Waldhausen's obstruction to connectivity

Introduction

The goal of this note is to give a new perspective on a result of Waldhausen's [Wal78] which, given a highly connective map of ring spectra $R \to S$, provides an obstruction to the induced map on K-theory spectra to be more connective than it "should be", for general reasons.

This obstruction is phrased in terms of Hochschild homology, and so immediately makes one think of the Dundas–McCarthy theorem [DM94] relating topological Hochschild homology to the Goodwillie derivative of K-theory. The approach we follow here is to use this result to derive Waldhausen's computation (where he uses spectral sequences to prove his result).

We phrase everything in the language of ∞-categories as developed in [Lur09], [Lur12].

1 A connectivity result

In this section, we recall the following classical connectivity result concerning algebraic K-theory:

Theorem 1.1. *Suppose $f : R \to S$ is a map of connective ring spectra, which is k-connective for some $k \geq 1$.*

Then the induced morphism $K(R) \to K(S)$ is $(k+1)$-connective.

Remark 1.2. This only works for *connective* ring spectra, and the reason is that we use "direct sum K-theory", which is only equivalent to K-theory for connective ring spectra (it specializes to the K-theory of the connective cover in general).

Remark 1.3. We say that a morphism $f : X \to Y$ is n-connective if its fiber is n-connective.

Proof. Observe that for P, Q projective R-modules, $\mathrm{Map}_R(P, Q) \to \mathrm{Map}_S(f_! P, f_! Q)$ is k-connective.

Indeed, this map is natural in P, Q, and k-connectivity is preserved under direct sums and retracts; it therefore suffices to show this for $P = Q = R$, but then this map is equivalent to $f : R \to S$, for which this was the hypothesis.

In particular, the map of E_∞-spaces $\mathbf{Proj}_R^{\simeq} \to \mathbf{Proj}_S^{\simeq}$ is $(k+1)$-connective. Here we mean that it is an isomorphism on π_0 and on π_n at all points, for any $n < (k+1)$, as well as an epimorphism on π_{k+1} at any point.

Note that this is where we use that $k \geq 1$: we need this map to be a bijection on π_0, and it is so because idempotents in $M_n(R)$ are the same as idempotents in $M_n(\pi_0(R))$.

The final claim is that $(k+1)$-connective morphisms of E_∞-spaces are preserved under group completion. This follows from McDuff's group completion theorem (see [Nik17] for a modern account): because all spaces in sight are simple, it suffices (by the Hurewciz theorem) to check the claim on homology, where the morphism $H_*(\Omega^\infty K(R)) \to H_*(\Omega^\infty K(S))$ is identified with $H_*(\mathbf{Proj}_R^{\simeq})[\pi_0^{-1}] \to H_*(\mathbf{Proj}_S^{\simeq})[\pi_0^{-1}]$. $\square$

2 Land-Tamme excision for square zero extensions

In this section, we recall a theorem of Land-Tamme [LT19] concerning pullback squares on K-theory, and specialize it to the case of square zero extensions.

Theorem 2.1. *[LT19, Main theorem] Consider a pullback square of ring spectra:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

There exists a ring spectrum (the "circle-dot ring") $B \odot_A^D C$ together with a commutative diagram of ring spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr] & \\ C \arrow[r] \arrow[drr] & B \odot_A^D C \arrow[dr] & \\ & & D \end{tikzcd}$$

such that the homotopy filling in the outer square is the one from the start, such that, as a map of (B,C)-bimodule spectra, $B \odot_A^D C \to D$ is identified with $B \otimes_A C \to D$ and finally such that the inner square induces a pullback square of K-theory spectra:

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K(B) \arrow[d] \\ K(C) \arrow[r] & K(B \odot_A^D C) \end{tikzcd}$$

An important thing is that even when $B \odot_A^D C \to D$ is not an equivalence, if everything is connective, then connectivity estimates can be used to see how far the original square is from being a K-theory pullback. Indeed, if $B \odot_A^D C \to D$ is k-connective, then $K(B \odot_A^D C) \to K(D)$ is $(k+1)$-connective, by the main result of the previous section. This specializes to:

Corollary 2.2. *Suppose R is a connective ring spectrum, M a k-connective R-bimodule and $\tilde{R} \to R$ a square-zero extension of R by M.*

Then the pullback square

$$\begin{tikzcd}
\tilde{R} \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & R \oplus \Sigma M
\end{tikzcd}$$

induces a $2(k+1)$*-connective map* $K(\tilde{R}) \to \Omega_{K(R)} K(R \oplus \Sigma M)$*, where* $\Omega_{K(R)} K(R \oplus \Sigma M) := K(R) \times_{K(R \oplus \Sigma)} K(R)$.

