A note on the universal coefficient theorem with twisted coefficients

Introduction

The usual universal coefficient theorem in algebraic topology can be seen as a version of the natural equivalence

$$\hom(\mathrm{colim}_X \mathbb{Z}, A) \simeq \lim_X \hom(\mathbb{Z}, A)$$

at the level of homology groups.

In this note, we simply wish to point out that this equivalence holds if we replace $\mathbb{Z}$ by a different local coefficient system on our space X.

We'll start by pointing out how this works for the usual universal coefficient theorem (henceforth UCT), and see that this generalizes with little trouble to the twisted setting.

1 The usual UCT

Classically, the UCT is proved by studying the chain complex of singular chains and its dual, and using the fact that subgroups of free abelian groups are free abelian.

This approach is not entirely satisfactory for the following reasons:

- The proof itself is not "homotopy invariant" : it relies on picking a specific chain level model for the homotopy type of the singular chains, and on writing that as a direct sum of "two term complexes", again, a chain level analysis. We contend here that this statement is a statement about homotopy types and so ideally should have a "homotopy invariant" proof.
- It specializes to $\mathbb{Z}$ very early on, in particular it is not clear how to generalize it to other rings (not PIDs) - where we do, in fact, have analogous statements, though more subtle.
- The previous points specialize to the problem of the proof not being easily generalizable to twisted coefficient systems.

Our presentation (which, we must stress, is not new, or original in any way - and in fact a careful analysis shows that the constructions are the same as the classical proof, but they will be more clearly generalizable and "homotopy invariant") will circumvent these issues, and in particular generalize to other coefficient systems.

Remark 1.1. We will use ∞-categories in two ways : essentially as an organizing tool for the usual universal coefficient theorem where this language is obviously not necessary; but also as an essential ingredient for our definition of cohomology with twisted coefficients.

Needless to say, ∞-categories aren't necessary for any of this, but as will hopefully be apparent from the twisted coefficients story, they can help make the presentation smoother, and help sort out the formal parts of the story from the non formal parts.

The main observation that allows us to move completely to the derived setting is the following:

Proposition 1.2. *Let X be a topological space. Let $C^*(X;A)$ denote its usual singular cochain complex with coefficients in A, and C_* the singular chain complex. Then, in the derived category of abelian groups, we have*

$$C^*(X;A) \simeq \mathrm{RHom}(C_*(X;\mathbb{Z}), A)$$

Proof. This is almost immediate: by definition, it is Hom of the same thing, and because $C_*(X;\mathbb{Z})$ is a complex of free abelian groups, it is also RHom. □

Remark 1.3. This proposition can be reinterpreted if we have a homotopy theoretic definition of chains and cochains: chains in X with coefficients in A should be the (homotopy) colimit of the constant ∞-functor $hX \to \mathbf{D}(\mathbb{Z})$ from the homotopy type of X, with value A in the derived ∞-category of abelian groups, and cochains should be the (homotopy) limit of the same thing.

The proof of this fact is the combination of the fact that $C_*(-;\mathbb{Z})$ is a left Quillen functor $\mathbf{sSet} \to \mathrm{Ch}(\mathbb{Z})$ together with the fact that the constant ∞-functor $hX \to \mathbf{Spaces}$ with values a point has (homotopy) colimit (the homotopy type of) X. The latter is essentially a version of the ∞-categorical Grothendieck construction. It is instructive to work out why this should be true intuitively, using a naive understanding of homotopy colimits.

Now we're going to simply work in the derived (∞-)category $\mathbf{D}(\mathbb{Z})$ and prove the following more general (and completely algebraic) fact:

Theorem 1.4 (Universal coefficient theorem)**.** *Suppose R is a PID, and $C \in \mathbf{D}(R)$. Then for any n, and any (ordinary) R-module A, there is a natural short exact sequence:*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C); A) \to H^n(\mathrm{RHom}(C, A)) \to \hom(H_n(C), A) \to 0$$

Remark 1.5. Here H^n means H_{-n}; but we're thinking of $\mathrm{RHom}(C, A)$ as a "cochain complex", hence the notation.

