representation. That is, $\bar{\rho}$ is the kernel of the augmentation map $\epsilon : \mathbb{W}[\mathbb{F}_p] \rightarrow \mathbb{W}$.

Let $\Lambda := \text{Sym}(\bar{\rho})$ and $A := \text{Sym}(\rho)$. We consider the element $s_1 := ([1] + [\sigma] + \cdots + [\sigma^{(p-1)}])(y_0) = y_0 + \cdots + y_{p-1}$ in A , and call $s_1 A$ the ideal that s_1 generates in A .

The short exact sequence we will be working with is the following:

$$s_1 A \rightarrow A \rightarrow A/s_1 A \simeq \Lambda$$

We will now proceed to go through several lemmas that we will piece together to get Theorem D, the Tate $C_p \rtimes C_{(p-1)^2}$ -cohomology of Λ .

Lemma 10.32. *The Tate cohomology of the C_p action on A is of the form*

$$\hat{H}^*(C_p, A) \simeq \mathbb{Z}/p[b^\pm, d],$$

where b is in degree $(2, 0)$ and d is in degree $(0, -2p)$.

Proof. The orbit of each monomial of A under σ is free except for $d := y_0 \cdots y_{p-1}$ which is fixed (thus the orbit is trivial), and its corresponding cohomology class d lies in degree $(0, |y_i|p) = (0, -2p)$. Therefore, A splits as a sum of a G -module F with a free C_p -action and $\mathbb{W}[d]$ which has trivial C_p -action,

$$F \oplus \mathbb{W}[d].$$

Thus, $H^q(C_p, \mathbb{W}[d]) \simeq H^q(C_p, A)$.

Next, we take the ring map from $\mathbb{W} \rightarrow A$, this gives a map from $H^*(C_p, \mathbb{W})$ to $H^*(C_p, A)$. Recall that

$$H^*(C_p, \mathbb{W}) = \begin{cases} \mathbb{W} & \text{if } * = 0 \\ \mathbb{W}/p & \text{if } * \text{ even,} \\ 0 & \text{if } * \text{ odd} \end{cases}$$

i.e., $H^*(C_p, \mathbb{W}) \simeq \mathbb{W} \oplus \mathbb{W}/p[b]$, where b has bidegree $(2, 0)$.

Let $N : A \rightarrow H^0(C_p, A)$ be the norm map. Then, we have an exact sequence

$$A \xrightarrow{N} H^0(C_p, A) \rightarrow \mathbb{W}/p[b, d] \rightarrow 0.$$

The Tate cohomology of A is then

$$\hat{H}^*(C_p, A) \simeq H^*(C_p, A)[b^{-1}] \simeq \mathbb{Z}/p[b^\pm, d].$$

$\square$

Lemma 10.33. *The Tate cohomology of the C_p action on $\Lambda[d^{-1}] := \text{Sym}(\bar{\rho})[y_0 \cdots y_{p-1}]$ is*

$$\hat{H}^*(C_p, \Lambda)[d^{-1}] \simeq \mathbb{Z}/p[b^{\pm 1}, c, d^{\pm 1}]/(c^2),$$

where b is in degree $(2, 0)$, d is in degree $(0, -2p)$, and c is in degree $(1, |y_i|) = (1, -2)$.

Proof. Let $\tilde{b} := s_1 b$, and $\tilde{d} := s_1 d$. Similarly, the Tate cohomology of $s_1 A$ is $\hat{H}^*(C_p, s_1 A) \simeq \mathbb{Z}/p[\tilde{b}^\pm, \tilde{d}]$.

From our short exact sequence we get a long exact sequence, which is zero at the ends for degree reasons:

$$0 \rightarrow \hat{H}^{2k-1}(C_p, \Lambda) \rightarrow \hat{H}^{2k}(C_p, s_1 A) \rightarrow \hat{H}^{2k}(C_p, A) \rightarrow \hat{H}^{2k}(C_p, \Lambda) \rightarrow 0.$$

The middle map in this long exact sequence is zero, because it is induced by multiplication by s_1 , which is in the image of the additive norm on A , and norms are modded out by Tate cohomology.

It follows that

$$\hat{H}^*(C_p, \Lambda) \simeq \mathbb{Z}/p[b^{\pm 1}, c, d]/(c^2)$$

where c is the element of bidegree $(1, |s_1|) = (1, |y_i|) = (1, -2)$ which maps to $\tilde{b} \in \hat{H}^2(C_p, s_1 A)$.

The cohomology of $H^*(C_p, \Lambda[d^{-1}]) \simeq H^*(C_p, \Lambda)[d^{-1}]$ because finite group cohomology commutes with colimits. $\square$

Theorem D.1. *Let R be the graded ring*

$$R := \mathbb{W}/p[b^\pm, d^\pm, c]/(c^2).$$

The Tate cohomology of the C_p -module $(E_{p-1})_*$ is:

$$R \simeq \hat{H}^*(C_p, (E_{p-1})_*).$$

where $|b| = (2, 0)$, $|c| = (1, -2)$, and $|d| = (0, -2p)$.

Proof. This follows from Theorem 10.33 combined with Lemma 10.47. $\square$

10.7. Analyzing the $C_{(p-1)^2}$ Action.

Lemma 10.34 (the $C_{(p-1)^2}$ -action on R_* , without coordinates). *Write $R_* = R[u^{\pm 1}]$ for the graded Lubin–Tate ring, R the degree-0 part, as in Lemma 10.20. Then:*

- (1) *The G' -action on R factors through $G'/G = C_p \rtimes C_{p-1}$ and is the configuration action of Theorem B: σ permutes the branch coordinates cyclically and τ acts on the affine coordinate of the branch line $X/G \cong \mathbb{P}^1$ by the scalar $\vartheta(\tau)$, the Teichmüller lift of $\bar{\chi}(\tau) \in \mathbb{F}_p^\times$.*
- (2) *The grading generator can be chosen so that*

$$\sigma \cdot u = u, \quad \tau \cdot u = \zeta^{-1} u,$$

with ζ the primitive $(p-1)^2$ -th root of unity of Proposition 9.20(3) (the eigenvalue of τ on the Hodge line $k\omega_1$).