Proof. Consider a diagram of spectra as follows:

$$\begin{tikzcd}
& X \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[drr] & Y \arrow[dr] & \\
& & Z
\end{tikzcd}$$

Then if $Y \to Z$ is j-connective, the induced morphism $X \times_Y X \to X \times_Z X$ is $(j-1)$-connective. Indeed, fibers commute with pullbacks, so that the fiber of this morphism is $\Omega \mathrm{fib}(Y \to Z)$, which is $(j-1)$-connective as claimed.

We apply this to

$$\begin{tikzcd}
& K(R) \arrow[d] \arrow[ddr] & \\
K(R) \arrow[r] \arrow[drr] & K(R \odot_{\tilde{R}}^{R \oplus \Sigma M} R) \arrow[dr] & \\
& & K(R \oplus \Sigma M)
\end{tikzcd}$$

So it now suffices to show that the map on bottom right corners is $2(k+1)+1$-connective, and so it suffices to show that $R \otimes_{\tilde{R}} R \to R \oplus \Sigma M$ is $2(k+1)$-connective. But now observe that we have a morphism of fiber sequences from $R \otimes_{\tilde{R}} (M \to \tilde{R} \to R)$ to $M \to R \to R \oplus \Sigma M$ (this comes from base-changing the domain from $\tilde{R}$ to R).

The middle term is an equivalence, so the fiber of $R \otimes_{\tilde{R}} R \to R \oplus \Sigma M$ is $\Sigma \mathrm{fib}(R \otimes_{\tilde{R}} M \to M)$.

Now we do the same but on the left and observe that $R \otimes_{\tilde{R}} M \to M$ has a section given by $M \simeq \tilde{R} \otimes_{\tilde{R}} M \to R \otimes_{\tilde{R}} M$, with fiber $M \otimes_{\tilde{R}} M$, so the fiber of $R \otimes_{\tilde{R}} M \to M$ is $\Sigma M \otimes_{\tilde{R}} M$ and so the fiber of $R \otimes_{\tilde{R}} R \to R \oplus \Sigma M$ is $\Sigma^2 M \otimes_{\tilde{R}} M$.

Because $\tilde{R}$ is connective and M is k-connective, this fiber is $2k+2 = 2(k+1)$-connective, as claimed. $\square$

3 The quantitative Dundas–McCarthy theorem

We now observe that the previous corollary can be phrased in terms of the first Goodwillie derivative of K-theory.

Definition 3.1. Let R be a ring and M a bimodule. Define $\tilde{K}(R,M) := \mathrm{fib}(K(R\oplus M) \to K(R))$ (where $K(R\oplus M)$ is the trivial square zero extension).

We can then take fibers over $K(R)$ in the previous corollary and obtain the following:

Corollary 3.2. *Let R be a connective ring, M a k-connective R-bimodule. Then the canonical map $\tilde{K}(R,M) \to \Omega\tilde{K}(R,\Sigma M)$ is $2(k+1)$-connective.*

Proof. Apply the previous corollary to $\tilde{R} = R\oplus M$ and then take fibers over $K(R)$. □

Recall the Dundas–McCarthy theorem:

Theorem 3.3 ([DM94])**.** *There is a natural equivalence* $\mathrm{THH}(R,\Sigma M) \simeq \mathrm{colim}_n \Omega^n \tilde{K}(R,\Sigma^n M)$.

Compare now the following estimate to [DM94, Theorem 3.4]

Corollary 3.4. *For any connective ring spectrum R and k-connective bimodule M, the map $\tilde{K}(R,M) \to \mathrm{THH}(R,\Sigma M)$ coming from the Dundas–McCarthy theorem is $2(k+1)$-connective .*

Proof. The previous corollary implies that all the maps $\tilde{K}(R,M) \to \Omega\tilde{K}(R,\Sigma M) \to \Omega^2\tilde{K}(R,\Sigma^2 M) \to \dots$ are $2(k+1)$-connective, therefore so is the colimit, which is, by the Dundas–McCarthy theorem, identified with $\tilde{K}(R,M) \to \mathrm{THH}(R,\Sigma M)$. □

4 Waldhausen's obstruction

We now have all the tools in hand to formulate a version of Waldhausen's obstruction.

We first deal with a special case.