Corollary 1.6. *Let X be a topological space.*

Suppose R is a PID and A a module over R. For any n, we have a natural short exact sequence:

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R); A) \to H^n(X;A) \to \hom(H_n(X;R), A) \to 0$$

Proof of theorem 1.4. For the proof, we use the Postnikov filtration of C. Namely, note that $H^n(\mathrm{RHom}(C, A)) = H_0(\Sigma^n \mathrm{RHom}(C, A))$ and $\Sigma^n \mathrm{RHom}(C, A) \simeq \mathrm{RHom}(\Omega^n C, A)$. In other words, we are reduced to the case $n = 0$, and there we have a co/fiber sequence (also known as exact triangle, if we're working at the homotopy category level)

$$\tau_{\geq 0} C \to C \to \tau_{<0} C$$

$\tau_{\geq 0}$ and $\tau_{<0}$ are intrinsically defined functors on $\mathbf{D}(\mathbb{Z})$, but they also have chain level models, namely $\tau_{\geq 0} C$ is the same complex as C in positive degrees, in degree 0 it's $\ker(d_0 : C_0 \to C_{-1})$, and it's 0 in negative degrees.

For $\tau_{\leq 0}$, you do something dual and take $C_0/\mathrm{im}(d_1)$ instead (and $\tau_{<0} = \tau_{\leq -1}$).

This co/fiber sequence induces a co/fiber sequence after applying $\mathrm{RHom}(-, A)$, and so we get a co/fiber sequence

$$\mathrm{RHom}(\tau_{<0} C, A) \to \mathrm{RHom}(C, A) \to \mathrm{RHom}(\tau_{\geq 0} C, A)$$

which induces a long exact sequence of homology groups.

We're interested in the H_0-part of that, so we need to understand H_0 and H_{-1} of the left hand term, and H_0 and H_1 of the right hand term.

"Fortunately", A is concentrated in degree 0, so $H_1 \mathrm{RHom}(\tau_{\geq 0} C, A) \cong H_0 \mathrm{RHom}(\Sigma \tau_{\geq 0} C, A) \cong \hom_{D(\mathbb{Z})}(\Sigma \tau_{\geq 0} C, A) = 0$ (something concentrated in positive degrees has no maps to something concentrated in nonpositive degrees - the concrete way to see that is to think of projective resolutions of the source)

Furthermore (and not similarly !)

$$H_{-1} \mathrm{RHom}(\tau_{<0} C, A) \cong H_0 \mathrm{RHom}(\Omega \tau_{<0} C, A) \cong \hom_{D(\mathbb{Z})}(\Omega \tau_{<0} C, A) = 0$$

because R is a PID so there are no derived maps from an object concentrated in degrees ≤ -2 to an object concentrated in degrees ≥ 0. Therefore, we are reduced to understanding both H_0's.

For that, we use the Postnikov truncation again: we have a co/fiber sequence

$$H_{-1}(C)[-1] \to \tau_{<0} C \to \tau_{<-1} C$$

, which induces a co/fiber sequence upon applying $\mathrm{RHom}(-, A)$, and thus a long exact sequence on homology. In degree 0, we find

$$0 = H_0(\mathrm{RHom}(\tau_{<-1} C, A)) \to H_0(\mathrm{RHom}(\tau_{<0} C, A)) \to H_0(\mathrm{RHom}(H_{-1}(C)[-1], A)) \to H_{-1}(\mathrm{RHom}(\tau_{<-1} C, A))$$

. The last term is $H_0(\mathrm{RHom}(\Omega \tau_{<-1} C, A))$ which vanishes for the same reason, so that the desired H_0 is exactly $H_{-1}(\mathrm{RHom}(H_{-1} C, A)) \cong \mathrm{Ext}^1_R(C, A)$.

We do the same for $\mathrm{RHom}(\tau_{\geq 0} C, A)$: by using the Postnikov filtration

$$\tau_{>0} C \to \tau_{\geq 0} C \to H_0(C)$$

and using again the fact that A is concentrated in degree 0 and that there are no maps from something concentrated in positive degrees to something in nonpositive degrees, we find that its H_0 is $\hom(H_0(C), A)$.

The desired short exact sequence follows at once from this fact.

Every isomorphism here is clearly natural in C, so this also proves naturality in C.

□

Remark 1.7. A lot of these constructions are the same as classically, but this time they are clearly homotopy invariant.