- (3) *Consequently G' acts on R_* by $g' \cdot (f u^n) = (g' \cdot f) c(g')^n u^n$ with $c(\sigma) = 1$, $c(\tau) = \zeta^{-1}$; in particular the scalar η by which τ acts on a degree- (-2) generator $y_i = u \cdot e_i$ of $\bar{\rho}$ is*

$$\eta = \zeta^{-1} \vartheta(\tau),$$

a primitive $(p-1)^2$ -th root of unity. (This η is the scalar of this Part, not the generator $\eta = \zeta^{p-1}$ of G in Convention 3.)

- (4) *Explicitly: the branch-line coordinate is the y of Lemma 4.1, on which $\tau^* y = \zeta^{p-1} y$, so $\vartheta(\tau) = \zeta^{p-1}$, $a = \bar{\zeta}^{p-1}$ (as $\sigma: y \mapsto y+1$ gives $\tau \sigma \tau^{-1} = \sigma^{\bar{\vartheta}(\tau)}$), and*

$$\eta = \zeta^{p-2}, \quad \eta^{p-1} = \zeta^{-(p-1)} \equiv a^{-1} \pmod{p},$$

which is the scalar by which τ acts on the periodicity class b (Lemma 10.41). This is the consistency check between the grading character and the cohomological action: $\tau(b) = \eta^{p-1}b$.

Proof. (1) is Theorem B together with the observation that the central G acts trivially on $\text{Def}_{(X,G)}$ (Lemma 10.7). (2) By Corollary 10.24, for any generator u of ω_F over R one has $g' \cdot u = c(g')u$ with $c(g') \in R^\times$, $c(\tau) \equiv \zeta^{-1}$ and $c(\sigma) \equiv 1 \pmod{\mathfrak{m}_R}$; $g' \mapsto c(g')$ is a 1-cocycle of G' with values in $R^\times$ for the action of G' on R . On the prime-to- p part it is a coboundary by averaging: $u' := \frac{1}{(p-1)^2} \sum_j \zeta^j \tau^j \cdot u$ satisfies $\tau \cdot u' = \zeta^{-1}u'$ and $u' \equiv u \pmod{\mathfrak{m}_R}$, hence is a generator. On $\langle \sigma \rangle$ the cocycle has values in $1 + \mathfrak{m}_R$ and is a coboundary layer by layer: on each $\mathfrak{m}_R^n / \mathfrak{m}_R^{n+1}$, a k -vector space on which σ acts through the configuration action, a 1-cocycle of C_p that is the reduction of a cocycle trivial on the previous layer is killed by modifying u' by a unit $\equiv 1 \pmod{\mathfrak{m}_R^n}$, and the modification can be taken τ -equivariant by averaging again. The normalizations are compatible because $\tau\sigma\tau^{-1} = \sigma^a$ preserves the ζ^{-1} -eigenline. (3) is (1) and (2) combined: τ acts on $y_i = u e_i$ by ζ^{-1} on u and by $\vartheta(\tau)$ on e_i , up to the permutation of indices. Since ζ^{-1} has exact order $(p-1)^2$ and $\vartheta(\tau)^{p-1} = 1$, the product has exact order $(p-1)^2$. (4) The branch line is $X/G = \mathbb{P}_y^1$ with $\sigma: y \mapsto y+1$ and $\tau^*y = \zeta^{p-1}y$ (Lemma 4.1); then $\tau\sigma\tau^{-1}: y \mapsto y + \zeta^{p-1}$, so $a \equiv \zeta^{p-1}$. Hence $\eta = \zeta^{-1}\zeta^{p-1} = \zeta^{p-2}$, which is primitive as $\gcd(p-2, (p-1)^2) = 1$, and $\eta^{p-1} = \zeta^{(p-2)(p-1)} = \zeta^{(p-1)^2} \zeta^{-(p-1)} = \zeta^{-(p-1)} \equiv a^{-1}$. The action of τ on the periodicity class $b \in \hat{H}^2(C_p, A)$ is the contragredient of its action on $H_1(C_p, \mathbb{Z}) = C_p$, which is multiplication by a ; so $\tau(b) = a^{-1}b = \eta^{p-1}b$. $\square$

Remark 10.35. The identity $\eta = \zeta^{-1}\vartheta(\tau)$ is the one place in this Part where the geometry of the curve enters; it replaces what an earlier draft obtained by inspecting the explicit universal curve. Both factors are characters of τ on cotangent lines — of the block $(\zeta^{-1}, \text{Proposition 9.20(3)})$ and of the branch line $(\vartheta(\tau), \text{Lemma 3.4})$ — and the relation between them is the ramification row of the table in Subsection 0.4, $\chi = \theta^{-m}$ with $m = p-1$. Clause (4) is checked against the cohomological normalization of Lemma 10.41: both give $\eta^{p-1} \equiv a^{-1}$. At $p=3$ this cannot be distinguished from $\eta^{p-1} \equiv a$, since $a = a^{-1}$ in $\mathbb{F}_3^\times$; at $p=5$ it can, and it is the relation $\tau(b) = \eta^{p-1}b$, not the value of a , that makes $\beta = d^{-(p-1)}b$ invariant in Corollary 10.44.

We call $\rho := \mathbb{W}[C_p]$ the regular representation of C_p , and $\bar{\rho}$ the reduced regular representation. That is, $\bar{\rho}$ is the kernel of the augmentation map $\epsilon: \mathbb{W}[\mathbb{F}_p] \rightarrow \mathbb{W}$. Let $\Lambda := \text{Sym}(\bar{\rho})$ and $A := \text{Sym}(\rho)$. We will spend this section discussing in detail how to extend the short exact sequence of interest to a short exact sequence of $C_p \rtimes C_{(p-1)^2}$ -representations.

Lemma 10.36. *We may extend the short exact sequence of C_p -representations*

$$s_1 A \rightarrow A \rightarrow A/s_1 A \simeq \Lambda$$

to a short exact sequence of $C_p \rtimes C_{(p-1)^2}$ -representations. The basis of A may be represented by $\sigma^i \otimes y$. The generator $\tau \in C_{(p-1)^2}$ acts on this basis by $\tau(\sigma^i \otimes y) = \sigma^{ia} \otimes \eta y$, where $\eta = \zeta^{-1}\vartheta(\tau)$ is the primitive $(p-1)^2$ -th root of unity of Lemma 10.34(3), and a is a primitive root of C_p .

Proof. Let us consider $\bar{\rho}$ as a $C_p \rtimes C_{(p-1)}$ -representation, by defining it as the restriction of the standard Σ_p -representation λ . Let V be a rank 1 C_{p-1} -subrepresentation in $\bar{\rho}$ over $\mathbb{W}$.