Proposition 4.1. *Suppose $f : \tilde{R} \to R$ is a k-connective square zero extension of connective ring spectra, with $k \geq 1$. Then there is a $2(k+1)$-connective map* $\mathrm{fib}(K(\tilde{R}) \to K(R)) \to \mathrm{THH}(R, \Sigma\mathrm{fib}(f))$

Proof. Let $M = \mathrm{fib}(f)$.

By corollary 2.2, $K(\tilde{R}) \to \Omega_{K(R)} K(R\oplus\Sigma M)$ is $2(k+1)$-connective, therefore the same holds for $\mathrm{fib}(K(\tilde{R}) \to K(R)) \to \Omega\tilde{K}(R,\Sigma M)$, and the latter has a $2(k+2)$-connective map to $\mathrm{THH}(R,\Sigma M)$. □

Now, a second special case:

Proposition 4.2. *Suppose $f : \tilde{R} \to R$ is a k-connective map of connective ring spectra, with $k \geq 2$. Then $\pi_{k+1}\mathrm{fib}(K(\tilde{R}) \to K(R)) \cong \mathrm{HH}_0(\pi_0(R), \pi_k \mathrm{fib}(f))$.*

Remark 4.3. This fiber is known to be $(k+1)$-connective by the first section, so this is its lowest homotopy group.

Example 4.4. Consider a discrete ring R, and $R[t]$ the differential graded algebra with $|t| = 2$, viewed as a ring spectrum. Then we have a ring map $R[t] \to R, t \mapsto 0$.

This is 2-connective and so $\pi_3\mathrm{fib}(K(R[t]) \to K(R)) \cong \mathrm{HH}_0(R, R) = R/[R, R]$. If R is commutative and nonzero, this is nonzero and so $K(R[t]) \to K(R)$ is *not* an isomorphism : this shows that there is no "graded $\mathbb{A}^1$-invariance".

Example 4.5. There is a canonical morphism of ring spectra $ku \to \mathbb{Z}$ from connective k-theory. This is also 2-connective, and $\pi_2(\mathrm{fib}(ku \to \mathbb{Z}))$ is $\mathbb{Z}$, so $\pi_3\mathrm{fib}(K(ku) \to K(\mathbb{Z})) \cong \mathbb{Z}$.

Proof. We apply the previous proposition to $\tilde{R}_{\leq k+1} \to R_{\leq k+1}$: it is clearly k-connective, and we claim that it is also a square zero extension.

This is because of degree reasons: it is an n-small extension [Lur12, Definition 7.4.1.18.] (by [Lur12, Remark 7.4.1.20.], as its fiber is actually in degrees $< 2k$ - because $k + 1 < 2k$, because $k > 1$), and therefore by [Lur12, 7.4.1.23.], it is a square zero extension.

In particular, the previous proposition shows that $\mathrm{fib}(K(\tilde{R}_{\leq k+1}) \to K(R_{\leq k+1})$ is $\mathrm{THH}(R_{\leq k+1}, \Sigma\mathrm{fib}(f_{\leq k+1}))$ in degrees $< 2(k + 1)$.

Now the canonical morphism $\mathrm{fib}(K(\tilde{R}) \to K(R)) \to \mathrm{fib}(K(\tilde{R}_{\leq k+1}) \to K(R_{\leq k+1})$ is $(k+2)$-connective, because both morphisms $K(\tilde{R}) \to K(\tilde{R}_{\leq k+1})$ (similarly for R) are $(k+3)$-connective (as $\tilde{R} \to \tilde{R}_{\leq k+1})$ is $(k + 2)$-connective).

In particular, it is an isomorphism on π_{k+1}, and we now observe that $\pi_k\mathrm{fib}(f) \cong \pi_k\mathrm{fib}(f_{\leq k+1})$ to conclude that

$$\pi_{k+1}\mathrm{fib}(K(\tilde{R}) \to K(R)) \cong \pi_{k+1}\mathrm{THH}(R_{\leq k+1}, \Sigma\mathrm{fib}(f_{\leq k})) \cong \pi_0\mathrm{THH}(R_{\leq k}, \Omega^k\mathrm{fib}(f_{\leq k}))$$

$$\cong \mathrm{HH}_0(\pi_0(R), \pi_k\mathrm{fib}(f_{\leq k})) \cong \mathrm{HH}_0(\pi_0(R), \pi_k\mathrm{fib}(f))$$

(where now $k + 1 < 2(k + 1)$ with no assumptions on k). ☐

Remark 4.6. Observe that we only used $k > 1$ to prove that $\tilde{R}_{\leq k+1} \to \tilde{R}_{\leq k+1}$ is a square zero extension. So we can have a more precise statement where this is our assumption, rather than the assumption on k.