Remark 1.8. The classical UCT also states that these sequences are split, but not naturally.

It is not hard to prove this fact: over a PID, any chain complex is formal, that is, equivalent to $\bigoplus_n H_n(C)[n]$ (although not naturally so). In particular, because these sequences are natural, it suffices to prove that these sequences are split for these chain complexes, which is an easy exercise.

The proof of this extension of the above statement "necessarily" uses models, because the splitting is not natural, and is not independent of the chosen model for C.

Remark 1.9. From this proof, we can see that the main input of this theorem is the Postnikov truncation. If we are not over a PID, some Ext^2's (and higher ext-groups) might not vanish, so the proof does not go through as such, but we can still filter C by its Postnikov tower.

What we get is no longer a simple short exact sequence but rather an object that can package all these higher ext's along this Postnikov filtration. This object is a spectral sequence, and one can in fact construct a spectral sequence with E_2-term $\mathrm{Ext}^p(H_q(C), A)$ weakly converging to $H^{p+q}(\mathrm{RHom}(C, A))$. With suitable hypotheses, it actually converges.

As opposed to classical proof of the UCT over a PID where this extension to non PIDs is not obvious, here the way to approach it is clear: the Postnikov filtration gives rise to a spectral sequence, period. The convergence analysis is then the usual analysis of when filtrations give rise to convergent spectral sequences.

Exercise 1.10. Write a similar argument for the homological UCT, again using a Postnikov filtration of C to filter $C \otimes_R^L A$; and noting that $C_*(X; A) \simeq C_*(X; R) \otimes_R^L A$.

More generally, write down a UCT for the left derived functors of a right exact functor out of the category of R-modules for a PID R.

2 The twisted UCT

The argument for the twisted UCT will follow essentially the same reasoning.

There is one notable difference in the analogue of proposition 1.2. Namely, for this one the ∞-categorical interpretation is much more suitable.

We will take this point of view here. Consider a space X (and now by space we mean "object of the ∞-category of spaces", so something closer to a homotopy type than a topological space).

Definition 2.1. An R-valued local coefficient system on X is an ∞-functor $X \to \mathbf{D}(R)$.

Remark 2.2. How does that relate to the usual notion ?

Suppose our coefficient system has values in the heart of $\mathbf{D}(R)$, which is equivalent to the usual category $\mathbf{Mod}_R$ of R-modules. Then it facts uniquely through $\tau_{\leq 1}X$, also known as the fundamental groupoid of X. So usual local coefficient systems, which are functors $\Pi_1(X) \to \mathbf{Mod}_R$, are special cases of our notion of local system.

More generally, while local coefficient systems correspond to locally constant sheaves on the (topological space) X, our local systems correspond to locally constant sheaves with values in the derived ∞-category.

Remark 2.3. Even in the classical case, the picture "functors $\Pi_1(X) \to \mathbf{Mod}_R$" is not enough to define twisted co/homology, because the latter depends on the whole space X, rather than only its underlying 1-type. That's why unless X is a 1-type, you can't define them as derived functors.

Remark 2.4. Suppose we're only interested in discrete group co/homology. Then we'll look at spaces of the form BG, G a discrete group, which *are* 1-types. In particular, for those, $\mathrm{Fun}(BG, \mathbf{D}(R))$ is in fact the derived category of $\mathrm{Fun}(BG, \mathbf{Mod}_R)$, and for those, co/homology with local coefficients is in fact the derived functor of the fixed points/orbits functor, in other words the "homotopy limit" or "homotopy orbits" functor.

It can be proved that this last remark is a general phenomenon, namely if I have a classical local coefficient system $\Pi_1(X) \to \mathbf{Mod}_R$, then its co/homology with local coefficients is the homology of its (homotopy) limit/colimit when viewed as a functor $X \to \mathbf{D}(R)$.We will not prove this in detail here and instead take it as our definition:

Definition 2.5. Suppose $A : X \to \mathbf{D}(R)$ is a local coefficient system. We define

$$H_*(X; A) := H_*(\mathrm{colim}_X A)$$

and

$$H^*(X; A) := H^*(\lim_X A)(= H_{-*}(\lim_X A))$$

Remark 2.6. As we explained, the agreement of this definition with the classical one in the case of a discrete local coefficient system is not too hard to check when $X = BG$ is a 1-type, and it also holds but is slightly more subtle for a general space X (see exercise 2.15 and the preceeding discussion)

Remark 2.7. Take A to be a constant functor (and we suppress the notational distinction between it and its value for simplicity). Then remark 1.3 states that this definition agrees with the classical definition as well.