We may extend the inclusion map of C_{p-1} -representations $V \rightarrow \bar{\rho}$ to a map of Γ representations,

$$f : R[C_p \rtimes C_{p-1}] \otimes_{R[C_{p-1}]} V \rightarrow \Lambda_0.$$

Let $A_0 := R[C_p \rtimes C_{p-1}] \otimes_{R[C_{p-1}]} V$, then A_0 has rank p over R and is isomorphic to $\rho \simeq R[C_p]$ as a C_p representation. This is because there are $(p-1)$ one dimensional C_{p-1} -subrepresentations of Λ , where t in $\mathbb{F}_p^\times$ acts by multiplication with t^r , where $1 \leq r \leq p-1$.

Now, let's say we consider the subrepresentation V of Λ generated by a fixed y . We may present ρ as being generated by $y_i := \sigma^i \otimes y$, where y is the generator of V , thus the map f sends $\sigma^i \otimes y \mapsto \sigma^i(y)$.

In order to show that f is a surjective map, it is sufficient to check that the determinant of the spanning matrix $(y \ \sigma(y) \ \sigma^2(y) \ \cdots \ \sigma^{p-2}(y))$ is a unit in $\mathbb{Z}_p$, or equivalently, its reduction is a unit in $\mathbb{Z}/p$. This is sufficient because it shows that $\text{im}(f)$ generates $\bar{\rho}$ as a rank $p-1$ $\mathbb{W}$ -module. This is shown in Lemma 10.37. This shows the short exact sequence holds as an SES of $C_p \rtimes C_{(p-1)}$ -modules. We may extend it to an SES of $C_p \rtimes C_{(p-1)^2}$ -modules by applying $\text{Ind}_{C_p \rtimes C_{(p-1)}}^{C_p \rtimes C_{(p-1)^2}}$, which preserves the SES. $\square$

Lemma 10.37. *Let a be a fixed primitive root of $\mathbb{Z}/p$, and ζ be a $p-1$ root of unity. Let*

$$y := \sum_{k \in \mathbb{F}_p \setminus 0} \zeta^k t^{a^k}.$$

Then, the $\mathbb{Z}_p[G]$ submodule of $R[t]/(t^p - 1)$ generated by y is the kernel of the augmentation map $R[\mathbb{F}_p] \rightarrow R$.

Proof. Since we are working with local rings, by Nakayama's lemma it is sufficient to show that y generates $\Lambda_0 \bmod p$. That is, it suffices to show Lemma 10.38. $\square$

Lemma 10.38. *For p odd, the $\mathbb{F}_p[C_p \rtimes C_{(p-1)}]$ -submodule of $\mathbb{F}_p[t]/(t-1)^p$ generated by the element*

$$\bar{y} = \sum_{i \in \mathbb{F}_p^\times} i^{-1} t^i,$$

is isomorphic to $(t-1)\mathbb{F}_p[t]/(t-1)^p$, and is the kernel of the augmentation $\mathbb{F}_p[t]/(t-1)^p \rightarrow \mathbb{F}_p$.

Proof. We may consider kernel of the augmentation map as the ideal generated by $(t-1)$, since $\bar{y}$ goes to zero under the augmentation. The $\mathbb{F}_p[G]$ -submodule generated by $\bar{y}$ is contained in this ideal. Further, the submodule generated by $\bar{y}$ is an ideal of $\mathbb{F}_p[t]/(t-1)^p$, because we have the element σ in G which acts by multiplication by t .

To see that the submodule generated by $\bar{y}$ agrees with the ideal generated by $(t-1)$, it suffices to check that $\bar{y} = (t-1)u$, where u is a unit in $\mathbb{F}_p[t]/(t-1)^p$. Since our ring $\mathbb{F}_p[t]/(t-1)^p$ is complete and local, it suffices to check that this equivalence holds mod $(t-1)^2$.

$$\begin{aligned} t^i &\equiv (1 + (t-1))^i \pmod{(t-1)^2} \\ &\equiv 1 + i(t-1), \end{aligned}$$

therefore

$$\begin{aligned} \bar{y} &= \sum_{i \in \mathbb{F}_p \setminus 0} i^{-1} t^i \\ &= \sum_{i \in \mathbb{F}_p \setminus 0} i^{-1} + \sum_{i \in \mathbb{F}_p \setminus 0} i^{-1} i(t-1) \\ &= (p-1)(t-1). \end{aligned}$$

The first sum is zero, and the second is $(p-1)$. (this uses $p \neq 2$). Thus, $\bar{y} \equiv -(t-1) \pmod{(t-1)^2}$. This concludes the argument. $\square$

Recall that $\mathbb{W} := W(\mathbb{F}_{p^{p-1}}) \simeq \mathbb{Z}_p[\omega]$, where ω is a $p^{p-1} - 1$ root of unity. The $\mathbb{Z}/(p-1)^2$ sitting inside of the maximal finite subgroup G' is generated by $\eta := \omega^{(p^{p-1}-1)/(p-1)^2}$ (Buj12), which is a $(p-1)^2$ root of unity. Since $\hat{\mathcal{F}}^1$ is an $\text{Aut}(X)$ -equivariant equivalence, the Lubin-Tate action coincides with the manipulation above. For more on the full $C_{(p-1)^2}$ action, see Gorbounov-Symonds Prop 3.2 (GS98) and Nave (Nav10) Prop 5.2.

10.8. $C_p \rtimes C_{(p-1)^2}$ Conclusion and Known Classes. Let us now examine the invariants of the $C_{(p-1)^2}$ -action on $\hat{H}^*(C_p, A)$. Let's start with a reduction of complexity.

Lemma 10.39. *Let $G' := C_p \rtimes C_{(p-1)^2}$. Let R be a ring in which $(p-1)^2$ is a unit. For a G - R -module M , the spectral sequence collapses and*

$$H^*(G, M) \simeq H^*(C_p, M)^{C_{(p-1)^2}}.$$

Proof. Given a G - $\mathbb{W}$ -module M , the short exact sequence $C_p \xrightarrow{\triangleleft} G \rightarrow C_{(p-1)^2}$ gives rise to the spectral sequence

$$H^p(C_{(p-1)^2}; H^q(C_p; M)) \Rightarrow H^{p+q}(G; N).$$

The transfer map in group cohomology yields a commutative diagram

$$\begin{array}{ccc} H^p(C_{(p-1)^2}, H^q(C_p, M)) & \xrightarrow{(p-1)^2} & H^p(C_{(p-1)^2}, H^q(C_p, M)) \\ & \searrow \text{res} & \nearrow \text{cores} \\ & H^p(1, H^q(C_p, M)) & \end{array}$$

Because $(p-1)^2$ is a unit in $\mathbb{W}$, the horizontal map is an isomorphism and the spectral sequence collapses. $\square$

Remark 10.40. Note that the same holds for Tate cohomology.