Now, if $\tilde{R} \to R$ is 1-connective, *and* the map $\pi_1(\mathrm{fib}) \otimes \pi_1(\mathrm{fib}) \to \pi_2(\mathrm{fib})$ is nonzero (e.g. in the case of $\mathbb{S} \to \mathbb{Z}$), then we have to work somewhat differently.

We can still reduce to $\tilde{R}_{\leq k+1} \to R_{\leq k+1}$ because the connectivity estimates from the last part of the proof do not depend on k being ≥ 2, and so we may assume $\tilde{R}, R$ are $(k + 1)$-truncated.

In this case, we will introduce a different intermediary ring. We prove the following:

Proposition 4.7. *Let $k \geq 1$.*

Suppose $\tilde{R} \xrightarrow{f} R$ is a k-connective morphism of connective $(k+1)$-truncated ring spectra. Then there exists a connective $(k+1)$-truncated ring spectrum S together with a factorization of f as $\tilde{R} \to S \to R$ such that :

1. *Both $\tilde{R} \to S, S \to R$ are k-connective*
2. *On π_{k+1}, $\tilde{R} \to S$ is surjective and has the same kernel as $\tilde{R} \to R$. In other words it realizes the map $\pi_{k+1}(\tilde{R}) \to \mathrm{im}(\pi_{k+1}(f)))$.*
3. *$\tilde{R} \to S$ is an isomorphism on π_k.*

Let us first assume that this proposition holds, and let us see how this solves our issue.

Conditions 1,2, 3 imply that $\tilde{R} \to S$ is $(k+1)$-connective, so it induces a $(k+2)$-connective map on K-theory.

Furthermore, we have the following factorization of our map on K-theory : $K(\tilde{R}) \to K(S) \to K(R)$.

This implies that we have a fiber sequence of the form:

$$\mathrm{fib}(K(\tilde{R}) \to K(S)) \to \mathrm{fib}(K(\tilde{R}) \to K(R)) \to \mathrm{fib}(K(S) \to K(R))$$

Because the leftmost one is $(k+2)$-connective, it follows that the rightmost morphism induces an isomorphism on π_{k+1}. So we are in fact reduced to computing this.

But now, $S \to R$ is k-connective, both are $(k+1)$-truncated so π_{k+1} of its fiber is just the kernel of $\pi_{k+1}(S) \to \pi_{k+1}(R)$ which, by condition 2, is trivial. In particular, the fiber is concentrated in degree k, and so again by [Lur12, Theorem 7.4.1.23], because $k < 2k$ (this time even for $k = 1$!), we find that $S \to R$ is a square zero extension.

In particular, by proposition 4.1, we find a $2(k+1)$-connective morphism from the fiber on K-theory to $\mathrm{THH}(R, \Sigma\mathrm{fib}(S \to R))$.

In particular, the π_{k+1} is the same, so that $\pi_{k+1}(\mathrm{fib}(K(\tilde{R}) \to K(R)) \cong \mathrm{HH}_0(\pi_0(R), \pi_k(\mathrm{fib}(S \to R)))$.

The last check to do is that $\pi_k\mathrm{fib}(S \to R) \cong \pi_k\mathrm{fib}(\tilde{R} \to R)$, but this is clear from conditions 1,2,3. Note that this isomorphism is clearly induced from a map between the fibers, which is easily seen to be a morphism of $\tilde{R}$-bimodules, so that this isomorphism is also one of $\pi_0(\tilde{R})$-bimodules - in particular, because $\pi_0(\tilde{R}) \cong \pi_0(R)$ in a compatible way with this isomorphism of $\pi_0(\tilde{R})$-bimodules, we can write:

$$\pi_{k+1}(\mathrm{fib}(K(\tilde{R}) \to K(R)) \cong \mathrm{HH}_0(\pi_0(\tilde{R}), \pi_k\mathrm{fib}(\tilde{R} \to R)))$$

Remark 4.8. Note that here it is strictly better to write $\pi_0(\tilde{R})$, because the fiber does not have an R-bimodule structure if the extension is not a square-zero extension.

For the previous case, one might argue that $\pi_0(R)$ is better suited, as the more precise connectivity estimates involve $\mathrm{THH}(R, -)$, but in fact what we plug in there is $(k+1)$-connective, so with $\mathrm{THH}(\tilde{R}, -)$ it would almost be as good.