With this definition, the analogue of proposition 1.2 is immediate, namely:

Proposition 2.8. *Let R be a commutative ring, and suppose $A : X \to \mathbf{D}(R)$ is a local coefficient system, $Q \in \mathbf{D}(R)$, and define* $\mathrm{RHom}(A, Q)$ *to be the coefficient system obtained by postcomposing with* $\mathrm{RHom}(-, Q) : \mathbf{D}(R) \to \mathbf{D}(R)$*. Then* $\lim_X \mathrm{RHom}(A, Q) \simeq \mathrm{RHom}(\mathrm{colim}_X A, Q)$*.*

Proof. $\mathrm{RHom}(-, Q)$ sends colimits to limits. □

Remark 2.9. If we do not require R to be commutative, the same statement is true but we need to have $\mathrm{RHom}(-, Q)$ be a functor $\mathbf{D}(R) \to \mathbf{D}(\mathbb{Z})$, and the equivalence is an equivalence in $\mathbf{D}(\mathbb{Z})$.

And we can now proceed the same way to state a UCT for co/homology with local coefficients. Note the funny fact that if R is commutative, then the above proposition identifies $\lim_X A^*$ in terms of an RHom with values in a discrete R-module, so that the following statement is a direct *consequence* of the algebraic UCT, theorem 1.4:

Proposition 2.10 (Twisted UCT). *Let R be a PID, and $Q \in \mathbf{Mod}_R$ Suppose A is a local coefficient system. Then we have natural short exact sequences:*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;A);Q) \to H^n(X;\mathrm{RHom}(A,Q)) \to \hom(H_n(X;A),Q) \to 0$$

Proof. Apply theorem 1.4, propostion 2.8 as well as the definitions. □

In particular, if we want to stay in the world of classical local coefficient systems, we have to make some slightly stronger assumptions on either A or Q.

The following is our statement of the twisted UCT for classical local coefficient systems (compare with Exercise J.4 from [Spa89, Ch.5])

Corollary 2.11. *Suppose now that A is a classical local coefficient system. Suppose that either all its values are projective R-modules, or that Q is an injective R-module.*

Then $\hom(A,Q)$ is also a coefficient system, and we have short exact sequences:

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;A);Q) \to H^n(X;\hom(A,Q)) \to \hom(H_n(X;A),Q) \to 0$$

Proof. Apply the previous proposition, noting that if Q is injective or A projective, $\mathrm{RHom}(A,Q) = \hom(A,Q)$. □

We conclude this note by sketching a proof of the fact that for a classical coefficient system $A : \Pi_1(X) \to \mathbf{Mod}_R$, our definition of its co/homology agrees with the classical one.

Suppose for simplicity and clarity that X is connected (this is not necessary, but in any case one can also deduce the general case from the connected case quite easily) with basepoint x; then $\tau_{\leq 1}X \simeq B\pi_1(X,x)$. We therefore have a canonical map $t : X \to B\pi_1(X,x)$, whose homotopy fiber over the canonical point of the latter is identified with $\tilde{X}$, the universal cover of X, and the inclusion of the fiber $p : \tilde{X} \to X$ is the covering map.

So if we now start from a local coefficient system $A \in \mathrm{Fun}(X, \mathbf{D}(R))$, its colimit can be computed by first left Kan extending along $X \to B\pi_1(X,x)$, and then taking the colimit along $B\pi_1(X,x)$, i.e. the homotopy orbits.

Evaluated at the canonical point of $B\pi_1(X,x)$, the value of this left Kan extension is $\mathrm{colim}_{y\in\tilde{X}} A(p(y))$, and it carries a certain $\pi_1(X,x)$-action (in the homotopy coherent sense).

If A is a classical coefficient system, that is it factors through $B\pi_1(X,x)$, then it can be written as $A' \circ t$, so $A \circ p = A' \circ t \circ p$ has a canonical equivalence to the constant local system at A, so this colimit is (see remark 1.3) precisely $C_*(\tilde{X};R) \otimes^L_R A = C_*(\tilde{X};R) \otimes_R A$ (the usual tensor product is derived, because $C_*(\tilde{X};R)$ is a complex of free R-modules).