Lemma 10.41. *Let η be a $(p-1)^2$ root of unity. The action of a generator $\tau \in C_{(p-1)^2}$ on $\hat{H}^*(C_p, A) \simeq \mathbb{Z}/p[b^{\pm 1}, d]$ is as follows,*

$$\begin{aligned} \tau : b &\mapsto \eta^{(p-1)} b \\ d &\mapsto \eta^p d \end{aligned}$$

Proof. Let a be a chosen primitive root of $\mathbb{Z}/p$ such that $\tau\sigma\tau^{-1} = \sigma^a$. We begin with b the generator of $H^2(C_p, A)$. Recall that $H_1(C_p, \mathbb{Z}) \simeq C_p$ where τ acts by multiplication by a . Then, $H^1(C_p, \mathbb{Z}/p) \simeq \text{Hom}(H_1(C_p, \mathbb{Z}), \mathbb{Z}/p)$. The induced left action on the dual $\text{Hom}(H_1(C_p, \mathbb{Z}), \mathbb{Z}/p)$ is the contragredient, $(\tau\phi)(x) = \phi(\tau^{-1}x)$, i.e. multiplication by a^{-1} (not by a : the transpose of a 1×1 matrix is itself, but the contragredient is its inverse). Via the Bockstein map, $H^1(C_p, \mathbb{Z}/p) \simeq H^2(C_p, \mathbb{Z})$. Thus $\tau(b) = a^{-1}b$, and the normalization of η is $\eta^{(p-1)} = a^{-1}$; this is the same convention as the action on A below, where $\tau\sigma^i\tau^{-1} = \sigma^{ia}$ is used, and it is the normalization $\eta^{p-1} = \zeta^{-(p-1)}$, $\bar{\zeta}^{p-1} = a$, forced by Lemma 10.34(4). (Only the relation $\tau(b) = \eta^{p-1}b$ between the two scalars enters the invariant computation of Corollary 10.44, so the invariants are unaffected by the choice of primitive root.)

We now examine how τ acts on $d = y_0 y_1 \cdots y_{p-1}$, which we do using section 10.7 which allows us to analyze how τ acts on $y_i := \sigma^i \otimes y$.

$$\begin{aligned} \tau(y_i) &:= \tau(\sigma^i \otimes y) \\ &= \tau(\sigma^i) \otimes y \\ &= \sigma^{ia} \otimes \tau(y) \\ &= \sigma^{ia} \otimes \eta y \\ &= \eta(\sigma^{ia} \otimes y) = \eta y_{ia} \end{aligned}$$

Therefore, $\tau(d) = \eta^p(\prod_{i=0}^{p-1} y_{ia}) = \eta^p d$. □

Lemma 10.42. *Let η be a $(p-1)^2$ root of unity. The action of a generator $\tau \in C_{(p-1)^2}$ on $\hat{H}^*(C_p, s_1 A) \simeq \mathbb{Z}/p[\tilde{b}^{\pm 1}, \tilde{d}]$ is as follows:*

$$\begin{aligned} \tau : \tilde{b} &\mapsto \eta^p \tilde{b} \\ \tilde{d} &\mapsto \eta^{p+1} \tilde{d} \end{aligned}$$

Proof.

$$\begin{aligned} \tau : \tilde{b} = s_1 b &\mapsto (\eta s_1)(\eta^{(p-1)} b) = \tilde{b} \\ \tilde{d} = s_1 d &\mapsto (\eta s_1)(\eta^p d) = \eta^{p+1} \tilde{d}. \end{aligned}$$

□

Lemma 10.43. *The action of $C_{(p-1)^2}$ on $H^*(C_p, \Lambda) \simeq \mathbb{Z}/p[b^{\pm 1}, c, d]/c^2$, is*

$$\begin{aligned} \tau : b &\mapsto \eta^{(p-1)} b \\ d &\mapsto \eta^p d \\ c &\mapsto \eta^p c. \end{aligned}$$

where b is in degree $(2, 0)$, d is in degree $(0, -2p)$, and c is in degree $(1, -2)$.

Proof. The actions on the generators b and d are the same as in 10.41. Recall from Lemma 10.32 that c is the element in bidegree $(1, (p-1))$ which maps to $\tilde{b} \in \hat{H}^2(C_p, s_1 A)$ equivariantly. The action is the same as that of $\tilde{b}$ as shown in Lemma 10.42, that is:

$$\tau : c \mapsto \eta^p c.$$

□

Corollary 10.44. (*meat*) *Let R be the graded ring*

$$R := \mathbb{W}/p[\alpha, \beta, \Delta^\pm]/(\alpha^2),$$

where $\alpha := d^{-1}c$, $\beta := d^{-(p-1)}b$, and $\Delta := d^{-(p-1)^2}$. The Tate cohomology of the $C_p \rtimes C_{(p-1)^2}$ -module $\Lambda[d^{-1}] = \text{Sym}(\bar{\rho}[d^{-1}])$ is:

$$R \simeq \hat{H}^*(C_p \rtimes C_{(p-1)^2}, \Lambda[d^{-1}]).$$

where $|\alpha| = (1, 2(p-1))$, $|\beta| = (2, 2p(p-1))$, and $|\Delta| = (0, 2p(p-1)^2)$.