In any case, we obtain Waldhausen's result:

Corollary 4.9. *Suppose $f : \tilde{R} \to R$ is a k-connective map of connective ring spectra, with $k \geq 1$. Then $\pi_{k+1}\mathrm{fib}(K(\tilde{R}) \to K(R)) \cong \mathrm{HH}_0(\pi_0(R), \pi_k\mathrm{fib}(f))$.*

We conclude with a proof of proposition 4.7:

Proof. Observe that $\tilde{R} \to \tilde{R}_{\leq k}$ is $(k+1)$-connective, and because R is $(k+1)$-truncated, this implies again for degree reasons that it is a square zero extension [Lur12, Theorem 7.4.1.23.].

In particular we can find a pullback square of ring spectra as follows:

$$\begin{tikzcd}
\tilde{R} \arrow[r] \arrow[d] & \tilde{R}_{\leq k} \arrow[d] \\
\tilde{R}_{\leq k} \arrow[r] & \tilde{R}_{\leq k} \oplus \Sigma^{k+1}\pi_{k+1}\tilde{R}
\end{tikzcd}$$

We can do the same for R, and in fact, because [Lur12, 7.4.1.23.] establishes an equivalence of ∞-categories between n-small extensions and square-zero extensions, this pullback square is natural in the extension we started with , so that we can compare the two squares. Furthermore, we obtain in particular that the canonical map $\pi_{k+1}\tilde{R} \to \pi_{k+1}R$ is a map of $\tilde{R}_{\leq k}$-bimodules (although this is easy to check, as they are both discrete and so it suffices to show that it is a map of $\pi_0\tilde{R}$-bimodules, which is clear).

This allows us to construct a similar pullback square, which we can compare to both, where we put $\mathrm{im}(\pi_{k+1}(f)) \cong \pi_{k+1}\tilde{R}/\ker(\pi_{k+1}(f))$ instead of $\pi_{k+1}\tilde{R}$, namely, we can define S to fit in a pullback square

$$\begin{tikzcd}
S \arrow[rr] \arrow[d] & & \tilde{R}_{\leq k} \arrow[d] \\
\tilde{R}_{\leq k} \arrow[r] & \tilde{R}_{\leq k} \oplus \Sigma^{k+1}\mathrm{im}(\pi_{k+1}(f)) &
\end{tikzcd}$$

It follows from our discussion that we get a factorization $\tilde{R} \to S \to R$ as desired and conditions 1,2,3, are easily checked at the additive level. $\square$

5 A bit more

Proposition 4.1 allows for stronger results than Waldhausen's, if we have more information about our ring map $\tilde{R} \to R$, e.g. in the case $F[t] \to F, |t| > 0$.

Let us be explicit about this example (which prompted this whole thing anyways). In this case, if $|t| = k > 1$, then $F[t]_{\leq 2k-1} \to F$ is still a square zero extension, and it is k-connective. In particular, we find a $2(k+1)$-connective map from $\mathrm{fib}(K(F[t]_{\leq 2k-1} \to F[t]))$ to $\mathrm{THH}(F, \Sigma\Sigma^k F) = \Sigma^{k+1}\mathrm{THH}(F)$.

But also, $K(F[t]) \to K(F[t]_{\leq 2k-1})$ is now $2k+1$-connective, as $F[t] \to F[t]_{\leq 2k-1}$ is $2k$-connective. In particular, in the fiber sequence

$$\mathrm{fib}(K(F[t]) \to K(F[t]_{\leq 2k-1})) \to \mathrm{fib}(K(F[t]) \to K(F)) \to \mathrm{fib}(K(F[t]_{\leq 2k-1}) \to K(F))$$

in degrees $\leq 2k$ we find $\pi_j(\mathrm{fib}(K(F[t]) \to K(F))) \cong \pi_{j-k-1}\mathrm{THH}(F)$ which is better than just π_{k+1} (note that $\mathrm{THH}(F)$ is understood when F is perfect of positive characteristic, and in characteristic 0 it is the same as Hochschild homology, hence it is related to de Rham cohomology).

In degree $2k+1$, we can apply Waldhausen's obstruction to get an exact sequence

$$F \to \pi_{2k+1}\mathrm{fib}(K(F[t]) \to K(F)) \to \pi_k\mathrm{THH}(F) \to 0$$

(e.g. when $F = \mathbb{F}_p$, this fiber is all p-torsion up to degree $\leq 2k+1$, while $K(F)$ is mostly prime-to-p torsion (except in degree 0).

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