Now this is an object of $\mathrm{Fun}(B\pi_1(X,x), \mathbf{D}(R))$ so an object of $\mathbf{D}(R)$ with a $\pi_1(X,x)$-action, and one can work out this action to be exactly the diagonal action on $C_*(\tilde{X};R) \otimes_R A$.

Furthermore, $C_*(\tilde{X};R)$ is free as an $R[\pi_1(X,x)]$-module, so that the orbits of this action coincide with the homotopy orbits, therefore the homotopy orbits of $C_*(\tilde{X};R) \otimes_R A$, are just the orbits, i.e. $C_*(\tilde{X};R) \otimes_{R[\pi_1(X,x)]} A$. We thus see that this is the complex that defines classical homology with (classical) local coefficients.

But we said that the homotopy orbits of this action was exactly the colimit over X, i.e. this recovers our definition of homology with local coefficients. Therefore the two definitions agree.

Remark 2.12. This is almost a complete and formal proof. The only really sketchy point is the identification of the action on $C_*(\tilde{X}; R) \otimes_R A$, which requires a bit more care than what we did.

If one does not wish to enter such subtleties, one can take this as an argument for why we should *define* homology with local coefficients the way we did, rather than an actual proof that they coincide.

Exercise 2.13. Provide a similar sketch for *co*homology with local coefficients. (or look at the next exercise)

Exercise 2.14. Assuming the case of homology, prove (formally rather than just a sketch) the case of cohomology.

Exercise 2.15. Provide a formal proof of the homology case if $X = BG$, for G a discrete group, accepting the following (hopefully believable) facts:

i. ∞-categorical co/limits coincide with homotopy co/limits for nice enough model categories (these include $\mathrm{Ch}(S)$ for any ring S),

ii. for a small category I and a nice enough model category C, the ∞-category associated to $\mathrm{Fun}(I, C)$ is $\mathrm{Fun}(I, C_\infty)$, where C_∞ is the ∞-category associated to C.

iii. All the things we call BG agree in all possible senses.

(Indication : you will need to prove, among other things, that the classical derived functors for homotopy orbits agree with homotopy colimits in the model category $\mathrm{Fun}(BG, \mathrm{Ch}(R))$; and you will want to relate algebraic group homology with homology of BG with local coefficients)

Question 2.16. Is it possible to deduce formally, without much work, the general case from the BG case?

Remark 2.17. The answer sounds like it might be yes, but there is still the difficulty of working out the action on $C_*(\tilde{X}; R) \otimes_R A$.

The main issue is that there is no simple model category whose associated ∞-category is $\mathrm{Fun}(X, \mathbf{D}(R))$ when X is not a 1-type.

Remark 2.18. We hope our use of model categories in these last paragraphs makes our philosophical understanding of these objects clear: one should resort to them only to relate classical, or 1-categorical constructions to homotopical or derived or ∞-categorical constructions.

They are *models* for the ("correct") homotopical objects.

Let us end this note by giving argument for our claim about the action on $C_*(\tilde{X}; R) \otimes_R A$. The point is the following : A is pulled back along t, i.e; it is of the form t^*A. Therefore its left Kan extension $t_!$ satisfies the projection formula: $t_! t^* A \simeq t_!(R \otimes_R t^* A) \simeq t_! R \otimes_R A$. This reduces the claim about the action to a claim about the constant local system with value R.

(to prove the projection formula, observe that there is a natural morphism going from the left hand side to the right hand side, so that we can then just check that it's an equivalence by evaluating, using the colimit formula from before)

It then follows from the fact that the equivalence $\mathrm{colim}_Y R \simeq C_*(Y; R)$ is natural in Y - because the action on both sides (for $Y = \tilde{X}$) is induced by the deck transformations, naturality implies that $t_! R$ has the same $\pi_1(X, x)$-action as $C_*(\tilde{X}; R)$.

To make this fully precise, one would need to state this naturality precisely; which can be done, but let us not get into it.

References

[Spa89] Edwin H Spanier. *Algebraic topology.* Springer Science & Business Media, 1989.