Proof. The C_{p-1} action on $H^*(C_p, \Lambda) \simeq \mathbb{Z}/p[b^\pm, c, d]/c^2$ is described by Lemma 10.43. The fixed points of this action are precisely $\alpha := d^{-1}c$, $\beta := d^{-(p-1)}b$, and $\Delta := d^{-(p-1)^2}$. The fact that this is the Tate cohomology of Λ follows from part (ii) Lemma 10.39. $\square$

Remark 10.45 (Ravenel's detection, in these coordinates). Ravenel (Rav78) proved that the odd-primary Arf invariant classes $b_i \in \text{Ext}_{BP_*BP}^{2, 2(p-1)p^{i+1}}(BP_*, BP_*)$ are nonzero by restricting to an order- p subgroup of $\mathbb{G}_{p-1}$ and showing $b_i \mapsto -c^{p^{i+1}}b$ in $H^*(\mathbb{Z}/p; \mathbb{F}_{p^{p-1}}) = E(h) \otimes P(b)$, $c \in \mathbb{F}_{p^{p-1}}^\times$; his b is the class b here and his h is our c . The restriction factors through $E_{p-1}^{h\mathbb{G}} \rightarrow E_{p-1}^{hG'} \rightarrow E_{p-1}^{hC_p}$, and Corollary 10.44 identifies the middle term: in cohomological degree 2 the ring R is spanned by the classes $\beta\Delta^m$, of internal degree $2p(p-1) + 2p(p-1)^2m$, and

$$2p(p-1)(1 + (p-1)m) = 2(p-1)p^{i+1} \iff m = \frac{p^i - 1}{p-1}.$$

Hence

$$b_i \mapsto u_i \beta \Delta^{(p^i-1)/(p-1)}, \quad u_i \in (\mathbb{W}/p)^\times,$$

the unit being nonzero because the further restriction to C_p is Ravenel's $-c^{p^{i+1}}b \neq 0$. The exponent $(p^i-1)/(p-1)$ is a_i/p for the modulus $\ker b_0^{a_i}$, $a_i = p(p^i-1)/(p-1)$, of his inductive differential $d_{2p-1}b_{i+1} \equiv h_0b_i^p$: the power of Δ carrying b_i is the power of b_0 modulo which the differential is computed. The scalar c , which he shows generates $\mathbb{F}_{p^{p-1}}$ and which is why the detecting map needs that field, is on our side the residue of the tame character ζ of Proposition 9.20(3), which lives in $\mathbb{F}_{p^{p-1}}$ for the same reason.

We've done most of the hard work. Now, let's translate this back to $(E_{p-1})_*$.

Lemma 10.46. *Let $x := \prod_{j=0}^{p-1} \sigma^j(\sigma-1)(y_0)$. Then, multiplication by x is zero on $H^*(C_p, A)$.*

Proof. It suffices to show that x is zero on $b \in H^*(C_p, A)$. Let X be the polynomial algebra over the abstract symbols $X := \mathbb{W}[y_0, \sigma(y_0), \dots, \sigma^{(p-1)}(y_0)]$, and give X the permutation C_p action. Let $[x] \in H^2(X)$ be the cohomology class represented by x in X . We explain x into a sum of monomials and see that the coefficient of $x\sigma(x) \cdots \sigma^{(p-1)}(y_0)$ is zero. This implies that $[x] = 0$, so $x = (1 + \sigma + \cdots + \sigma^{p-1})z$ for some $z \in X$. The map $X \rightarrow \mathbb{W}[y_0, \dots, y_{p-1}]$ sending $\sigma^j(y_0) \mapsto y_j$, exhibits $[x]$ as a coboundary. Thus, the action is zero as desired. $\square$

Lemma 10.47. (*ideal hop*)

$$\hat{H}^*(C_p \rtimes C_{(p-1)^2}, (E_{p-1})_*) \simeq \hat{H}^*(C_p \rtimes C_{(p-1)^2}, \text{Sym}(\bar{\rho}))[\Delta^{-1}].$$

Proof. We will abbreviate $H^*(-) := H^*(C_p \rtimes C_{p-1}, -)$. By Theorem D, where $I = (p, y_i)$, it suffices to show that

$$\begin{aligned} \hat{H}^*(\text{Sym}(\bar{\rho})[d^{-1}]_I^\wedge) &\simeq \hat{H}^*(\text{Sym}(\bar{\rho}))[\Delta^{-1}]_I^\wedge \\ &\simeq \hat{H}^*(\text{Sym}(\bar{\rho}))[\Delta^{-1}] \end{aligned}$$

We will call $\Lambda' := \Lambda[d^{-1}]$. The commutation of inversion with cohomology follows from inversion being a colimit and cohomology commuting with colimits. Now let's move on to the tricky ideal, and why it both commutes and dies. Consider the diagram:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \Lambda' & \xrightarrow{I^2} & \Lambda' & \longrightarrow & \Lambda'/I^2 \longrightarrow 0 \\ & & \downarrow I & & \downarrow \text{id} & & \downarrow I \\ 0 & \longrightarrow & \Lambda' & \xrightarrow{I} & \Lambda' & \longrightarrow & \Lambda'/I \longrightarrow 0 \end{array}$$

Multiplication by p and y_i is zero on $\hat{H}^*(A)$ by Lemma 10.46, which means by naturality multiplication by p and y_i^k are zero on $\hat{H}^*(\Lambda')$ as well. Therefore, we have a short exact sequence of towers

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ H^i(\Lambda') & \longrightarrow & H^i(\Lambda'/I^2) & \longrightarrow & H^{i+1}(\Lambda') \\ \downarrow \text{id} & & \downarrow H^i(I) & & \downarrow 0 \\ H^i(\Lambda') & \longrightarrow & H^i(\Lambda'/I) & \longrightarrow & H^{i+1}(\Lambda') \end{array}$$

and a corresponding 6-term exact sequence involving $\lim$ and $\lim^1$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \lim_k \{H^i(\Lambda')\}_k & \longrightarrow & \lim_k \{H^i(\Lambda'/I^k)\}_k & \longrightarrow & \lim_k \{H^{i+1}(\Lambda')\}_k \\ & & & & & & \searrow \\ & & & & & & \lim_k^1 \{H^i(\Lambda')\}_k \longrightarrow \lim_k^1 \{H^i(\Lambda'/I_k)\}_k \longrightarrow \lim_k^1 \{H^{i+1}(\Lambda')\}_k \longrightarrow 0 \end{array}$$

Now, $\lim^1$ of the first tower is 0 because the maps involved are the identity. $\lim^1$ of the third tower is 0 because the maps are trivial. Therefore, $\lim^1 H^*(\Lambda'/I^k) = 0$. Since the limit over identity is itself, and the limit over the 0 map is 0, our 6 term exact sequence collapses severely to just the first two $\lim$ terms:

$$H^i(\Lambda') \simeq \lim_k H^i(\Lambda'/I^k).$$

Now, let's examine the latter $\lim_k H^i(\Lambda'/I^k)$. For each fixed k , there's a cochain complex computing the finite group cohomology of Λ'/I^k . (For ease of notation we write the complex for the C_p -group cohomology, but it works for the whole group $C_p \rtimes C_{p-1}$.)

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Lambda'/I^k & \xrightarrow{\sigma-1} & \Lambda'/I^k & \xrightarrow{1+\dots+\sigma^{p-1}} & \Lambda'/I^k \xrightarrow{\sigma-1} \dots \\
& \downarrow I & & \downarrow I & & \downarrow I & \\
0 & \longrightarrow & \Lambda'/I^{k-1} & \xrightarrow{\sigma-1} & \Lambda'/I^{k-1} & \xrightarrow{1+\sigma+\dots+\sigma^{p-1}} & \Lambda'/I^{k-1} \xrightarrow{\sigma-1} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

The limit of this tower of complexes is $H^*(\Lambda_I'^\wedge)$. The tower in each cohomological degree consists of surjections, which means that the $\lim^1$ term vanishes. So, we get that

$$H^i(\Lambda_I'^\wedge) \simeq \lim_k \{H^i(\Lambda'/I^k)\}_k,$$

where the latter is a limit taken over $H^*(I)$. Putting this together with our conclusion from the analysis of our first lim sequence, we get

$$H^i(\Lambda') \simeq \lim_k H^i(\Lambda'/I^k) \simeq H^i(\Lambda_I'^\wedge).$$

Thus, the cohomology of Λ' is unaffected by the I completion as claimed. $\square$

Remark 10.48. The preceeding proof is based on Nave's proof of theorem 5.7 in (Nav99).

Theorem E. *Let R be the graded ring*

$$R := \mathbb{W}/p[\alpha, \beta, \Delta^\pm]/(\alpha^2).$$

The Tate cohomology of the $C_p \rtimes C_{(p-1)^2}$ -module $(E_{p-1})_*$ is:

$$R \simeq \hat{H}^*(C_p \rtimes C_{(p-1)^2}, (E_{p-1})_*).$$

where $|\alpha| = (1, 2(p-1))$, $|\beta| = (2, 2p(p-1))$, and $|\Delta| = (0, 2p(p-1)^2)$.

Proof. This follows from Theorem 10.44 combined with Lemma 10.47. $\square$

Remark 10.49. The classes c and bd represent α and β in the homotopy groups of spheres, up to a unit. That is why we notate them as such in Theorem E.

APPENDIX A. IDENTIFICATION OF THE ACTION

A.1. Identification of the action. Write $\text{Aut}(F) = \mathcal{O}_D^\times$ for D the division algebra of invariant $1/(p-1)$, presented as $D = \mathbb{Q}_{p^{p-1}}\langle \Pi \rangle$ with $\Pi a = a^\varphi \Pi$, $\Pi^{p-1} = p$.

Lemma A.1 (the normalizer). *Choose $\omega_0 \in \mu_{p^{p-1}-1}$ with $\omega_0^{(p^{p-1}-1)/(p-1)} = -1$ and set $\pi := \omega_0 \Pi$, so $\pi^{p-1} = -p$ and $L := \mathbb{Q}_p(\pi) \cong \mathbb{Q}_p(\zeta_p)$ is a maximal totally ramified subfield. For $\rho \in \mu_{p^{p-1}-1}$,*

$$\rho \pi \rho^{-1} = \rho^{1-p} \pi,$$

which lies in L if and only if $\rho \in \mu_{(p-1)^2}$. The subgroup $j(G') := \langle \zeta_p \rangle \rtimes \mu_{(p-1)^2} \subset \mathcal{O}_D^\times$ is isomorphic to G' , with surjective conjugation character $\rho \mapsto \overline{\rho^{-(p-1)}}$.

Proof. The commutation rule follows from $\Pi\rho^{-1} = \rho^{-p}\Pi$; normalization requires $\rho^{p-1} \in \mu(L)_{p'} = \mu_{p-1}$ (L totally ramified), i.e. $\rho \in \mu_{(p-1)^2}$, using $(p-1)^2 \mid p^{p-1} - 1$. Conjugation sends $\zeta_p \mapsto \zeta_p^q$ with $q \equiv \rho^{-(p-1)}$ modulo π ; $(p-1)$ -th powering maps $\mu_{(p-1)^2}$ onto $\mu_{p-1} = \mathbb{F}_p^\times$. $\square$

Proposition A.2 (uniqueness of the class). *Any subgroup $\Gamma \subset \mathcal{O}_D^\times$ abstractly isomorphic to $C_p \rtimes_\chi C_{(p-1)^2}$ with χ surjective is conjugate to $j(G')$ by an element of $\mathcal{O}_D^\times$.*

Proof. Let s generate the Sylow p -subgroup, t a complement generator.

Step 1. Skolem–Noether provides $u \in D^\times$ with $usu^{-1} = \zeta_p$; since $v_D(\pi) = 1$, multiplying u by a power of a uniformizer of $\mathbb{Q}_p(s)$ (which centralizes s) makes u a unit. Assume $\langle s \rangle = \langle \zeta_p \rangle$.

Step 2. t normalizes L ; by surjectivity of χ , after replacing t by a power we may assume $t\zeta_p t^{-1} = \zeta_p^q$ with q the conjugation value of a fixed generator ρ_1 of $\mu_{(p-1)^2}$. Then $c := t\rho_1^{-1}$ centralizes ζ_p , hence $c \in \mathcal{O}_L^\times$ (L maximal).

Step 3. Let $g \in \text{Gal}(L/\mathbb{Q}_p)$ be induced by t and ρ_1 alike. Telescoping $t^{p-1} = N^g(c)\rho_1^{p-1}$, both outer terms lie in $\mu_{p-1} \subset \mathbb{Z}_p^\times$, so $N^g(c) \in \mu_{p-1}$; total ramification makes g trivial on residues, so $N^g(c) \equiv \bar{c}^{p-1} = 1$ in $\mathbb{F}_p^\times$, and $\mu_{p-1} \cap (1 + p\mathbb{Z}_p) = 1$: $N^g(c) = 1$.

Step 4. Hilbert 90: $c = \ell g(\ell)^{-1}$, and writing $\ell = \pi^n v$ with $v \in \mathcal{O}_L^\times$, $c = \delta^n v g(v)^{-1}$ with $\delta = \pi g(\pi)^{-1} \in [q]^{-1}(1 + \mathfrak{m}_L)$ of norm one. Principal units form a pro- p module over the order- $(p-1)$ group $\langle g \rangle$, hence are cohomologically trivial; so $[q]\delta = w g(w)^{-1}$ with $w \in 1 + \mathfrak{m}_L$. Conjugating by the unit $v' := w^n v$ carries t to $[q]^{-n}\rho_1$, whose $(p-1)$ -th power is ρ_1^{p-1} : a generator of $\mu_{(p-1)^2}$. Done. $\square$

Corollary A.3. There exist $u \in \mathcal{O}_D^\times$ and $\alpha \in \text{Aut}(G')$ with $u^{-1}\iota_{\text{aut}}(g')u = j(\alpha(g'))$; consequently the twist action of $\iota_{\text{aut}}(G')$ on $\text{Def}^* F$ is isomorphic, via an automorphism of $\text{Def}^* F$, to the canonical Lubin–Tate action of G' restricted along α .

REFERENCES

- [Bea17] Agnès Beaudry. The chromatic splitting conjecture at $n = p = 2$. *Geom. Topol.*, 21(6):3213–3230, 2017.
- [Beh06] Mark Behrens. A modular description of the $K(2)$ -local sphere at the prime 3. *Topology*, 45:343–402, 2006.
- [Beh20] Mark Behrens. Topological modular and automorphic forms. In *Handbook of homotopy theory*, CRC Press/Chapman Hall Handb. Math. Ser., pages 221–261. CRC Press, Boca Raton, FL, [2020] ©2020.
- [BG18] Irina Bobkova and Paul G. Goerss. Topological resolutions in $K(2)$ -local homotopy theory at the prime 2. *J. Topol.*, 11:918–957, 2018.
- [BL10] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Mem. Amer. Math. Soc.*, 204(958):xxiv+141, 2010.
- [BM00] José Bertin and Ariane Mézard. Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. *Invent. Math.*, 141(1):195–238, 2000.
- [Boy99] Pascal Boyer. Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale. *Invent. Math.*, 138:573–629, 1999.
- [BR11] José Bertin and Matthieu Romagny. Champs de Hurwitz. *Mém. Soc. Math. Fr. (N.S.)*, 125–126, 2011.

- [BSS20] Tobias Barthel, Tomer M. Schlank, and Nathaniel Stapleton. Chromatic homotopy theory is asymptotically algebraic. *Invent. Math.*, 220:737–845, 2020.
- [Buj12] Cédric Bujard. Finite subgroups of the extended morava stabilizer groups. *arXiv: Algebraic Topology*, 2012.
- [Car90] Henri Carayol. Nonabelian lubin-tate theory, in automorphic forms, shimura varieties, and l-functions, vol ii. *Perspect. Math., vol. 11*, pages p. 15–39., 1990.
- [CK03] Gunther Cornelissen and Fumiharu Kato. Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic. *Duke Math. J.*, 116(3):431–470, 2003.
- [Col88] Robert F Coleman. On the frobenius endomorphisms of fermat and artin-schreier curves. *Proceedings of the American Mathematical Society*, 102(3):463–466, 1988.
- [CS64] Pierre Colmez and Jean-Pierre Serre. Correspondance serre–tate. *SMF 2015*, 2, 1964.
- [DH04] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.
- [DM86] Pierre Deligne and George Daniel Mostow. Monodromy of hypergeometric functions and non-lattice integral monodromy. *Publications Mathématiques de l’IHÉS*, 63:5–89, 1986.
- [FN62] Edward Fadell and Lee Neuwirth. Configuration spaces. *Mathematica Scandinavica*, 10:111–118, 1962.
- [GH04] Paul G. Goerss and Michael J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, 2004.
- [GHMR05] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk. A resolution of the $K(2)$ -local sphere at the prime 3. *Ann. of Math. (2)*, 162(2):777–822, 2005.
- [GJF23] Davide Gaiotto and Theo Johnson-Freyd. Mock modularity and a secondary elliptic genus. *J. High Energy Phys.*, 2023(8):023, 2023.
- [GJFW19] Davide Gaiotto, Theo Johnson-Freyd, and Edward Witten. A note on some minimally supersymmetric models in two dimensions. *arXiv:1902.10249*, 2019.
- [GKP⁺17] Steven S. Gubser, Johannes Knaute, Sarthak Parikh, Andreas Samberg, and Przemek Witaszczyk. p -adic AdS/CFT. *Comm. Math. Phys.*, 352:1019–1059, 2017.
- [GM00] V. Gorbounov and M. Mahowald. Formal completion of the Jacobians of plane curves and higher real K -theories. *J. Pure Appl. Algebra*, 145(3):293–308, 2000.
- [Goe08] Paul G Goerss. Quasi-coherent sheaves on the moduli stack of formal groups. *arXiv preprint arXiv:0802.0996*, 2008.
- [GS98] Vassily Gorbounov and Peter Symonds. Toward the homotopy groups of the higher real k -theory eo2. *Contemporary Mathematics*, 220:103–116, 1998.
- [Har66] William J. Harvey. Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math. Oxford Ser. (2)*, 17:86–97, 1966.

- [Har77] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.
- [Hen98] Hans-Werner Henn. Centralizers of elementary abelian p -subgroups and mod- p cohomology of profinite groups. *Duke Math. J.*, 91(3):561–585, 1998.
- [Hew95] T. Hewett. Finite subgroups of division algebras over local fields. *Journal of Algebra*, 173(3):518–548, 1995.
- [HG94] Michael J. Hopkins and Benedict H. Gross. Equivariant vector bundles on the Lubin–Tate moduli space. In *Topology and representation theory*, volume 158 of *Contemp. Math.*, pages 23–88. Amer. Math. Soc., 1994.
- [HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.
- [Hil06] Michael Hill. Computational methods for higher real k-theory with applications to tmf. *Ph.D. Thesis*, 2006.
- [HL10] Michael Hill and Tyler Lawson. Automorphic forms and cohomology theories on Shimura curves of small discriminant. *Adv. Math.*, 225(2):1013–1045, 2010.
- [HMSS18] Matthew Heydeman, Matilde Marcolli, Ingmar Saberi, and Bogdan Stolica. Tensor networks, p -adic fields, and algebraic curves: arithmetic and the $\text{AdS}_3/\text{CFT}_2$ correspondence. *Adv. Theor. Math. Phys.*, 22:93–176, 2018.
- [HS99] Mark Hovey and Neil P Strickland. *Morava K -theories and localisation*, volume 666. American Mathematical Soc., 1999.
- [HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, 2001.
- [Hui18] Jeroen Huijben. Deformation of curves with a group action. Master’s thesis, Utrecht University, 2018.
- [Ill72] Luc Illusie. Cotangent complex and deformations of torsors and group schemes. In *Toposes, algebraic geometry and logic*, volume 274 of *Lecture Notes in Math.*, pages 159–189. Springer, 1972.
- [IS05] Ippei Ichigi and Katsumi Shimomura. Homotopy groups of generalized $E(2)$ -local Moore spectra at the prime three. *Hiroshima Math. J.*, 35(1):125–142, 2005.
- [Kat81] Nicholas M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Institute of Fundamental Research, Bombay, 1981.
- [Kha23] Adeel A Khan. Lectures on algebraic stacks. *arXiv preprint arXiv:2310.12456*, 2023.
- [KO68] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968.
- [Kob21] Andrew Kobin. Artin–schreier root stacks. *Journal of Algebra*, 586:1014–1052, 2021.
- [LMPT18] Wanlin Li, Elena Mantovan, Rachel Pries, and Yunqing Tang. Newton polygon stratification of the Torelli locus in PEL-type Shimura varieties.

- arXiv preprint arXiv:1811.00604*, 2018.
- [LT65] Jonathan Lubin and John Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.
 - [LT66a] J. Lubin and J. Tate. Formal moduli for one-parameter formal Lie groups. *Bulletin de la Société Mathématique de France*, 94:49–59, 1966.
 - [LT66b] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal Lie groups. *Bull. Soc. Math. France*, 94:49–59, 1966.
 - [Man63] Ju. I. Manin. Theory of commutative formal groups over fields of finite characteristic. *Uspehi Mat. Nauk*, 18(6 (114)):3–90, 1963.
 - [Mes72] William Messing. *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, volume 264 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.
 - [Mor89a] Jack Morava. Forms of K -theory. *Math. Z.*, 201(3):401–428, 1989.
 - [Mor89b] Jack Morava. Stable homotopy and local number theory. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 291–305. Johns Hopkins Univ. Press, Baltimore, MD, 1989.
 - [MRT14] Ariane Mézard, Matthieu Romagny, and Dajano Tossici. Sekiguchi–Suwa theory revisited. *J. Théor. Nombres Bordeaux*, 26:163–200, 2014.
 - [MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams–Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.
 - [Nau08] Niko Naumann. Arithmetically defined dense subgroups of Morava stabilizer groups. *Compos. Math.*, 144:247–270, 2008.
 - [Nav99] Lee Stewart Nave. *The cohomology of finite subgroups of Morava stabilizer groups and Smith–Toda complexes*. University of Washington, 1999.
 - [Nav10] Lee S. Nave. The Smith–Toda complex $V((p+1)/2)$ does not exist. *Ann. of Math. (2)*, 171(1):491–509, 2010.
 - [OP06] A. Okounkov and R. Pandharipande. Gromov–Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math. (2)*, 163(2):517–560, 2006.
 - [Pst21] Piotr Pstragowski. Chromatic homotopy theory is algebraic when $p > n^2 + n + 1$. *Adv. Math.*, 391:Paper No. 107958, 37, 2021.
 - [Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.
 - [Rav78] Douglas C. Ravenel. The non-existence of odd primary Arf invariant elements in stable homotopy. *Math. Proc. Cambridge Philos. Soc.*, 83(3):429–443, 1978.
 - [Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.
 - [Rav07] Douglas C. Ravenel. Toward higher chromatic analogs of elliptic cohomology. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 286–305. Cambridge Univ. Press, Cambridge, 2007.
 - [Rav08] Douglas C. Ravenel. Toward higher chromatic analogs of elliptic cohomology. II. *Homology Homotopy Appl.*, 10(3):335–368, 2008.
 - [Ray25] Rin Ray. Modeling group actions on stacks (especially the Lubin–Tate action). *arXiv:2507.00309*, 2025.

- [Rez98] Charles Rezk. Notes on the Hopkins–Miller theorem. In *Homotopy theory via algebraic geometry and group representations*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., 1998.
- [Rez06] Charles Rezk. The units of a ring spectrum and a logarithmic cohomology operation. *J. Amer. Math. Soc.*, 19:969–1014, 2006.
- [Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898), 2008.
- [RZ96] M. Rapoport and Th. Zink. *Period Spaces for “ p ”-divisible Groups (AM-141)*. Princeton University Press, 1996.
- [SS01] Tsutomu Sekiguchi and Noriyuki Suwa. A note on extensions of algebraic and formal groups. IV. Kummer–Artin–Schreier–Witt theory of degree p^2 . *Tohoku Math. J. (2)*, 53:203–240, 2001.
- [ST11] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., 2011.
- [Sti09] Henning Stichtenoth. *Algebraic function fields and codes*, volume 254 of *Graduate Texts in Mathematics*. Springer-Verlag, second edition, 2009.
- [Sto12] Vesna Stojanoska. Duality for topological modular forms. *Doc. Math.*, 17:271–311, 2012.
- [Sub75] Doré Subrao. The p -rank of Artin–Schreier curves. *Manuscripta Math.*, 16:169–193, 1975.
- [SW13] Peter Scholze and Jared Weinstein. Moduli of p -divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.
- [TY23] Yuji Tachikawa and Mayuko Yamashita. Topological modular forms and the absence of all heterotic global anomalies. *Comm. Math. Phys.*, 402:1585–1620, 2023.
- [VM80] Robert C. Valentini and Manohar L. Madan. A Hauptsatz of L.E. Dickson and Artin–Schreier extensions. *J. Reine Angew. Math.*, 318:156–177, 1980.
- [Wew05] Stefan Wewers. Formal deformation of curves with group scheme action. *Ann. Inst. Fourier (Grenoble)*, 55(4):1105–1143, 2005.
- [Wim95] Anders Wiman. Über die hyperelliptischen Curven und diejenigen vom Geschlechte $p = 3$, welche eindeutige Transformationen in sich zulassen. *Bihang Kongl. Svenska Vetenskaps-Akademiens Handlingar*, 21, 1895.
- [Wit87] Edward Witten. Elliptic genera and quantum field theory. *Comm. Math. Phys.*, 109:525–536, 1987.
- [Zhu19] Yifei Zhu. Semistable models for modular curves and power operations for Morava E -theories of height 2. *Adv. Math.*, 354, 2019.